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A criterion of Γ-nullness and differentiability of convex and quasiconvex functions

by

Jaroslav Tišer and Luděk Zajíček (Praha)

Abstract. We introduce a criterion for a set to be Γ-null. Using it we give a shorter proof of the result that the set of points where a continuous convex function on a separable Asplund space is not Fréchet differentiable is Γ-null. Our criterion also implies a new result about Gâteaux (and Hadamard) differentiability of quasiconvex functions.

1. Introduction. Our paper deals with two closely related topics:

- Γ-null sets in Banach spaces, which were introduced by J. Lindenstrauss and D. Preiss [LP] in the theory of Fréchet differentiability of Lipschitz functions on Banach spaces.
- Differentiability of convex and quasiconvex functions on Banach spaces.

It was proved in [LP, Corollary 3.11] (or [LPT, Corollary 6.3.10]) that every continuous convex function on a separable Asplund space is Γ-a.e. Fréchet differentiable. This result, however, follows from a more general and very deep theorem [LP, Theorem 3.10] (or [LPT, Theorem 6.3.9]) with a long and sophisticated proof.

In the present article we introduce a new criterion for a set to be Γ-null (Proposition 3.6). Its proof is not short but is considerably more straightforward and much easier than the proof of the above mentioned theorem. Using that criterion we give a quite different and much simpler proof of the Γ-a.e. Fréchet differentiability of convex functions. (We note, however, that the main application of [LP, Theorem 3.10] concerns general Lipschitz functions, to which our method does not apply.) In this proof we observe that the set N_F of points of Fréchet nondifferentiability of a continuous convex function f on a separable Asplund space has a relatively easily formulated property (of being a countable union of $\mathcal{P}^{dc}$-sets, see Theorem 4.2), which permits the use of our Γ-nullness criterion. Also, Theorem 4.2 may be of

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some independent interest, since there exist a number of results (in [K], [M], [LP], [PZ1], [PZ2], [Z3]) concerning smallness of the set N_F, but our Theorem 4.2 does not seem to follow from the known results.

A further application of our Γ-nullness criterion is a new observation in Proposition 5.1 that the boundary of every closed and convex subset of *any separable* Banach space is Γ-null. We use it to prove that every continuous quasiconvex function on any separable Banach space is Γ-a.e. Hadamard (in particular Gâteaux) differentiable.

2. Preliminaries. The symbol $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$, and $B(t,r)$ is the closed ball centred at t with radius $r>0$. We will use the same symbol for balls in different spaces; it will be clear from the context which space is intended. For a Lipschitz map f between metric spaces we denote by $\mathrm{Lip}(f)$ the smallest Lipschitz constant of f.

Γ-null and Γ_n-null sets are defined as follows (see [LPT]). Let X be a Banach space and let $[0,1]^{\mathbb{N}}$ be equipped with the product topology and the product Lebesgue measure $\mathscr{L}^{\mathbb{N}}$. We denote by $\Gamma(X)$ the space of all continuous mappings

$$\gamma\colon [0,1]^{\mathbb{N}} \to X$$

which have continuous partial derivatives $D_k\gamma$. (At the points where the kth coordinate is 0 or 1 we consider one-sided derivatives.) The topology on $\Gamma(X)$ is generated by the countable family of pseudonorms

$$\|\gamma\|_\infty \quad \text{and} \quad \|D_k\gamma\|_\infty,\ k\ge 1.$$

The space $\Gamma_n(X) := C^1([0,1]^n, X)$ is equipped with the norm

$$\|f\|_{C^1} = \max\{\|f\|_\infty, \|f'\|_\infty\}.$$

For the sake of completeness we recall that the derivative $\gamma'(t)$ for t belonging to the boundary $\partial([0,1]^n)$ is a bounded linear operator $L\colon \mathbb{R}^n \to X$ such that

$$\|\gamma(s)-\gamma(t)-L(s-t)\| = o(|s-t|), \qquad s\to t,\ s\in[0,1]^n.$$

The same convention will be used for C^1 mappings defined on closed convex sets. Moreover, by a C^1 map defined on a noncompact subset of $\mathbb{R}^n$ we mean a *bounded* map with a bounded and continuous derivative. The norm of such a derivative will always refer to the Euclidean norm, although some other norms in $\mathbb{R}^n$ will appear, too.

Definition 2.1. A Borel set $A\subset X$ is called Γ-*null* if

$$\mathscr{L}^{\mathbb{N}}\{t\in[0,1]^{\mathbb{N}} \mid \gamma(t)\in A\} = 0$$

for residually many $\gamma\in\Gamma(X)$. Analogously, a Borel set $A\subset X$ is called Γ_n-*null* if

$$\mathscr{L}^{n}\{t\in[0,1]^{n} \mid \gamma(t)\in A\} = 0$$

for residually many $\gamma\in\Gamma_n(X)$.

It is easy to verify that the families of Γ-null and of Γ_n-null sets are each closed under countable unions. Also, the notions of Γ-null and Γ_n-null are incomparable in general. Nevertheless, for sets of low Borel class we will use the following two properties (see [LPT, Theorem 5.4.2] and [LPT, Theorem 5.4.3]).

LEMMA 2.2. *If $A \subset X$ is a $G_{\delta\sigma}$ set which is Γ_n-null for infinitely many $n \in \mathbb{N}$, then A is Γ-null. If A is an F_σ set which is Γ-null, then A is Γ_n-null for all $n \in \mathbb{N}$.*

We will also need the following notion.

DEFINITION 2.3. Let X be a Banach space. A set $H \subset X$ is called a *d.c.-hypersurface* if X can be decomposed as $X = X_0 \oplus \mathbb{R}v$, $v \in X \setminus \{0\}$, and there exists a *d.c.*-function $g\colon X_0 \to \mathbb{R}$ such that $H = \{x + g(x)v \mid x \in X_0\}$. Here a *d.c.-function* is a difference of two continuous convex functions.

REMARK 2.4. The above notion is important in the theory of differentiability of convex continuous functions. Namely (see [Z1] or [BL, Theorem 4.20]), if f is a continuous convex function on a separable Banach space then the set N_G of points of Gâteaux nondifferentiability of f can be covered by countably many *d.c.*-hypersurfaces.

3. Criterion. We will need the following fact concerning extensions of C^1 mappings. The idea of the proof is the same as the idea of the proof of a more special statement in [LPT, Lemma 5.3.1]. It is based on the following extension result for Lipschitz functions (see [JLS]): Given a Lipschitz map γ from a subset F of $\mathbb{R}^n$ into a Banach space X, there exists a Lipschitz extension $\widetilde{\gamma}\colon \mathbb{R}^n \to X$ of γ such that

$$\sup_{t\in\mathbb{R}^n} \|\widetilde{\gamma}(t)\| = \sup_{t\in F} \|\gamma(t)\|, \qquad \mathrm{Lip}(\widetilde{\gamma}) \le c_n \,\mathrm{Lip}(\gamma), \tag{3.1}$$

where c_n depends only on the dimension n.

LEMMA 3.1. *Let X be a Banach space, $q \ge 1$, and let $\|\cdot\|$ be an equivalent norm on $\mathbb{R}^n$ satisfying*

$$\frac{1}{q}|\cdot| \le \|\cdot\| \le q|\cdot|.$$

Let $r > 0$ and denote $U = \{x \in \mathbb{R}^n \mid \|x\| \le r\}$. Then for every $\varepsilon > 0$ and every C^1 map $\gamma\colon U \to X$ there exists an extension $\widetilde{\gamma} \in C^1(\mathbb{R}^n, X)$ such that

- *$\widetilde{\gamma} = 0$ outside $(1+\varepsilon)U$;*
- *$\|\widetilde{\gamma}\|_{C^1} \le 2\|\gamma\|_\infty + K(\|\gamma'\|_\infty + \|\gamma\|_\infty/(\varepsilon r))$, where $K \ge 0$ depends only on n and q.*

Proof. In the first step we observe that for every C^1 map $\gamma\colon U \to X$ one can find $\widehat{\gamma} \in C^1(\mathbb{R}^n, X)$ such that

(i) $\widehat{\gamma} = 0$ outside $(1+\varepsilon)U$;
(ii) $\|\gamma - \widehat{\gamma}\|_\infty \le \frac{1}{2}\|\gamma\|_\infty$, $\|\gamma' - \widehat{\gamma}'\|_\infty \le \frac{1}{2}\|\gamma'\|_\infty$;
(iii) $\|\widehat{\gamma}\|_{C^1} \le \|\gamma\|_\infty + \kappa(\|\gamma'\|_\infty + \|\gamma\|_\infty/(\varepsilon r))$,

where κ depends only on n and q. To this end, choose $0 < \delta < \frac{1}{4}\varepsilon$ small enough to guarantee that

$$\begin{aligned} \left\|\gamma(t) - \gamma\left(\frac{1}{1+\delta}s\right)\right\| &\le \frac{1}{2}\|\gamma\|_\infty, \\ \left\|\gamma'(t) - \frac{1}{1+\delta}\gamma'\left(\frac{1}{1+\delta}s\right)\right\| &\le \frac{1}{2}\|\gamma'\|_\infty \end{aligned} \tag{3.2}$$

for any $t \in U$, $s \in (1+\delta)U$ and $|s-t| < r\delta/q$. Define

$$\gamma^*(t) = \begin{cases} \gamma\left(\dfrac{t}{1+\delta}\right) & \text{if } t \in (1+\delta)U, \\ 0 & \text{if } t \notin (1+\varepsilon-\delta)U. \end{cases}$$

Then clearly $\|\gamma^*\|_\infty = \|\gamma\|_\infty$ and the Lipschitz constant of γ^* can be estimated by

$$\operatorname{Lip}\gamma^* \le \max\left\{\|\gamma'\|_\infty, \frac{q\|\gamma\|_\infty}{(\varepsilon - 2\delta)r}\right\} \le \|\gamma'\|_\infty + \frac{2q\|\gamma\|_\infty}{\varepsilon r}.$$

Indeed, the first term in the maximum follows from the mean value theorem for γ^* on $(1+\delta)U$. Since the $\|\cdot\|$-distance between the sets $(1+\delta)U$ and $\mathbb{R}^n \setminus (1+\varepsilon-\delta)U$ is $(\varepsilon-2\delta)r$, the Euclidean distance between these sets is at least $(r/q)(\varepsilon - 2\delta)$, which gives the second term in the maximum. Using (3.1) we can extend γ^* to a Lipschitz map on $\mathbb{R}^n$ preserving the $\|\cdot\|_\infty$-norm and with the Lipschitz constant at most

$$c_n\left(\|\gamma'\|_\infty + \frac{2q\|\gamma\|_\infty}{\varepsilon r}\right).$$

Convolving now this Lipschitz extension with a real C^1 mollifier M supported in the $(r\delta/q)$-neighbourhood of the origin, we obtain the desired $\widehat{\gamma}$: The condition (i) is obviously true. To verify (ii), notice first that if $t \in U$ and $|t-s| < r\delta/q$, then $s \in (1+\delta)U$. Now

$$\begin{aligned} \gamma(t) - \int \gamma^*(s)M(t-s)\,ds &= \int (\gamma(t) - \gamma^*(s))M(t-s)\,ds \\ &= \int \left(\gamma(t) - \gamma\left(\frac{1}{1+\delta}s\right)\right)M(t-s)\,ds. \end{aligned}$$

The first inequality in (3.2) implies the first estimate in (ii). Similarly we obtain the second estimate. Since convolution can increase neither the sup-norm nor the Lipschitz constant, we obtain (iii) with e.g. $\kappa = 2qc_n$.

In the second step of the proof we will be recursively applying the extensions just established. Denote $\gamma_0 = \gamma$ and then define $\gamma_{k+1} = \gamma_k - \widehat{\gamma}_k$ on U for all $k \ge 0$. Condition (ii) implies $\|\gamma_k\|_\infty \le 2^{-k}\|\gamma\|_\infty$ and $\|\gamma_k'\|_\infty \le 2^{-k}\|\gamma'\|_\infty$. Employing also (iii) we get

$$\begin{aligned}\sum_{k=0}^{\infty} \|\widehat{\gamma}_k\|_{C^1} &\le \sum_{k=0}^{\infty}\bigg(\|\gamma_k\|_\infty + \kappa\bigg(\|\gamma_k'\|_\infty + \frac{\|\gamma_k\|_\infty}{\varepsilon r}\bigg)\bigg)\\ &\le 2\|\gamma\|_\infty + 2\kappa\bigg(\|\gamma'\|_\infty + \frac{\|\gamma\|_\infty}{\varepsilon r}\bigg).\end{aligned}$$

It follows that the series $\sum_{k=0}^{\infty}\widehat{\gamma}_k$ converges in the space $C^1(\mathbb{R}^n, X)$ to some $\widetilde{\gamma}$. Obviously, $\widetilde{\gamma} = 0$ outside $(1+\varepsilon)U$ and for $t \in U$ we obtain

$$\widetilde{\gamma}(t) = \sum_{k=0}^{\infty} \widehat{\gamma}_k(t) = \sum_{k=0}^{\infty}(\gamma_k(t) - \gamma_{k+1}(t)) = \gamma(t).$$

Finally,

$$\|\widetilde{\gamma}\|_{C^1} \le \sum_{k=0}^{\infty}\|\widehat{\gamma}_k\|_{C^1} \le 2\|\gamma\|_\infty + 2\kappa\bigg(\|\gamma'\|_\infty + \frac{\|\gamma\|_\infty}{\varepsilon r}\bigg),$$

and we obtain the statement of the lemma with the constant $K = 2\kappa$. ∎

Let $T\colon \mathbb{R}^n \to X$ be an affine injective map of $\mathbb{R}^n$ into a Banach space X. Consider the image $T[\mathscr{L}^n]$ of the Lebesgue measure $\mathscr{L}^n$ under T. Then both $T[\mathscr{L}^n]$ and the n-dimensional Hausdorff measure $\mathscr{H}^n$ are uniformly distributed measures on the affine subspace $T(\mathbb{R}^n)$ and so the former is a constant multiple of the latter,

$$T[\mathscr{L}^n] = c(T)\mathscr{H}^n \tag{3.3}$$

(see e.g. [MT, Theorem 3.4]). The exact value of the constant $c(T) > 0$ will not be important for us.

The next definition introduces our key notion.

Definition 3.2. Let $A \subset X$ be a subset of the Banach space X, let $a \in A$ and $\lambda \in [0,1)$. We say that A is *P_λ-small* at the point a if the following property holds:

For every finite-dimensional subspace $V \subset X$ there are sequences $(y_k)_{k\in\mathbb{N}}$ of points of X and $(r_k)_{k\in\mathbb{N}}$ of positive reals such that

(i) $r_k \searrow 0$;
(ii) $\|y_k - a\| = o(r_k)$, $k \to \infty$;
(iii) for every k,

$$\mathscr{H}^m\big(B(y_k, r_k) \cap (y_k + V) \cap A\big) \le \lambda\mathscr{H}^m\big(B(y_k, r_k) \cap (y_k + V)\big),$$

where $m = \dim V \ge 1$.

In case the sequences (r_k) and (y_k) satisfy (i) and (ii), but the last condition (iii) is strengthened to

$$B(y_k, r_k) \cap (y_k + V) \cap A = \emptyset,$$

we say that A is *P-small* at a.

REMARK 3.3. (i) The name "P-small" comes from the fact that the set A is relatively small on many finite-dimensional planes close to the point a.

(ii) One can easily notice that A is P-small at $a \in A$ iff for every $\delta > 0$ and every finite-dimensional subspace $V \subset X$ there is $y \in X$ with $0 < \|y - a\| < \delta$ such that

$$B(y, \|y - a\|/\delta) \cap (y + V) \cap A = \emptyset.$$

(iii) The property of P_λ-smallness is not so symmetric. If $\lambda \in [0,1)$ and $a \in A$ have the property that for every $\delta > 0$ and every finite-dimensional subspace $V \subset X$ there is $y \in X$ with $0 < \|y - a\| < \delta$ such that

$$\mathscr{H}^m\big(B(y, \|y - a\|/\delta) \cap (y + V) \cap A\big) \leq \lambda \mathscr{H}^m\big(B(y, \|y - a\|/\delta) \cap (y + V)\big),$$

then A is P_λ-small at the point A. However, the opposite implication does not hold.

The next two notions follow the same pattern: after disregarding an exceptional subset, the set under consideration is P_λ-small (P-small, resp.) at all remaining points.

DEFINITION 3.4. Let $A \subset X$ be a closed subset of a separable Banach space X.

(i) The set A is called a *$\mathcal{P}_\lambda^\Gamma$-set*, $\lambda \in [0,1)$, if there is a Borel subset $A_0 \subset X$ which is Γ_n-null for infinitely many n and A is P_λ-small at all points of $A \setminus A_0$.

(ii) The set A is called a *$\mathcal{P}^{dc}$-set* if there is a subset $A_0 \subset X$ which is a countable union of $d.c.$-hypersurfaces and such that A is P-small at all points of $A \setminus A_0$.

REMARK 3.5. Observe that each $\mathcal{P}^{dc}$-set is also a $\mathcal{P}_\lambda^\Gamma$-set for all $\lambda \in [0,1)$. Indeed, any countable union of $d.c.$-hypersurfaces is an F_σ set which is Γ-null (see [Z2, p. 157]). By Lemma 2.2, it is a Γ_n-null set even for all n.

We now formulate a criterion for Γ-nullness.

PROPOSITION 3.6. *Let $A \subset X$ be a $\mathcal{P}_\lambda^\Gamma$-set, $\lambda \in [0,1)$, in a separable Banach space X. Then A is Γ-null.*

Proof. Let $n \in \mathbb{N}$ be such that the corresponding A_0 is Γ_n-null and let $\alpha > 0$. The set

$$S_\alpha = \{\gamma \in \Gamma_n(X) \mid \mathscr{L}^n \gamma^{-1}(A) \geq \alpha\}$$

is closed in $\Gamma_n(X)$ by [LPT, Lemma 5.4.1]. We show that S_α is nowhere dense for all $\alpha > 0$.

Suppose not and set

$$\alpha_0 = \sup\{\alpha > 0 \mid S_\alpha \text{ is not nowhere dense}\}.$$

We can find positive numbers α, α_1 such that $0 < \alpha < \alpha_0 < \alpha_1$ and

$$\frac{1+\lambda}{2}\,\alpha_1 < \alpha. \tag{3.4}$$

Recall now that by [LPT, Lemma 5.3.5] the set of surfaces $\gamma \in \Gamma_n(X)$ for which $\operatorname{rank}\gamma'(t) = n$ for almost all $t \in [0,1]^n$ is residual. Let $A^* = A \setminus A_0$ for short. By our assumption, the set

$$\{\gamma \in \Gamma_n(X) \mid \mathscr{L}^n\gamma^{-1}(A^*) = \mathscr{L}^n\gamma^{-1}(A)\} \tag{3.5}$$

and the set of all surfaces with $\mathscr{L}^n\gamma^{-1}(A) < \alpha_1$ are both residual. We denote by S the residual set of surfaces possessing all three properties.

Let $\gamma \in S \cap \operatorname{Int} S_\alpha$ and choose $\tau > 0$ so that $B(\gamma,\tau) \subset S_\alpha$. To achieve the desired contradiction it suffices to find a surface $\gamma_0 \in B(\gamma,\tau) \setminus S_\alpha$. We start by choosing $\eta > 0$ small enough that

$$\lambda(\mathscr{L}^n\gamma^{-1}(A^*) + \eta) + 2\eta < \frac{1+\lambda}{2}\,\alpha_1. \tag{3.6}$$

For every $t \in (0,1)^n$ with $\operatorname{rank}\gamma'(t) = n$ we denote $V_t = \operatorname{Im}\gamma'(t)$ and $L_t = \gamma'(t)$, a linear isomorphism of $\mathbb{R}^n$ onto V_t. It is easy to show that the set of points in $(0,1)^n$ with $\operatorname{rank}\gamma'(t) = n$ is open, and the mappings $t \mapsto \|L_t\|$ and $t \mapsto \|L_t^{-1}\|$ are continuous on this set. Find $q > 0$ such that the set

$$H := \{t \in (0,1)^n \mid \operatorname{rank}\gamma'(t) = n,\ \|L_t\| \le q,\ \|L_t^{-1}\| \le q\}$$

has measure at least $1-\eta$. Using (3.6) we now choose $\varepsilon > 0$ to satisfy both

$$\bigl(2\bigl(\tfrac{1}{2}q+1\bigr)\varepsilon + K\bigl(1+2q\bigl(\tfrac{1}{2}q+1\bigr)\bigr)\bigr)\varepsilon < \tau, \tag{3.7}$$

$$\frac{(1+\varepsilon)^n - 1 + \lambda}{(1+\varepsilon)^n}(\mathscr{L}^n\gamma^{-1}(A^*) + \eta) + 2\eta \le \frac{1+\lambda}{2}\,\alpha_1, \tag{3.8}$$

where $K = K(n,q)$ is from Lemma 3.1. Since γ' is uniformly continuous on $[0,1]^n$, there is $\delta \in (0,1]$ such that for any $s,t \in [0,1]^n$ with $|s-t| < \delta$ we have $\|\gamma'(s) - \gamma'(t)\| < \varepsilon$.

Further, let $G \subset (0,1)^n$ be an open set containing $H \cap \gamma^{-1}(A^*)$ with

$$\mathscr{L}^n\bigl(G \setminus (H \cap \gamma^{-1}(A^*))\bigr) < \eta. \tag{3.9}$$

Since the set A is P_λ-small at each point of A^*, by Definition 3.2 one can find, for any $a_t := \gamma(t)$ with $t \in H \cap \gamma^{-1}(A^*)$ and the finite-dimensional subspace V_t, the corresponding sequences $(y_k(t))$ of points of X and $(r_k(t))$

of positive numbers with

$$\mathscr{H}^n\big(B(y_k(t), r_k(t)) \cap (V_t + y_k(t)) \cap A\big) \le \lambda \mathscr{H}^n\big(B(y_k(t), r_k(t)) \cap (V_t + y_k(t))\big). \tag{3.10}$$

For such t and $k \in \mathbb{N}$, denote

$$U_k(t) := t + L_t^{-1} B(0, r_k(t)),$$
$$\widetilde{U}_k(t) := t + L_t^{-1} B(0, (1+\varepsilon) r_k(t)).$$

Clearly, $U_k(t) \subset \widetilde{U}_k(t)$. Note also that by the definition of H,

$$B(t, r_k(t)/q) \subset U_k(t) \subset B(t, q r_k(t)), \quad \text{hence} \quad \frac{2r_k}{q} \le \operatorname{diam} U_k \le 2q r_k. \tag{3.11}$$

In particular, $\operatorname{diam} U_k(t) \to 0$ as $k \to \infty$. Assume without loss of generality that for every $t \in H \cap \gamma^{-1}(A^*)$ and for all $k \in \mathbb{N}$ we have, together with (3.10), also the following:

$$\operatorname{diam} U_k(t) < \delta \quad \text{and} \quad \widetilde{U}_k(t) \subset G; \tag{3.12}$$

$$\|y_k(t) - \gamma(t)\| < \varepsilon^2 r_k(t); \tag{3.13}$$

$$\|\gamma(s) - \gamma(t) - \gamma'(t; s-t)\| \le \varepsilon^2 |s-t| \qquad \text{for } s \in U_k(t). \tag{3.14}$$

Form a Vitali system by assigning to every $t \in H \cap \gamma^{-1}(A^*)$ the sequence $(\widetilde{U}_k(t))_{k\in\mathbb{N}}$ of closed sets. Due to the regularity condition (3.11) the Vitali Covering Theorem is applicable (see e.g. [F, Lemma 471**O**]) and it yields a finite disjoint subfamily of the Vitali system covering the set $H \cap \gamma^{-1}(A^*)$ up to a measure η,

$$\mathscr{L}^n\Big(H \cap \gamma^{-1}(A^*) \setminus \bigcup_{i=1}^{m} \widetilde{U}_{k_i}(t_i)\Big) < \eta. \tag{3.15}$$

We simplify the notation by setting $y_i = y_{k_i}(t_i)$, $U_i = U_{k_i}(t_i)$, $\widetilde{U}_i = \widetilde{U}_{k_i}(t_i)$, $L_i = L_{t_i}$, and $r_i = r_{k_i}(t_i)$. In order to apply Lemma 3.1 introduce new norms $\|t\|_i := \|L_i t\|$ on $\mathbb{R}^n$, $i = 1, \dots, m$. Since $t_i \in H$, we have

$$\frac{1}{q}|t| \le \|t\|_i \le q|t|, \qquad i = 1, \dots, m.$$

Then the sets U_i and $\widetilde{U}_i$ are balls in the norm $\|\cdot\|_i$ with centre t_i and radius r_i and $(1+\varepsilon) r_i$, respectively. Thus

$$\mathscr{L}^n \widetilde{U}_i = (1+\varepsilon)^n \mathscr{L}^n U_i.$$

For every $i = 1, \dots, m$ define $\gamma_i \colon U_i \to X$ by

$$\gamma_i(t) = y_i + \gamma'(t_i; t - t_i) - \gamma(t) = y_i + L_i(t - t_i) - \gamma(t), \quad t \in U_i.$$

With the help of Lemma 3.1 we can find extensions $\widetilde{\gamma}_i \in \Gamma_n(X)$ such that

- $\widetilde{\gamma}_i = 0$ outside $\widetilde{U}_i$;
- $\|\widetilde{\gamma}_i\|_{C^1} \le 2\|\gamma_i\|_\infty + K(\|\gamma_i'\|_\infty + \|\gamma_i\|_\infty/(\varepsilon r_i))$.

Set $\gamma_0(t) = \gamma(t) + \sum_{i=1}^m \widetilde{\gamma}_i(t)$. To estimate $\|\gamma - \gamma_0\|_{C^1}$ we notice that by (3.13), (3.14) and (3.11) we have

$$\begin{aligned}\|\gamma_i\|_\infty &= \|y_i + L_i(\cdot - t_i) - \gamma\|_\infty \\ &\le \|y_i - \gamma(t_i)\| + \|\gamma(t_i) + L_i(\cdot - t_i) - \gamma\|_\infty \\ &\le \varepsilon^2 r_i + \varepsilon^2 \operatorname{diam} U_i \\ &\le \big(\tfrac12 q + 1\big)\varepsilon^2 \operatorname{diam} U_i.\end{aligned}$$

Since $\operatorname{diam} U_i < \delta$ (see (3.12)), by the choice of δ we have

$$\|\gamma_i'\|_\infty = \|\gamma'(t_i) - \gamma'\|_\infty < \varepsilon.$$

Hence using also (3.11), (3.12), (3.7) and the fact that $\delta \le 1$, we get

$$\begin{aligned}\|\widetilde{\gamma}_i\|_{C^1} &\le 2\big(\tfrac12 q + 1\big)\varepsilon^2 \operatorname{diam} U_i + K\bigg(\varepsilon + \frac{2q\big(\frac12 q + 1\big)\varepsilon^2 \operatorname{diam} U_i}{\varepsilon \operatorname{diam} U_i}\bigg) \\ &\le \Big(2\big(\tfrac12 q + 1\big)\varepsilon\delta + K\big(1 + 2q\big(\tfrac12 q + 1\big)\big)\Big)\varepsilon < \tau.\end{aligned}$$

It follows that $\|\gamma - \gamma_0\|_{C^1} \le \max_{1\le i\le m} \|\widetilde{\gamma}_i\|_{C^1} < \tau$ and so $\gamma_0 \in B(\gamma, \tau)$.

It remains to show that γ_0 does not belong to S_α, i.e. $\mathscr{L}^n \gamma_0^{-1}(A) < \alpha$. Since

$$\mathscr{L}^n \gamma_0^{-1}(A) = \sum_{i=1}^m \mathscr{L}^n(\widetilde{U}_i \cap \gamma_0^{-1}(A)) + \mathscr{L}^n\Big(\gamma_0^{-1}(A) \setminus \bigcup_{i=1}^m \widetilde{U}_i\Big), \tag{3.16}$$

we have to estimate both summands. For the first one, notice that γ_0 is an affine map on each U_i, $\gamma_0 = \gamma + \gamma_i = y_i - L_i(t_i) + L_i$. Denoting for a moment $T_i = y_i - L_i(t_i) + L_i$, one easily checks that

$$T_i(U_i) = y_i + B(0, r_i) \cap V_{t_i} = B(y_i, r_i) \cap (y_i + V_{t_i}).$$

Using now (3.3) and (3.10) we obtain

$$\begin{aligned}\mathscr{L}^n(U_i \cap \gamma_0^{-1}(A)) &= \mathscr{L}^n\big(T_i^{-1}(B(y_i, r_i) \cap A)\big) \\ &= T_i[\mathscr{L}^n](B(y_i, r_i) \cap A) \\ &= c(T_i)\, \mathscr{H}^n\big(B(y_i, r_i) \cap (y_i + V_{t_i}) \cap A\big) \\ &\le \lambda c(T_i) \mathscr{H}^n\big(B(y_i, r_i) \cap (y_i + V_{t_i})\big) = \lambda \mathscr{L}^n U_i.\end{aligned}$$

Thus we have

$$\begin{aligned}\sum_{i=1}^m \mathscr{L}^n(\widetilde{U}_i \cap \gamma_0^{-1}(A)) &\le \sum_{i=1}^m \mathscr{L}^n(\widetilde{U}_i \setminus U_i) + \sum_{i=1}^m \mathscr{L}^n(U_i \cap \gamma_0^{-1}(A)) \\ &\le \bigg(1 - \frac{1}{(1+\varepsilon)^n}\bigg) \sum_{i=1}^m \mathscr{L}^n \widetilde{U}_i + \lambda \sum_{i=1}^m \mathscr{L}^n U_i\end{aligned}$$

$$= \frac{(1+\varepsilon)^n - 1 + \lambda}{(1+\varepsilon)^n} \sum_{i=1}^{m} \mathscr{L}^n \widetilde{U}_i$$
$$\le \frac{(1+\varepsilon)^n - 1 + \lambda}{(1+\varepsilon)^n} \mathscr{L}^n G$$
$$\le \frac{(1+\varepsilon)^n - 1 + \lambda}{(1+\varepsilon)^n} (\mathscr{L}^n \gamma^{-1}(A^*) + \eta),$$

where the last inequality follows from (3.9).

The second summand in (3.16) is easy to control. Since the surface γ satisfies $\mathscr{L}^n\gamma^{-1}(A) = \mathscr{L}^n\gamma^{-1}(A^*)$ by (3.5) and $\gamma_0 = \gamma$ outside $\bigcup_{i=1}^k \widetilde{U}_i$, we conclude that (3.15) holds for γ_0 and A as well:

$$\mathscr{L}^n\Big(H \cap \gamma_0^{-1}(A) \setminus \bigcup_{i=1}^{m} \widetilde{U}_{k_i}(t_i)\Big) < \eta.$$

By the choice of H,

$$\mathscr{L}^n\Big(\gamma_0^{-1}(A) \setminus \bigcup_{i=1}^{k} \widetilde{U}_i\Big) \le \mathscr{L}^n\Big((H \cap \gamma_0^{-1}(A)) \setminus \bigcup_{i=1}^{k} \widetilde{U}_i\Big) + \eta < 2\eta.$$

Using these estimates in (3.16) together with (3.8) we obtain

$$\mathscr{L}^n\gamma_0^{-1}(A) \le \frac{(1+\varepsilon)^n - 1 + \lambda}{(1+\varepsilon)^n}(\mathscr{L}^n\gamma^{-1}(A^*) + \eta) + 2\eta \le \frac{1+\lambda}{2}\,\alpha_1 < \alpha$$

by the condition (3.4).

In this way we have proved that all sets S_α, $\alpha > 0$, are nowhere dense. Consequently, A is Γ_n-null. Since n belongs to an infinite subset of positive integers, A is Γ-null by Lemma 2.2. ∎

Corollary 3.7. *Let A be a Borel subset of a separable Banach space and let $A \subset \bigcup_{n=1}^\infty A_n$. If for every n the set A_n is a $\mathcal{P}^{\Gamma}_{\lambda_n}$-set, then A is Γ-null. In particular, every $\mathcal{P}^{dc}$-set is Γ-null.*

Proof. The last statement follows immediately from Remark 3.5. ∎

4. Convex functions. In this section we prove that for every convex continuous function f on a Banach space with separable dual the set N_F of all points of Fréchet nondifferentiability of f is a countable union of $\mathcal{P}^{dc}$-sets. This result on smallness of N_F is new. Moreover, combining it with Corollary 3.7 we obtain a simpler proof of the known result of [LP] that N_F is Γ-null.

Lemma 4.1. *Let X be a Banach space and let $f\colon X \to \mathbb{R}$ be a convex Lipschitz function. Let $0 < \varepsilon \le 1$ and $\varphi \in X^*$. Denote by $A = A(f,\varepsilon,\varphi)$ the* *set of all $x \in X$ for which there exists $\varphi_x \in \partial f(x)$ such that*

$$\|\varphi_x - \varphi\| \le \varepsilon, \tag{4.1}$$

$$\limsup_{h\to 0} \frac{|f(x+h) - f(x) - \varphi_x(h)|}{\|h\|} \ge 6\varepsilon. \tag{4.2}$$

Assume that f is Gâteaux differentiable at a point $a \in A$. Then A is P-small at a.

Proof. Choose $K \ge 1$ such that f is Lipschitz with constant K. For the proof it will be convenient to use the alternative definition of P-smallness mentioned in Remark 3.3(ii). Let $0 < \delta \le 1$ and a finite-dimensional subspace $V \subset X$ be given. We show that there is $y \in X$ with $0 < \|y - a\| < \delta$ satisfying

$$B(y, \|y-a\|/\delta) \cap (y + V) \cap A = \emptyset. \tag{4.3}$$

Since f is Gâteaux differentiable at $a \in A$, we have $\|\varphi - f'_G(a)\| \le \varepsilon$. To simplify the notation we set

$$g(x) := f(x) - f(a) - f'_G(a)(x-a), \qquad x \in X.$$

Then clearly $g(a) = 0$, $g'_G(a) = 0$ and

$$\limsup_{h\to 0} \frac{|g(a+h)|}{\|h\|} \ge 6\varepsilon. \tag{4.4}$$

The function $\xi(t) := g(a+t)$, $t \in V$, is a Lipschitz function on a finite-dimensional space. For such functions the Gâteaux and Fréchet differentiability are equivalent. Since $\xi'_G(0) = 0$ we can choose $0 < r \le 1$ so that

$$|g(a+t)| \le \delta\varepsilon\|t\| \quad \text{whenever } t \in V \cap B(0,r). \tag{4.5}$$

By (4.4) there is $h \in X$ with $0 < \|h\| < \delta\varepsilon r/(2K)$ such that

$$|g(a+h)| \ge 5\varepsilon\|h\|. \tag{4.6}$$

To prove (4.3) with $y := a + h$ suppose the contrary: There is a point x such that

$$x \in B(y, \|y-a\|/\delta) \cap (y+V) \cap A.$$

Denote $M = 2K/\varepsilon$. The restriction imposed upon $\|h\|$ implies

$$M\|x-y\| \le M\frac{\|y-a\|}{\delta} = M\frac{\|h\|}{\delta} < \frac{2K}{\varepsilon}\,\frac{\varepsilon r}{2K} = r.$$

Noticing that $M \ge 1$ we also have, in particular, $\|x-y\| < r$. Since $x-y \in V$, the condition (4.5) gives

$$|g(a + (x-y))| \le \varepsilon\|y-a\|, \tag{4.7}$$

$$|g(a - M(x-y))| \le M\varepsilon\|y-a\|. \tag{4.8}$$

Let $\varphi_x \in \partial f(x)$ be as in the definition of the set A. Then $\|\varphi_x - \varphi\| \le \varepsilon$, $\varphi_x - f'_G(a) \in \partial g(x)$ and

$$\|\varphi_x - f'_G(a)\| \le \|\varphi_x - \varphi\| + \|\varphi - f'_G(a)\| \le 2\varepsilon.$$

Consequently, using also (4.7) we obtain

$$\begin{aligned} g(x) &\le g(a + (x - y)) + (\varphi_x - f'_G(a), y - a) \\ &\le \varepsilon\|y - a\| + \|\varphi_x - f'_G(a)\| \, \|y - a\| \\ &\le \varepsilon\|y - a\| + 2\varepsilon\|y - a\| = 3\varepsilon\|y - a\|. \end{aligned} \tag{4.9}$$

Since g is clearly a Lipschitz function with constant $2K$, we have

$$|g(y - M(x - y)) - g(a - M(x - y))| \le 2K\|y - a\|,$$

which combined with (4.8) implies

$$|g(y - M(x - y))| \le M\varepsilon\|y - a\| + 2K\|y - a\| = 2M\varepsilon\|y - a\|. \tag{4.10}$$

Finally, the point y is a convex combination

$$y = \frac{M}{M+1}x + \frac{1}{M+1}(y - M(x - y)).$$

Thus, using convexity of g, (4.6), (4.9) and (4.10), we obtain

$$\begin{aligned} 5\varepsilon\|y - a\| \le g(y) &\le \frac{M}{M+1}g(x) + \frac{1}{M+1}g(y - M(x - y)) \\ &\le \frac{M}{M+1}\, 3\varepsilon\|y - a\| + \frac{M}{M+1}\, 2\varepsilon\|y - a\| \\ &< 5\varepsilon\|y - a\|, \end{aligned}$$

which is a contradiction. ∎

We are ready for the main result of this section.

THEOREM 4.2. *Let X be a Banach space with X^* separable, $G \subset X$ an open convex set and $f : G \to \mathbb{R}$ a continuous convex function. Then the set of points where f is not Fréchet differentiable is a countable union of $\mathcal{P}^{dc}$-sets. Consequently, f is Fréchet differentiable Γ-almost everywhere in G.*

Proof. Since f is locally Lipschitz and X is separable, we may and will assume that f is Lipschitz and convex, and that it is defined on the whole space. (We use the well known and easy fact that every Lipschitz convex function on an open convex set can be extended to a Lipschitz convex function on the whole space.) Denote by N_F and N_G the sets of points of Fréchet and Gâteaux nondifferentiability of f, resp. The set N_G is contained in a set A_0 which is a countable union of *d.c.*-hypersurfaces (see Remark 2.4). We are going to show that N_F is a countable union of $\mathcal{P}^{dc}$-sets.

Let $\{\varphi_k \mid k \in \mathbb{N}\}$ be a dense sequence in X^*. For every $m \in \mathbb{N}$ we define

$$F_m = \left\{ x \in X \;\middle|\; \sup_{\|h\| \le \delta} |f(x+h) + f(x-h) - 2f(x)| \ge \frac{\delta}{m} \text{ for every } \delta > 0 \right\}.$$

Then $N_F = \bigcup_{m=1}^{\infty} F_m$ and every F_m is closed (see [BL, proof of Proposition 4.16]). We further set

$$F_{m,k} = \left\{ x \in F_m \;\middle|\; \text{there is } \varphi_x \in \partial f(x) \text{ such that } \|\varphi_k - \varphi_x\| \le \frac{1}{24m} \right\}.$$

Clearly, $N_F = \bigcup_{m,k=1}^{\infty} F_{m,k}$. We are going to prove that each $F_{m,k}$ is closed and P-small at the points of $F_{m,k} \setminus A_0$.

Recall that the mapping $x \mapsto \partial f(x)$ is norm-weak* upper semicontinuous by [P, Proposition 2.5]. Since the ball $B(\varphi_k, 1/(24m))$ is weak* closed, we conclude that $F_{m,k}$ is a closed set.

Now fix m, k and set $\varphi := \varphi_k$ and $\varepsilon := 1/(24m)$. We prove that

$$F_{m,k} \subset A(f, \varepsilon, \varphi), \tag{4.11}$$

where $A(f, \varepsilon, \varphi)$ is as in Lemma 4.1. Let $x \in F_{m,k}$. Choose $\varphi_x \in \partial f(x)$ such that $\|\varphi_k - \varphi_x\| \le 1/(24m)$. So the condition (4.1) of Lemma 4.1 holds. To verify (4.2), suppose the contrary. Then there exists $\delta > 0$ such that

$$|f(x+h) - f(x) - \varphi_x(h)| < 6\varepsilon\|h\| \quad \text{whenever } \|h\| \le \delta. \tag{4.12}$$

By the definition of F_m there exists $h \in X$ with $\|h\| \le \delta$ such that

$$|f(x+h) + f(x-h) - 2f(x)| \ge \frac{\delta}{2m} = 12\delta\varepsilon. \tag{4.13}$$

Using (4.12), we obtain

$$\begin{aligned} &|f(x+h) + f(x-h) - 2f(x)| \\ &\qquad \le |f(x+h) - f(x) - \varphi_x(h)| + |f(x-h) - f(x) - \varphi_x(-h)| \\ &\qquad < 12\varepsilon\delta, \end{aligned}$$

which contradicts (4.13). Hence $F_{m,k} \subset A(f, \varepsilon, \varphi)$.

Let $x \in F_{m,k} \setminus A_0$. By (4.11) the point x belongs to $A(f, \varepsilon, \varphi)$ and it is also a point of Gâteaux differentiability of f. Lemma 4.1 now shows that $F_{m,k}$ is P-small at x. Therefore N_F is a countable union of $\mathcal{P}^{dc}$-sets and so it is Γ-null by Corollary 3.7. ∎

5. Quasiconvex functions. We recall that a function $f\colon X \to \mathbb{R}$ is *quasiconvex* if

$$f(\tau x + (1-\tau)y) \le \max\{f(x), f(y)\}$$

for all $x, y \in X$ and $\tau \in [0,1]$. This is equivalent to the requirement that the sets $\{x \in X \mid f(x) \le r\}$ are convex for all $r \in \mathbb{R}$.

In [R] it is proved (besides other results) that a continuous quasiconvex function on a separable reflexive Banach space is Hadamard differentiable off a Haar null set. In what follows we show that a continuous quasiconvex function on any separable Banach X space is Hadamard differentiable Γ-a.e. For X reflexive our result and that of [R] are incomparable, since the notions of Haar null set and Γ-null set are not comparable in general. However, our result is the first result on almost everywhere differentiability of quasiconvex functions with respect to a certain σ-ideal in general separable Banach spaces (even for Gâteaux differentiability).

Our proof is a combination of well-known methods and the following result.

Proposition 5.1. *Let $A \subset X$ be a closed convex subset of a separable Banach space X. Then the boundary ∂A is $P_{1/2}$-small at all of its points. Consequently, ∂A is Γ-null. In particular, a closed convex nowhere dense subset of X is Γ-null.*

Proof. Here we use the formulation given in Remark 3.3(iii).

Let $a \in \partial A$ and assume that $\delta > 0$ and a finite-dimensional space $V \subset X$ are given. Choose an open ball $B \subset B(a,\delta) \setminus A$ and use the Hahn–Banach theorem to separate the ball B from A by a closed hyperplane H. Let $y \in H \cap B(a,\delta) \setminus A$. We set $F = B(y, \|y-a\|/\delta) \cap (y+V)$ and denote by $\sigma\colon X \to X$ the symmetry map with respect to the point y, $\sigma(x) = 2y - x$. Then $F = (F \setminus A) \cup \sigma(F \setminus A)$, which implies

$$\mathscr{H}^m(F \cap \partial A) \le \mathscr{H}^m(F \cap A) \le \tfrac{1}{2}\mathscr{H}^m(F),$$

where $m = \dim V$.

Having verified that ∂A is $P_{1/2}$-small at all its points, we apply Proposition 3.6 to complete the proof. ∎

Remark 5.2. It follows from an unpublished observation of D. Preiss that Γ-nullness of ∂A can be proved in a much shorter way without using Proposition 3.6.

Before the statement of the last theorem let us recall one fact that will be needed in the proof. Several systems of exceptional sets were introduced in [PZ3]. We will refer to one of them, denoted by $\widetilde{C}$, and to the property that $\widetilde{C}$ is (strictly) smaller than the system of Γ-null sets (see [Z3]).

Theorem 5.3. *Let $f\colon X \to \mathbb{R}$ be a continuous quasiconvex function on a separable Banach space X. Then f is Hadamard differentiable Γ-a.e.*

Proof. Let $A = \{x \in X \mid f(x) \le \inf f\}$. Then A is a (possibly empty) closed convex set. The function f is locally cone monotone outside A (see [R, Lemma 4.1]). By [D], f is Hadamard differentiable on $X \setminus A$ except possibly on a set from $\widetilde{C}$. Since any such set is Γ-null, we may conclude that f is Hadamard differentiable at Γ-a.a. points of $X \setminus A$.

If A is nowhere dense, it is Γ-null by Proposition 5.1 and we are done. If A has nonempty interior then the boundary is again Γ-null and f is even constant and so Fréchet differentiable in the interior of A. ∎

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Jaroslav Tišer
Department of Mathematics
Faculty of Electrical Engineering
Czech Technical University
Technická 2
166 27 Praha 6, Czech Republic
E-mail: tiser@math.feld.cvut.cz

Luděk Zajíček
Charles University
Faculty of Mathematics and Physics
Sokolovská 83
186 75 Praha 8, Czech Republic
E-mail: zajicek@karlin.mff.cuni.cz

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