## 3.3 Binary relations

In mathematics, as in everyday situations, we often speak about a relationship between objects, which means an idea of two objects being related or associated one to another in some way. The notion of a binary relation makes this precise. Let us start with some examples.

1. To be a grandfather. Objects of our consideration are people; a person $a$ is associated with a person $b$ if $a$ is a grandfather of $b$.
2. To be of the same length. Objects of our consideration are sticks; a stick $a$ is associated with another stick $b$ if both sticks have the same length.
3. To be a subset. Objects of our consideration are subsets of a given set $U$; a subset $X$ is related to a subset $Y$ if $X$ is a subset of $Y$.
4. To be greater or equal. Objects of our consideration are numbers; a number $n$ is related to a number $m$ if $n$ is greater than or equal to $m$.
5. To be a student of a study group. Objects of our consideration are first year students and study groups; a student $a$ is related to a study group number $K$ if student $a$ belongs to study group $K$.
6. The sine function. Consider real numbers; a number $x$ is related to a number $y$ if $y = \sin x$.

**3.3.1 Definition.** A *relation* (more precisely a *binary relation*) *from a set A into a set B* is any set of ordered pairs $R \subseteq A \times B$. If $A = B$ we speak about a *relation on a set A*. □

We can construct new relations from others. Since a relation is a set of ordered pairs, we can use set operations for construction of new relations. But there are also specific operations – inverse relation and composition of relations. First we start with set operations.

### 3.3.2 Set Operations with Relations.

**Definition.** We say that a relation $R$ is a *sub relation* of a relation $S$ if $R \subseteq S$; i.e. if $a\,R\,b$, then also $a\,S\,b$. □

**Definition.** Let $R$ and $S$ be two relations from a set $A$ into a set $B$. The *intersection* of relations $R$ and $S$ is the relation $R \cap S$; the *union* of $R$ and $S$ is the relation $R \cup S$; the *complement* of $R$ is the relation $\overline{R} = (A \times B) \setminus R$. □

For example, let $T$ be equality on the set of all real numbers $\mathbb{R}$, and $S$ be the relation "smaller than" on $\mathbb{R}$. Then $T \cap S = \emptyset$ and $T \cup S$ is the relation to be smaller than or equal to. The complement of the relation $T$ is non-equality on $\mathbb{R}$; i.e. the relation $\overline{T} = \{(a,b) \mid a,b \in \mathbb{R}, a \neq b\}$.

### 3.3.3 Inverse Relation.

**Definition.** Let $R$ be a relation from a set $A$ into a set $B$. Then the *inverse relation* of the relation $R$ is the relation $R^{-1}$ from set $B$ into set $A$, defined by:

$$x\,R^{-1}y \quad \text{if and only if} \quad y\,R\,x.$$

□

Notice that the inverse relation $R^{-1}$ to $R$ always exists. So if $R$ is a function then the relation $R^{-1}$ exists; on the other hand, $R^{-1}$ does not need to be a function.

### 3.3.4 Composition of Relations.

**Definition.** Let $R$ be a relation from a set $A$ into a set $B$ and $S$ be a relation from the set $B$ into a set $C$. Then the *composition of the relations* (sometimes also called the *product*), $R \circ S$, is the relation from the set $A$ into the set $C$ defined by:

$$a\,(R \circ S)\,c \quad \text{if and only if there is an element } b \in B \text{ such that } \; a\,R\,b \; \text{ and } \; b\,S\,c.$$

□

**3.3.5 Properties of Composition of Relations.** We will show some properties of composition of relation. First, we prove that a composition of relations is associative, then that it is not commutative. (Roughly speaking, we do not need to use parentheses, but we cannot change the order.)

**Proposition.** The composition of relations is associative. More precisely, if $R$ is a relation from $A$ to $B$, $S$ is a relation from $B$ to $C$, and $T$ is a relation from $C$ to $D$ then

$$R \circ (S \circ T) \;=\; (R \circ S) \circ T\,.$$

□

*Justification:* It is not difficult to show hat for all elements $a \in A$, $d \in D$ it holds: $a\,R\circ(S\circ T)\,d$ if and only if there exist $b \in B$, $c \in C$ such that $a\,R\,b$, $b\,S\,c$ and $c\,T\,d$. And this means that $a\,(R \circ S) \circ T\,d$.

**Proposition.** The composition of relations is not commutative. It is not the case that $R \circ S \;=\; S \circ R$ holds for all relations $R$ and $S$. □

*Justification.* To show the above proposition it suffices to find two relations $S$ and $T$ for which $R \circ S \;=\; S \circ R$ does not hold despite of the fact that both compositions exist.

Here is an example: Let $A$ be the set of all people in the Czech Republic. Consider the following two relations $R$, $S$ defined on $A$:

$$\begin{aligned} &a\,R\,b \quad \text{if and only if } a \text{ is a brother or a sister of } b \text{ and } a \neq b \\ &c\,S\,d \quad \text{if and only if } c \text{ is a child of } d. \end{aligned}$$

To show that $R \circ S \;\neq\; S \circ R$ it suffices to find two people $x$, $y$ such that $x\,R \circ S\,y$ holds and $x\,S \circ R\,y$ does not hold. Consider any pair of a nephew $a$ and an uncle $b$. We have $a\,S \circ R\,b$ since a parent of $a$ is a brother or a sister of the uncle $b$. On the other hand, $a\,R \circ S\,b$ does not hold. Indeed, it would mean that one of the brothers or sisters of $a$ were a parent of uncle $b$.

**3.3.6 Relations on a Set.** In applications an important role play relations $S \subseteq A \times B$ where $A = B$. Recall that such relations are called *relations on A*.

**3.3.7 Different Types of Relations on $A$.** Relations on a set $A$ may have different properties. We will be mainly interested in four of them: reflexivity, symmetry, antisymmetry and transitivity. Here are the definitions:

**Definition.** We say that relation $R$ on set $A$ is

1. *reflexive* if for every $a \in A$ we have $a\,R\,a$;
2. *symmetric* if for every $a, b \in A$ it holds that: $a\,R\,b$ implies $b\,R\,a$;
3. *antisymmetric*, if for every $a, b \in A$ it holds that: $a\,R\,b$ and $b\,R\,a$ imply $a = b$;
4. *transitive*, if for every $a, b, c \in A$ it holds that: if $a\,R\,b$ and $b\,R\,c$ then $a\,R\,c$.

□

**Examples.** Consider the relation of non-equality $R$ on the set of all natural numbers $\mathbb{N}$; (i.e. $n\,R\,m$ if and only if $n$ and $m$ are different natural numbers). This relation is not reflexive because for no $n \in \mathbb{N}$ do we have $n \neq n$. It is symmetric: If $n \neq m$ then also $m \neq n$. Relation $R$ is not antisymmetric because e.g. $2 \neq 3$, $3 \neq 2$, and 2 and 3 are different numbers. (That is $2\,R\,3$ and $3\,R\,2$ and at the same time $2 \neq 3$.) This relation is not transitive because for example we have $2 \neq 3$ and $3 \neq 2$, and at the same time $2 = 2$ (i.e. $2\,R\,3$ and $3\,R\,2$ and it is not true $2\,R\,2$).

Relation "to be smaller than or equal to" $\leq$ on set $\mathbb{R}$ is reflexive, since $a \leq a$ for every $a$. It is also antisymmetric, since whenever for two numbers $a, b$ we have $a \leq b$ and $b \leq a$, then $a = b$. It is also transitive, since if $a \leq b$ and $b \leq c$, then also $a \leq c$.

**3.3.8 Equivalence Relations.** One of the most important type of relations on $A$ is so called equivalence relation. Let us recall the tautological equivalence of propositional formulas. It is one example of equivalence relation on the set of all propositional formulas. Have in mind that an "equivalence relation" on $A$ is some sort of "generalized equality" of elements of $A$.

**Definition.** A relation $R$ on a set $A$ is called an *equivalence*, if it is reflexive, symmetric and transitive. □

**Example.** The following relation $R$ on the set of all integers $\mathbb{Z}$, defined by:

$$m\,R\,n \quad \text{if and only if} \quad m - n \text{ is divisible by } 12,\ (m, n \in \mathbb{Z}),$$

is an equivalence relation.

*Justification.* Relation $R$ is reflexive. Indeed, for every $m \in \mathbb{Z}$ we have $m - m = 0$, and zero is divisible by 12. Hence $m\,R\,m$.

Relation $R$ is also symmetric. Indeed, if $m\,R\,n$, i.e., $m - n = 12k$ for some $k$, then also $n - m$ is divisible by 12 ($n - m = -12k$). Therefore $n\,R\,m$.

Moreover, $R$ is transitive: Take any numbers $m, n, p \in \mathbb{Z}$ such that $m\,R\,n$ and $n\,R\,p$. This means $m - n = 12k$ and $n - p = 12l$ for some $k$ and $l$. Then $m - p = (m - n) + (n - p) = 12k + 12l = 12(k + l)$. Hence we have $m\,R\,p$.

**3.3.9 Equivalence Classes.** Every equivalence relation $R$ on $A$ "divides" $A$ into the sets of equivalent elements. These sets are called equivalence classes. We will see the importance of equivalence classes later when we introduce so called residue classes.

**Definition.** Let $R$ be an equivalence relation on a set $A$. An *equivalence class* of $R$ corresponding to the element $a \in A$ is the set $R\,[a] = \{b \in A \mid a\,R\,b\}$. □

**Example:** Given the equivalence relation from 3.3.8. There are 12 different equivalence classes of $R$; namely

$$R\,[i] = \{j \mid j = i + 12k, k \in \mathbb{Z}\}, \ \text{ for } i = 0, \ldots, 11.$$

**Definition.** Let $R$ be an equivalence relation on $A$. The set

$$\{R[a] \mid a \in A\}$$

is called the *quotient set* and denoted by $A/R$.

**3.3.10 Properties of the Set of Equivalence Classes.** The next proposition gives properties that sets of equivalence classes have.

**Proposition.** Let $R$ be an equivalence relation on a set $A$. The set $\{\, R\,[a] \mid a \in A\}$ has the following properties:

1. Every element $a \in A$ belongs to $R\,[a]$ and hence $\bigcup\{\, R\,[a] \mid a \in A\} = A$.

2. Equivalence classes $R[a]$ are pairwise disjoint. That is, if $R[a] \cap R[b] \neq \emptyset$, then $R[a] = R[b]$.

□

*Justification.* Since every element $a \in A$ is related to itself (reflexivity), we get $a \in R[a]$. Thus $A \subseteq \bigcup\{ R[a] \mid a \in A\}$. Moreover, each equivalence class is a subset of $A$, and therefore $\bigcup\{ R[a] \mid a \in A\} \subseteq A$. We have shown the first property.

Let us verify the second property. Assume that there are two classes with non-empty intersection. We will show that they coincide. Take an element $z \in R[a] \cap R[b]$. Then $a\,R\,z$ and $b\,R\,z$. Since $R$ is symmetric, we have $z\,R\,b$. Furthermore, since $a\,R\,z$ and $z\,R\,b$, it follows from transitivity of $R$ that $a\,R\,b$. We have shown: If two classes $R[a]$, $R[b]$ have non-empty intersection, then the elements $a$ and $b$ are equivalent. Now, take any element $c \in R[a]$. Then $c\,R\,a$. From transitivity of $R$ and from $a\,R\,b$ we get that $c\,R\,b$. Hence $c \in R[b]$. Analogously one can show that every element $c \in R[b]$ also belongs to $R[a]$. Therefore $R[a] = R[b]$.

**3.3.11 Partition.** The properties stated in the above proposition characterize another mathematics notion – a partition of a set. Here is the formal definition.

**Definition.** Let $A$ be a non-empty set. A set $\mathcal{S}$ of non-empty subsets of $A$ is called a *partition* of set $A$ provided the following conditions hold:

1. Every element $a \in A$ belongs to some member of $\mathcal{S}$, i.e. $\bigcup \mathcal{S} = A$.

2. Elements of the set $\mathcal{S}$ are pairwise disjoint. In other words, if $X \cap Y \neq \emptyset$ then $X = Y$ for all $X, Y \in \mathcal{S}$.

□

In the above proposition we have shown that the quotient set modulo an equivalence relation forms a partition of the underlying set. On the other hand, we can associate an equivalence relation to any partition.

**3.3.12** We already know that for every equivalence relation its set of equivalence classes forms a partition of the underlying set. On the other hand, every partition creates an equivalence relation. This is what the next proposition states.

**Proposition.** Let $\mathcal{S}$ be a partition of a set $A$. Then the relation $R_{\mathcal{S}}$ defined by:

$$a\,R_{\mathcal{S}}b \quad \text{if and only if} \quad a, b \in X \text{ for some } X \in \mathcal{S}$$

is an equivalence relation on set $A$. □

*Justification.* We need to show that relation $R_{\mathcal{S}}$ is reflexive, symmetric, and transitive. Since every element $a \in A$ belongs to some set $X \in \mathcal{S}$, we have $a\,R_{\mathcal{S}}a$, and relation $R_{\mathcal{S}}$ is reflexive. The symmetry of $R_{\mathcal{S}}$ is clear. Whenever we have $a\,R_{\mathcal{S}}b$ then also $b\,R_{\mathcal{S}}a$, and relation $R_{\mathcal{S}}$ is symmetric.

We show the transitivity: Let $a\,R_{\mathcal{S}}b$ and $b\,R_{\mathcal{S}}c$, i.e., let $a, b \in X$, $b, c \in Y$ for some $X, Y \in \mathcal{S}$. This means that $b \in X \cap Y$. Therefore the sets $X$ and $Y$ have a non-empty intersection. Hence they coincide. Thus we get $a, c \in X$, and $a\,R_{\mathcal{S}}c$ follows.

**3.3.13 Remark.** Notice that if we start with an equivalence relation $R$, then we form a corresponding partition into classes of $R$, and finally we make the equivalence relation that corresponds to the partition (according to the above proposition), we get precisely the equivalence relation $R$, with which we have started. Similarly, if we start with a partition, then form its corresponding equivalence relation, and finish with the partition into classes of the equivalence relation, we get the original partition.

**3.3.14 Partial Order, a Poset.** Apart from equivalence relations there is another type of relations that plays a special role in mathematics. And it is a so called partial order, or a partial ordered set, shortly a poset.

**Definition.** A relation $R$ on a set $A$ is called an *order (partial order)*, if it is reflexive, antisymmetric and transitive. A set $A$ together with a partial order is often called a *poset.* □

### 3.3.15 Examples of Posets.

1. The well-known ordering of real numbers is a partial order in the above sense. Hence, $(\mathbb{R}, \leq)$ is a poset. Indeed, for all real numbers $a, b, c \in \mathbb{R}$ we have: $a \leq a$; if $a \leq b$ and $b \leq a$ then necessarily $a = b$; if $a \leq b$ and $b \leq c$ then also $a \leq c$.

2. Denote by $A$ the set of all subsets of the set $U$. Then the relation $\subseteq$, "to be a subset", is a partial order on $A$. Hence, $(P(U), \subseteq)$ is a poset. Verification of reflexivity, antisymmetry and transitivity is left to the reader.

3. Let $A = \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. The the relation of divisibility defined by $m \,|\, n$ if and only if $m$ is a divisor of $n$ (i.e. if $n = k \cdot m$ for some $k \in \mathbb{N}$) is a partial order. Hence $(\mathbb{N}, |)$ is a poset. Indeed, for all natural numbers $m, n, k$ we have $m \,|\, m$; if $m \,|\, n$ and $n \,|\, m$ then $m = n$; if $m \,|\, n$ and $n \,|\, k$ then also $m \,|\, k$.