5 Tutorial 5 – October 31st, 2017

5.1 Using the Euclid's Algorithm find the greatest common divisor of 346 and 36.

Solution. 1. We set $u := 346$, $t := 36$.

2. We divide u by t:

$$346 = 9 \cdot 36 + 22,$$

and we set $u := 36$, $t := 22$.

3. Since $t \neq 0$, we divide u by t:

$$36 = 1 \cdot 22 + 14,$$

and we set $u := 22$, $t := 14$.

4. Since $t \neq 0$, we divide u by t:

$$22 = 1 \cdot 14 + 8,$$

and we set $u := 14$, $t := 8$.

5. Since $t \neq 0$, we divide u by t:

$$14 = 1 \cdot 8 + 6.$$

and we set $u := 8$, $t := 6$.

6. Since $t \neq 0$, we divide u by t:

$$8 = 1 \cdot 6 + 2,$$

and we set $u := 6$, $t := 2$.

7. Since $t \neq 0$, we divide u by t:

$$6 = 3 \cdot 2 + 0,$$

and we set $u := 2$, $t := 0$.

8. Since $t = 0$, we return $c := t = 2$.

The greatest common divisor $\gcd(346, 36) = 2$.

5.2 Find all the solutions of the following Diophantic equation

$$319\,x + 473\,y = 0.$$

Solution. First, we use the Euclid's algorithm to find $\gcd(319, 473)$. We proceed faster than in the first exercise.

$$\begin{array}{rcrcr} 319 & = & 0 \cdot 473 & + & 319 \\ 473 & = & 1 \cdot 319 & + & 154 \\ 319 & = & 2 \cdot 154 & + & 11 \\ 154 & = & 14 \cdot 11 & + & 0 \end{array}$$

We have obtained $\gcd(319, 473) = 11$.

We divide the equation by $\gcd(319, 473) = 11$ and we get

$$29\,x + 43\,y = 0.$$

This equation has the following general solution: $x = 43\,k$, $y = -29\,k$, $k \in \mathbb{Z}$.

5.3 Find all the pairs of integers x and y for which

$$167\,x + 32\,y = 1.$$

Solution. We use the extended Euclid's algorithm.

1. We set $u := 167$, $x_u := 1$, $y_u := 0$, $t := 32$ $x_t := 0$, $y_t := 1$.

2. We divide u by t:

$$167 = 5 \cdot 32 + 7,$$

and we set $x_r := 1 - 5 \cdot 0$, $y_r := 0 - 5 \cdot 1$. Further, $u := 32$, $x_u := 0$, $y_u := 1$, $t := 7$, $x_t := 1$, $y_t := -5$.

3. Since $t \neq 0$, we divide u by t:

$$32 = 4 \cdot 7 + 4,$$

and we set $x_r := 0 - 4 \cdot 1$, $y_r := 1 - 4 \cdot (-5)$. Further, $u := 7$, $x_u := 1$, $y_u := -5$, $t := 4$, $x_t := -4$, $y_t := 21$.

4. Since $t \neq 0$, we divide u by t:

$$7 = 1 \cdot 4 + 3,$$

and we set $x_r := 1 - 1 \cdot (-4)$, $y_r := -5 - 1 \cdot 21$. Further, $u := 4$, $x_u := -4$, $y_u := 21$, $t := 3$, $x_t := 5$, $y_t := -26$.

5. Since $t \neq 0$, we divide u by t:

$$4 = 1 \cdot 3 + 1,$$

and we set $x_r := -4 - 1 \cdot 5$, $y_r := 21 - 1 \cdot (-26)$. Further, $u := 3$, $x_u := 5$, $y_u := -26$, $t := 1$, $x_t := -9$, $y_t := 47$.

6. Since $t \neq 0$, we divide u by t:

$$3 = 3 \cdot 1 + 0,$$

and we set $x_r := 5 - 3 \cdot (-9)$, $y_r := -26 - 3 \cdot 47$. Further, $u := 1$, $x_u := -9$, $y_u := 47$, $t := 0$, $x_t := 32$, $y_t := -167$.

Since $t = 0$, we set $\gcd(176, 32) := 1$, $x := -9$, $y := 47$.

The calculations of x_u, x_t and y_u, y_t can be arranged in the following table (which reproduces the procedure above).

	167	32
$\mid$ 7	1	-5
32	0	1
$-4 \cdot 7$	-4	20
$\mid$ 4	-4	21
7	1	-5
$-1 \cdot 4$	4	-21
$\mid$ 3	5	-26
4	-4	21
$-1 \cdot 3$	-5	26
$\mid$ 1	-9	47

We can end the table because now we know that

$$1 = -9 \cdot 167 + 47 \cdot 32,$$

and one of the solutions of $167\,x + 32\,y = 1$ is $x_0 = -9$ and $y_0 = 47$.

To get the general solution of $167\,x + 32\,y = 1$ we need to solve the homogeneous equation $167\,x + 32\,y = 0$. Since 167 and 32 are relatively prime, the general solution of the homogeneous equation is: $x = 32\,k$, $y = -167\,k$ for any $k \in \mathbb{Z}$. Hence the general solution of $167\,x + 32\,y = 1$ is

$$x = -9 + 32\,k, \; y = 47 - 167\,k \quad \text{where } k \in \mathbb{Z}.$$

5.4 Find all the solutions of the following Diophantic equation

$$712\,x + 36\,y = 2.$$

5.5 Find all the pairs of integers x and y for which

$$654\,x + 234\,y = 12.$$

5.6 Find all the pairs of integers x and y for which

$$512\,x + 355\,y = 6.$$

5.7 Find all pairs of integers x and y for which

$$32\,x + 590\,y = 16.$$

Solution. We know that the equation $32\,x + 590\,y = 16$ has a solution if and only if $\gcd(32, 590)$ divides 16. To find $\gcd(32, 590)$ it is convenient to use the Euclid's algorithm.

$$\begin{array}{rcrcr} 590 & = & 18 \cdot 32 & + & 14 \\ 32 & = & 2 \cdot 14 & + & 4 \\ 14 & = & 3 \cdot 4 & + & 2 \\ 4 & = & 2 \cdot 2 & + & 0 \end{array}$$

We have obtained $\gcd(590, 32) = 2$. Because $16 = 8 \cdot 2$, the equation has a solution.

To find on solution of the equation $32\,x + 590\,y = 16$ we extend the Euclid's algorithm.

	32	590
| 14	−18	1
32	1	0
$-2 \cdot 14$	36	−2
| 4	37	−2
14	−18	1
$-3 \cdot 4$	−111	6

Since 16 is a multiple of 4, and since $4 = 37 \cdot 32 - 2 \cdot 590$, we have got

$$16 = 148 \cdot 32 - 8 \cdot 590.$$

Hence one of solutions is $x = 148$, $y = -8$.

To get the general solution, we have to solve the homogeneous equation $32\,x + 590\,y = 0$. Since $\gcd(32, 590) = 2$, we divide the equation by 2 and get

$$16\,x + 295\,y = 0,$$

where 16 and 295 are relatively prime. So the general solution is $x = 295\,k$, $y = -16\,k$ for $k \in \mathbb{Z}$.

The general solution of $32\,x + 590\,y = 8$ is $x = 148 + 259\,k$, $y = -8 - 16\,k$, where $k \in \mathbb{Z}$.

If we do another step of the extended Euclid's algorithm we get $2 = -129\,x + 7\,590$ and hence $16 = -1032\,x + 28\,y$. with the solution $x = -1032$, $y = 28$. It is correct, indeed, $-1032 + 4 \cdot 295 = 148$ and $28 - 4 \cdot 16 = -8$.

Notice that if we were looking for one solution of the non-homogeneous equation we could divide the equation $32\,x + 590\,y = 16$ by $\gcd(32, 590) = 2$ and solve $16\,x + 295\,y = 8$ instead. But then we would need to use the Euclid's algorithm to the pair $16, 295$.

5.8 Find all pairs of integers x and y for which

$$121\,x + 531\,y = 6.$$

5.9 Find all pairs of integers x and y for which

$$141\,x + 531\,y = 6.$$

Answers

5.4 Since the greatest common divisor of 712 and 36 is 4 and 2 is not divisible by 4, the equation has no solution.

5.5 The general solution is: $x = -10 + 39\,k$, $y = 28 - 109\,k$ where $k \in \mathbb{Z}$.

5.6 The general solution is: $x = 43 - 355\,k$, $y = -62 + 512\,k$ where $k \in \mathbb{Z}$.

5.8 The general solution is: $x = 123 - 513\,k$, $y = -29 + 121\,k$ where $k \in \mathbb{Z}$.

6.2 The general solution is: $x = 49 - 177\,k$, $y = -13 + 47\,k$ where $k \in \mathbb{Z}$.