12 Tutorial 12 – December 19th, 2017

12.1 Give an example of a simple directed graph which has 9 vertices, 12 directed edges, 2 components of connectivity, and 4 strongly connected components.

12.2 Give an example of a simple directed graph which has 9 vertices, 12 directed edges, 2 components of connectivity, and 2 strongly connected components.

12.3 Is it possible that a simple directed graph has more connected components than strongly connected components?

12.4 Let G be a simple strongly connected directed graph without loops, which has n vertices. Give the smallest and biggest numbers of edges that G can have. Justify your answers.

12.5 Given a simple directed graph G with the set of vertices $V = \{1, \ldots, 8\}$, and the set of edges is given by the following table.

IV	1	1	1	2	2	3	3	3	4	5	6	6	7	8
TV	2	3	5	1	3	1	4	6	1	2	3	7	8	6

Decide whether G is an Euler graph; in other words, either find a closed directed Euler trail, or justify that such trail does not exist.

Solution. First of all we calculate the in-degrees and out-degrees of all vertices in G. If there is a vertex v for which $d^+(v) \neq d^-(v)$, then a closed directed Euler trail does not exist.

We have

	1	2	3	4	5	6	7	8
$d^-(v)$	3	2	3	1	1	2	1	1
$d^+(v)$	3	2	3	1	1	2	1	1

Since $d^+(v) = d^-(v)$ for all vertices of G, we can start the algorithm for finding a closed directed Euler trail.

We start in an arbitrary vertex, say 1, and we randomly form a maximal trail T (the trail T will be given as a sequence of edges only). Then T is

$$T := (1,2), (2,1), (1,3), (3,1), (1,5), (5,2), (2,3), (3,4), (4,1).$$

The trail T cannot be extended, since there is no edge with its initial vertex 1 which is not contained in T.

Since T does not contain all edges of G, there must be a vertex w on T for which not all edges incident to w are contained in T. (If this was not true, then G would be disconnected and a closed Euler trail would not exist). All edges incident to 1 and 2 are contained in T, hence the first vertex with edges incident to it and not in T is 3. We randomly form a directed trail T_1 starting in 3 and containing only edges not in T. T_1 is

$$T_1 := (3,6), (6,3).$$

We insert T_1 into T and get a new closed trail T:

$$T := (1,2), (2,1), (1,3), (3,6), (6,3), (3,1), (1,5), (5,2), (2,3), (3,4), (4,1).$$

Now, all edges incident to 1,2,3 are contained in T, and we choose 6 with the edge $(6,7)$ not in T. We randomly construct a maximal trail T_2 starting in 6 consisting of edges not in T. We have

$$T_2 := (6,7), (7,8), (8,6).$$

We insert T_2 in T and get a new trail

$$T := (1,2), (2,1), (1,3), (3,6), (6,7), (7,8), (8,6), (6,3), (3,1), (1,5), (5,2), (2,3), (3,4), (4,1).$$

Now, T contains all edges of G, hence it is a closed directed Euler trail (and G is an Euler graph).

12.6 Given a simple directed graph G with the set of vertices $V = \{1, \ldots, 8\}$, and the set of edges is given by the following table.

IV	1	1	1	2	3	4	5	5	6	7	7	8	8
TV	5	6	8	1	7	2	1	2	8	1	5	3	7

Decide whether G contains an open directed Euler trail, or justify that such trail does not exist.

12.7 Give an example of a directed graph with 10 vertices and 12 edges that

1. is a Hamiltonian graph;
2. is not a Hamiltonian graph.

Answers

12.3 It is not possible.

12.4 The smallest number is n, the biggest number is $n\,(n-1)$.

12.6 Yes, there exists an open directed Euler trail T.

$$T := (4,2), (2,1)(1,5), (5,1), (1,6)(6,8), (8,3), (3,7), (7,1), (1,8), (8,7), (7,5), (5,2).$$