4 Tutorial 4 – October 24th, 2017

4.1 Write the following relations on a set A as sets of ordered pairs:

a) A is the set of all subsets of the set $\{1,2\}$, relation R is “to be a proper subset”. This means that for $X, Y \in A$ we have $X\,R\,Y$ if and only if $X \subseteq Y$ and $X \neq Y$.

b) $A = \{2,4,5,8,45,60\}$, R is the relation of divisibility; i.e. $m\,R\,n$ if and only if m divides n.

4.2 A relation R on a closed interval $A = [0,4]$ is given by:

$$x\,R\,y \quad \text{if and only if} \quad x^2 + y^2 + 7 \leq 4x + 4y.$$

Decide a) whether $2(R \circ R)2$ and b) whether $0\,(R^{-1} \circ R)\,3$.

Solution of a). $2(R \circ R)2$ means that there exists $z \in [0,4]$ for which

$$2\,R\,z \quad \text{and} \quad z\,R\,2.$$

From the definition of R we know that $z\,R\,2$ holds if and only if $z^2 + 4 + 7 \leq 4z + 8$. And this reduces to the following quadratic equation $z^2 - 4z + 3 \leq 0$. Similarly, $2\,R\,z$ yields to $4 + z^2 + 7 \leq 8 + 4z$ which represents the same quadratic equation.

Since $z^2 - 4z + 3 = (z-3)\cdot(z-1)$, we get that $z^2 - 4z + 7 \leq 0$ if and only if $z \in [1,3]$. Therefore, $2\,R\,\frac{3}{2}$ holds. Hence, $2(R \circ R)2$ holds.

4.3 A relation R on a closed interval $A = [0,1]$ is given by: $x\,R\,y$ if and only if $y = 2|x - \frac{1}{2}|$. Sketch in a plane (as a set of ordered pairs) the relations R, R^{-1} and $R \circ R^{-1}$.

4.4 Give the properties of the following relations on the set of all natural numbers $\mathbb{N}$:

a) $m\,R\,n$ if and only if m divides n;

b) $m\,R\,n$ if and only if $m + n \geq 50$;

c) $m\,R\,n$ if and only if $m + n$ is even;

d) $m\,R\,n$ if and only if $m \cdot n$ is even;

e) $m\,R\,n$ if and only if $m = n^k$ for some $k \in \mathbb{N}$;

f) $m\,R\,n$ if and only if $m + n$ is a multiple of 3;

g) $m\,R\,n$ if and only if $m > n$.

Solution of d). R is not reflexive. Indeed, R is reflexive if for every natural number n we have $n\,R\,n$. This means that $n \cdot n$ is an even number. This is not true, because for example $3 \cdot 3 = 9$ which is an odd number.

R is symmetric. Indeed. assume that $n\,R\,m$ for some $n, m \in \mathbb{N}$. Then $n \cdot m$ is an even number. Because $m \cdot n = n \cdot m$ we also have $m\,R\,n$.

R is not antisymmetric; indeed, we have $2\,R\,3$ and $3\,R\,2$, but $2 \neq 3$.

R is not transitive. Indeed, for example $3\,R\,2$ (because $3 \cdot 2 = 6$ which is an even number), and so is $2\,R\,3$ but $3\,R\,3$ does not hold (because $3 \cdot 3 = 9$ is an odd number).

4.5 In the following examples S is a relation on a set A and x, y are elements of set A. Decide whether S is reflexive, symmetric, antisymmetric, transitive. Is it an equivalence, an order relation?

a) A is the set of all complex numbers, $x\,S\,y$ if and only if $|x| = |y|$.

b) A is the set of all complex numbers, $x\,S\,y$ if and only if $|x| < |y|$.

c) A is the set of all real numbers, $x\,S\,y$ if and only if $x - y$ is a rational number.

d) A is the set of all triangles of a given plane, two triangles are related in S if and only if they are congruent.

e) A is the set of all triangles of a given plane, two triangles are related in S if and only if they are similar.

f) A is the set of all subsets of a set B, two subsets X, Y of the set B are related in S if and only if they have the same cardinality; i.e., if and only if there exists an injective mapping of X onto Y.

Solution of a). S is reflexive; indeed, for every complex number z we have $|z| = |z|$, so $z\,S\,z$.

S is symmetric; indeed, if for two complex numbers x and y we have $|x| = |y|$, then so $|y| = |x|$, hence $y\,S\,x$.

S is not antisymmetric; indeed, we have $|1| = |i|$ (i is the imaginary unit), hence $1\,S\,i$ and $i\,S\,1$ but $1 \neq i$.

S is transitive; indeed, if for three complex numbers x, y, z we have $|x| = |y|$ and $|y| = |z|$, then $|x| = |z|$.

Therefore, S is an equivalence relation on the set of all complex numbers.

4.6 Given two relations R and S from a set A into a set B. Decide whether the following is true:

a) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$;

b) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$.

Solution of a) To show the statement, it is convenient to look at relations as a set of ordered pairs. We know that $R, S \subseteq A \times B$ and $R^{-1}, S^{-1} \subseteq B \times A$. Moreover, $(a, b) \in R$ if and only if $(b, a) \in R^{-1}$.

Hence we have, $(b, a) \in R^{-1} \cup S^{-1}$ if and only if $(b, a) \in R^{-1}$ or $(b, a) \in S^{-1}$ and it is if and only if $(a, b) \in R$ or $(a, b) \in S$. It means that $(a, b) \in R \cup S$ hence $(b, a) \in (R \cup S)^{-1}$.

Therefore, the equality holds.

4.7 Given two relations R and S on a set A. Decide whether it is true:

a) If R and S are reflexive, then so is $R \circ S$.

b) If R and S are symmetric, then so is $R \circ S$.

c) If R and S are antisymmetric, then so is $R \circ S$.

d) If R and S are transitive, then so is $R \circ S$.

Solution of b).

It does not hold. By the symmetry of R and S we can only say: If $x\, R\circ S\, y$, then $y\, S\circ R\, x$. Let us give an example of two relations R and S on the set of all real numbers $\mathbb{R}$, which are symmetric and such that $R\circ S$ is not symmetric. Let $x\, R\, y$ if and only if $|x-y|=1$, and $x\, S\, y$ if and only if $x^2+y^2=2$. Then $1\, R\, 0$ and $0\, S\, \sqrt{2}$; hence $0\, R\circ S\, \sqrt{2}$. There exist only two real numbers z for which $\sqrt{2}\, R\, z$: $z_1=1+\sqrt{2}$ and $z_2=-1+\sqrt{2}$. For none of them do we have $z_i\, S\, 1$, since $z_i^2+1\neq 2$. Therefore the relation $R\circ S$ is not symmetric.

Answers

4.1

a) $R=\{(\emptyset,\{1\}),(\emptyset,\{2\}),(\emptyset,\{1,2\}),(\{1\},\{1,2\}),(\{2\},\{1,2\})\}$.

b) $R=\{(2,2),(2,4),(2,8),(2,60),(4,4),(4,8),(4,60),(5,5),(5,45),(5,60),(8,8),(45,45),(60,60)\}$.

4.2 b) $0\ (R^{-1}\circ R)\ 3$ does not hold.

4.3

- The set of ordered pairs (x,y) for which $x\, R\, y$ is the graph of the function $y=2|x-\frac{1}{2}|$ for $x\in[0,1]$.
- The set of ordered pairs (x,y) for which $x\, R^{-1}\, y$ consists of the graphs of $y=\frac{1}{2}(1-x)$ and of $y=\frac{1}{2}(1+x)$ both for $x\in[0,1]$.
- The set of ordered pairs (x,y) for which $x\, R\circ R^{-1}\, y$ consists of the graphs of $y=x$ and of $y=1-x$ both for $x\in[0,1]$.

4.4

a) It is reflexive, antisymmetric, and transitive; i.e., a partial order.

b) It is only symmetric.

c) It is reflexive, symmetric, and transitive; i.e., an equivalence relation.

d) It is only symmetric.

e) It is reflexive, antisymmetric, and transitive; i.e., a partial order.

f) It is only symmetric.

g) It is only antisymmetric and transitive.

4.5

b) Antisymmetric, transitive; it is neither an equivalence nor a partial order.

c) Reflexive, symmetric, and transitive; an equivalence relation.

d) Reflexive, symmetric, and transitive; an equivalence relation.

e) Reflexive, symmetric, and transitive; an equivalence relation.

f) Reflexive, symmetric, and transitive; an equivalence relation.

4.6 b) It holds.

4.7

a) It holds.

c) It does not hold. For instance, consider the following antisymmetric relations on the set $\mathbb{N}$ of all natural numbers: $x\,R\,y$ if and only if $x \leq y$, and $x\,S\,y$ if and only if x divides y. Then $2\,R \circ S\,0$, since $2 \leq 3$ and 3 divides 0. Further, $0\,R \circ S\,2$, since $0 \leq 1$ and 1 is a divisor of 2. On the other hand, $0 \neq 2$.

d) It does not hold. A simple counter-example is the following: $A = \{1,2,3,4,5\}$, $R = \{(1,2),(3,4)\}$ (i.e. there is only $1\,R\,2$ and $3\,R\,4$), $S = \{(2,3),(4,5)\}$. Then $1\,R \circ S\,3$ and $3\,R \circ S\,5$, although it is not true $1\,R \circ S\,5$. We get a similar result if we consider relations R and S on the set of complex numbers, where $x\,R\,y$ if and only if $x - y$ is a rational number, $x\,S\,y$ if and only if $|x| = |y|$.