3 Tutorial 3 – October 17th, 2017

3.1 Which conditions must be satisfied by sets A, B, C to guarantee that:

a) $(A \setminus C) \setminus B = A \setminus (C \setminus B)$;

b) $A \cap (B \cup C) = (A \cap B) \cup C$;

c) $A \cup (B \oplus C) = (A \cup B) \oplus (A \cup C)$;

d) $A \setminus (B \cup C) = (A \setminus B) \setminus C$;

e) $(A \cap B) \setminus C = A \cap (B \setminus C)$;

f) $A \cap (B \setminus C) = (A \setminus C) \cap B$;

g) $A \cup (B \setminus C) = (A \cup B) \setminus (A \cup C)$.

Solution of b). Take any $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$, which means $x \in A$ and ($x \in B$ or $x \in C$). From the propositional logic we know that it is equivalent to ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$).

On the other hand, $x \in (A \cap B) \cup C$ means that ($x \in A$ and $x \in B$) or $x \in C$. Hence for the two conditions to be the same we must have $x \in A$ and $x \in C$ is the same as $x \in C$. And this is just when all $x \in C$ belong to A as well, i.e. $C \subseteq A$.

3.2 Decide whether or not the following assertions hold; give arguments for your answers.

a) $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.

b) For all sets A, B we have $A \times B = B \times A$.

c) For all sets A, B, C the following holds: If $B \subseteq C$, then $A \times B \subseteq A \times C$.

d) If $A \times B \subseteq A \times C$, then $B \subseteq C$.

e) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

f) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

g) $(B \oplus C) \times A = (B \times A) \oplus (C \times A)$.

h) If $A \oplus B = A \oplus C$, then $B = C$.

i) $A \setminus (B \oplus C) = (A \setminus B) \oplus (A \setminus C)$.

Solution of e). First we show that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Consider any pair $(x, y) \in A \times (B \cup C)$. Then $x \in A$, and y is at least in of the sets B, C. Hence, $x \in A$ and $y \in B$, or $x \in A$ and $y \in C$. It means that $(x, y) \in A \times B$, or $(x, y) \in A \times C$.

Now we show that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Consider any $(x, y) \in (A \times B) \cup (A \times C)$. Then $x \in A$ and $y \in B$, or $x \in A$ and $y \in C$. In both cases $x \in A$, and y is in at least one of B and C. Hence, $(x, y) \in A \times (B \cup C)$.

Therefore, $A \times (B \cup C) = (A \times B) \cup (A \times C)$ holds.

3.3 List all the subsets of the set $\{1, 2, 3, 4\}$. How many are there?

Solution. $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 2, 3, 4\}$. There are $2^4 = 16$ subsets.

3.4 There are 200 students in a school, 140 of them can speak French, 80 students can speak German, and 20 students do not know either of these languages. How many students speak both languages?

Solution. Denote by F the set of all students who speak French, and by G the set of all students who speak German. The union $F \cup G$ has $200 - 20 = 180$ students. Moreover, if we add the number of students in F and the number of students in G we the number of students in $F \cup G$ plus the number of students in $F \cap G$. Hence, in $F \cap G$ there are $140 + 80 - 180 = 40$ students.

3.5 Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. List all mappings from A into B. How many maps are there? Which of them are injective? Which of them are surjective?

Solution. There are $2^3 = 8$ mappings; indeed, 0 can be mapped to either of a and b, the same hold for 1 and 2 as well.

None is injective; there exists no injective mapping from A to B whenever both are finite and A has more elements than B (see Pigeonhole Principle).

There are only two mappings that are not surjective; indeed, they are f and g, where $f(0) = f(1) = f(2) = a$ and $g(0) = g(1) = g(2) = b$. So, there are $8 - 2 = 6$ surjective mapping from A to B.

3.6 Find an example of a mapping $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that

a) f is injective but not surjective,
b) f is surjective but not injective,
c) f is injective and surjective,
d) f is neither injective nor surjective.

3.7 Show that the rule

$$(m, n) \mapsto 2^m(2n + 1) - 1 \quad (m, n \in \mathbb{N})$$

defines an injective mapping of the set $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$.

3.8 Show that the set of all binary words is countable. (A binary word is a finite sequence of 0's and 1's.)

3.9

a) Show that any two non-empty open intervals (a, b) and (c, d) of real numbers have the same cardinality.
b) Show that the set $\mathbb{R}$ and the set of all positive real numbers $(0, \infty)$ have the same cardinality.

Answers

3.1 a) $A \cap B = \emptyset$; b) $C \subseteq A$; c) $A = \emptyset$; d) holds for arbitrary sets A, B, C; e) holds for arbitrary sets A, B, C; f) holds for arbitrary sets A, B, C; g) $A = \emptyset$.

3.2 a) True. b) False, for instance, $\{1\} \times \{2\} = \{(1, 2)\} \neq \{(2, 1)\} = \{2\} \times \{1\}$. c) True. d) False; for instance, $\emptyset \times \{2\} = \emptyset \subseteq \emptyset = \emptyset \times \{3\}$ and $\{2\} \not\subseteq \{3\}$. e) True. f) True.
g) True. h) True. i) False; for example, if $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3, 4\}$ then $A \setminus (B \oplus C) = \{1\}$, $(A \setminus B) \oplus (A \setminus C) = \{2\}$.

3.6 There are many functions for each task, we give always one possible solution.

a) For example, $f(n) = n + 1$. Indeed, there is no natural number i for which $i + 1 = 0$.

b) For example, $f(0) = 0$ and $f(n) = n - 1$ for all $n \geq 1$. ($f(0) = f(1)$, so f is not injective.)

c) For example, $f(n) = n+1$ for n even, and $f(n) = n-1$ for n odd. Indeed, the image of an even number is always odd and the image of an odd number is always even. Moreover, no distinct even numbers are mapped on the same odd number, as well no distinct odd numbers are mapped on the same even number. Hence f is injective. Take any $m \in \mathbb{N}$. Then m is either odd or even. If m is odd then $f(m - 1) = m$, if m is even then $f(m + 1) = m$. This shows that f is surjective.

d) For example, $f(n) = (n-3)^2$ is neither injective nor surjective. Indeed, $f(1) = 4 = f(5)$, hence f is not injective. Moreover, there is no $n \in \mathbb{N}$ for which $f(n) = (n - 3)^2 = 2$. Hence, f is not surjective.

3.9 a) If we have two non-empty intervals of real numbers (a, b) and (c, d) then the linear function $f(x) = \frac{d-c}{b-a}(x - a) + c$ maps the interval (a, b) bijectively to (c, d).

b) Consider the function $f(x) = 2^x$. It is an injective function (increasing) and it maps the set of all real numbers $\mathbb{R}$ onto $(0, \infty)$.