2 Tutorial 2 – October 10th, 2017

2.1 Write formulas of predicate logic corresponding to the following sentences. Use the predicate symbols mentioned in the text.

a) Somebody has a musical ear (E) and somebody does not.

b) Some children (C) do not like chocolate (H).

c) Not every talented painter (P) exhibits his/her pictures in the National Gallery (G).

d) Only students (S) can buy cold suppers (C).

e) Not every person (P) who has expensive skis (E), is a bad skier (B).

2.2 Find predicate symbols, constant symbols, and functional symbols which are needed to formalize the following statements as formulas of predicate logic.

a) The square of an odd number is always odd.

b) If a natural number is divisible by six then it is also divisible by three.

c) There are numbers a, b, and c such that the sum of the squares of a and b equals the square of c.

d) Every tetragon whose diagonals are equal is a rhombus.

2.3 Given predicate symbols P, Q, functional symbols f, s, and constant symbols a, and b. Moreover, Q and f are binary, and P and s unary. Decide which of the following strings are well formed formulas of predicate logic. If a string is a formula draw its derivation tree.

a) $Q(f(a), s(b))$;

b) $P(f(x, s(x)))$;

c) $\forall x\,(Q(f(x, a), b) \Rightarrow P(f(a, b)))$;

d) $(\forall x\, P(f(x, b)) \Rightarrow (\exists y\, Q(f(y), P(y)))$;

e) $(P(x) \wedge Q(f(x, y)) \Rightarrow (\exists y\,(P(y) \vee P(f(y)))$;

f) $\exists x\,(P(Q(x, y)) \Rightarrow Q(a, b))$;

g) $\forall x(P(x) \Rightarrow (\exists y\, Q(x, y)))$.

2.4 A predicate logic language has the following special symbols: predicate symbols P, Q, and functional symbols f, g. All symbols are unary.

Given an interpretation $\langle U, [\![-]\!]\rangle$, where U is the set of all people,
f is interpreted as a father, i.e. $[\![f]\!]$ assigns to a person x his/her father,
g is interpreted as a mother, i.e. $[\![g]\!]$ assigns to a person x his/her mother,
P is interpreted as the property "to play piano", and Q is interpreted as the property "to play guitar".

Write the sentences that correspond to the following sentences in the given interpretation:

a) $\forall x\,(P(f(x)) \vee Q(g(x)))$;

b) $\exists x\,(P(g(x)) \wedge Q(f(x)))$;

c) $\forall x\,((P(f(x)) \vee Q(g(x))) \Rightarrow (P(x) \vee Q(x)))$;

d) $\exists x\,(P(g(f(x)))$;

e) $\exists y\,(P(y) \wedge \neg Q(f(g(y))))$;

2.5 For each of the following sentences, decide whether it is a tautology, a contradiction, or a satisfiable sentence which is not a tautology. (P is a unary, and Q is a binary predicate symbol.)

a) $(\exists x\, P(x)) \vee (\exists x\, \neg P(x))$;

b) $\forall x\, (P(x) \vee \neg P(x))$;

c) $(\exists x\, P(x)) \Rightarrow (\forall x\, P(x))$;

d) $(\forall x\, P(x)) \wedge (\exists x\, \neg P(x))$;

e) $\forall x\, [\exists y\, Q(x,y) \vee \forall z\, \neg Q(x,z)]$.

2.6 Decide whether the following sets of sentences are satisfiable or not. State the reasons. (P and R are unary predicate symbols, Q is a binary predicate symbol.)

a) $S = \{\forall x\, \exists y\, Q(x,y), \forall x\, \neg Q(x,x)\}$;

b) $S = \{\exists x\, \forall y\, Q(x,y), \forall x\, \neg Q(x,x)\}$;

c) $S = \{\forall x\, (P(x) \vee R(x)), \neg\exists x\, R(x), \neg P(a)\}$.

2.7 For the sentence φ find a sentence ψ which is tautologically equivalent to $\neg\varphi$ and such that ψ has the negations only before atomic formulas. (P is a unary predicate symbol, R is a binary predicate symbol, and a is a constant symbol.)

a) $\forall x\, [P(x) \Rightarrow (\exists y\, (P(y) \wedge R(x,y)))]$;

b) $P(a) \vee [\exists z\, (P(z) \wedge \forall y\, (R(y,z) \Rightarrow \neg P(y)))]$.

2.8 Show that the following inferences are valid.

1. $\{P(a), \forall x\, (\neg P(x) \vee Q(x))\} \models Q(a)$.
2. $\{\forall x\, (P(x) \Rightarrow Q(x)), \exists x\, P(x)\} \models \exists x\, Q(x)$.
3. $\{\exists x\, \neg Q(x), \{\forall x\, (P(x) \Rightarrow Q(x))\} \models \exists x\, \neg P(x)$.

2.9 Given a formula α of predicate logic. Write down the formula β tautologically equivalent to $\neg\alpha$ which has negation in front of atomic formulas only.

a) $\alpha = R(a) \wedge \forall x\, (Q(x,a) \Rightarrow \exists y\, (R(y) \wedge Q(x,y)))$.

b) $\alpha = \forall x\, (P(x) \Rightarrow \exists y\, (Q(x,y) \wedge P(y)))$.

Solution of a). We have

$$\neg\alpha = \neg(R(a) \wedge \forall x\, (Q(x,a) \Rightarrow \exists y\, (R(y) \wedge Q(x,y))).$$

Since the last logical connective is $\wedge$, we have

$$\neg\alpha \models\!\!\!\!\dashv \neg R(a) \vee \neg(\forall x\, (Q(x,a) \Rightarrow \exists y\, (R(y) \wedge Q(x,y)))).$$

Moreover,

$$\neg(\forall x\, (Q(x,a) \Rightarrow \exists y\, (R(y) \wedge Q(x,y)))) \models\!\!\!\!\dashv \exists x\, \neg(Q(x,a) \Rightarrow \exists y\, (R(y) \wedge Q(x,y))),$$

and

$$\neg(Q(x,a) \Rightarrow \exists y\, (R(y) \wedge Q(x,y)) \models\!\!\!\!\dashv Q(x,a) \wedge \neg(\exists y\, (R(y) \wedge Q(x,y))).$$

Finally,

$$\neg\exists y\,(R(y) \wedge Q(x,y)) \models\!\!\mid \forall y\,(\neg R(y) \vee \neg Q(x,y)).$$

Hence

$$\beta = \neg R(a) \vee \exists x\,(Q(x,a) \wedge \forall y\,(\neg R(y) \vee \neg Q(x,y))).$$

Answers

1.1 a) $(\exists x\, E(x)) \wedge (\exists x\, \neg E(x))$;
b) $\exists x\,(C(x) \wedge \neg H(x))$;
c) $\neg(\forall x\,(P(x) \Rightarrow G(x)))$;
d) $\forall x\,(C(x) \Rightarrow S(x))$;
e) $\neg[\forall x\,((P(x) \wedge E(x)) \Rightarrow B(x))]$.

1.2 The formulas can be formed in different ways. We give one of them.

a) Objects are natural numbers. The predicate symbol O represents the property "to be an odd number", the functional symbol f corresponds to the function which assigns to a number n its square n^2. Then the formula has the following form:
$\forall x\,(O(x) \Rightarrow O(f(x)))$.

b) Objects are natural numbers. The predicate symbol Q represents the relation of divisibility on $\mathbb{N}$, the constant symbol a is 6, and the constant symbol b is 3. The formula has the following form:
$\forall x\,(Q(a,x) \Rightarrow Q(b,x))$.

c) Objects are natural numbers. The predicate symbol $=$ represents a well known equality, the functional symbol $+$ is binary and represents addition of natural numbers, the functional symbol f is unary and it assigns to every natural number its square. The formula has the following form:
$\exists x\,\exists y\,\exists z\,(f(x) + f(y) = f(z))$.

1.3 a) It is not a formula, since $f(a)$ is not a term. b) A formula. c) A formula. d) It is not a formula, since $P(y)$ is not a term. e) It is not a formula, since $f(y)$ is not a term. f) It is not a formula, since $Q(x,y)$ is not a term. g) A formula.

1.4 a) For all people it holds that their father plays piano or their mother plays guitar. b) There is a person whose mother plays piano and whose father plays guitar. c) If a person's father plays piano or his/her mother plays guitar then the person himself/herself plays piano or guitar. d) Somebody has a grandmother from the father side who plays piano. e) Somebody plays piano even though his/her grandfather from the mother side does not play guitar.

1.5 a) A tautology. b) A tautology.
c) A satisfiable sentence which is not a tautology. Verification: The sentence $(\exists x\, P(x)) \Rightarrow (\forall x\, P(x))$ is true whenever the sentence $\exists x\, P(x)$ is false. Consider the following interpretation: U is the set of real numbers, P is interpreted as the property "to be a square root of -1". Since no real number has the property $I(P)$, our sentence is true in $\langle U, [\![-]\!]\rangle$.
On the other hand, consider the interpretation: U' is the set of natural numbers, P is interpreted as the property "to be even". Then the sentence $\exists x\, P(x)$ is true in U', I', because there exists an even number. On the other hand, the sentence $\forall x\, P(x)$ is, of course, false, since it is not the case that all natural numbers are even. Hence, we have shown that the sentence $(\exists x\, P(x)) \Rightarrow (\forall x\, P(x))$ is false in $\langle U', [\![-]\!]\rangle$.
d) A contradiction. e) A tautology.

1.6 a) S is satisfiable. Its model is, for instance, the following interpretation: $U = \mathbb{N}$, $[\![Q]\!]$ is the relation $<$ on the set $\mathbb{N}$, i.e. $[\![Q]\!] = \{(m,n) \mid m < n\}$. Then for every natural number n there exists a bigger number (e.g. $n+1$), and no natural number is bigger than itself.

b) S is unsatisfiable. Let us read the first sentence: "There exists an element, say d, such that for every element y the pair (d,y) has the property Q." If we substitute the element d for y, the pair (d,d) also has the property Q. Hence the second sentence cannot be true. Indeed, it says: "For no element x the pair (x,x) has the property Q."

Formally: Take any interpretation $\langle U, [\![-]\!]\rangle$, in which the sentence $\exists x\, \forall y\, Q(x,y)$ is true. Then there exists an element $d \in U$ such that for every element $d' \in U$ the pair (d,d') belongs to $[\![Q]\!]$. Therefore also $(d,d) \in [\![Q]\!]$. This means that the sentence $\forall x\, \neg Q(x,x)$ is false in $\langle U, [\![-]\!]\rangle$.

c) S is unsatisfiable. Let us read the first and the third sentences: "Every element has the property P or the property R." "The element a does not have the property P." If both the sentences are true in an interpretation, then the element a has the property R. It means that the second sentence: "No element has the property R." is false. Formally: Take any interpretation $\langle U, [\![-]\!]\rangle$, in which the first and the third sentences are true. Then there is an element $d \in U$ $(d = [\![a]\!])$ such that $d \notin [\![P]\!]$. Since the $\forall x\, (P(x) \vee R(x))$ is true in $\langle U, [\![-]\!]\rangle$, the sentence $P(a) \vee R(a)$ is true in $\langle U, [\![-]\!]\rangle$. It means that the sentence $R(a)$ is true. Thus $d = [\![a]\!] \in [\![Q]\!]$. Therefore the sentence $\forall x\, \neg R(x)$ is false in $\langle U, [\![-]\!]\rangle$.

1.7 a) $\exists x\, [P(x) \wedge \forall y\, (\neg P(y) \vee \neg R(x,y))]$.
b) $\neg P(a) \wedge \forall z\, (\neg P(z) \vee \exists y\, (R(y,z) \wedge P(y)))$.

1.8 a) Take any interpretation $\langle U, [\![-]\!]\rangle$ in which both sentences $P(a)$ and $\forall x\, (\neg P(x) \vee Q(x))$ are true. Since the element $d \in U$ corresponding to a has property corresponding to P, d must have the property corresponding to Q (otherwise the formula $\forall x\, (\neg P(x) \vee Q(x)$ will be false). This means that $Q(a)$ is true in $\langle U, [\![-]\!]\rangle$.

b) ake any interpretation $\langle U, [\![-]\!]\rangle$ in which both sentences $\forall x\, (P(x) \Rightarrow Q(x))$ and $\exists x\, P(x)$ are true. Since $\exists x\, P(x)$ is true, there is $d \in U$ which has the property corresponding to P. Substitute d for x. Then from the second sentence we get that d must also have the property corresponding to Q. Hence $\exists x\, Q(x)$ is true in $\langle U, [\![-]\!]\rangle$.

c) It is a special case of b), since $\forall x\, (P(x) \Rightarrow Q(x))$ is tautologically equivalent to $\forall x\, (\neg Q(x) \Rightarrow \neg P(x))$.

1.9 b) $\beta = \exists x\, (P(x) \wedge \forall y\, (\neg Q(x,y) \vee \neg P(y)))$.