Fréchet differentiability of Lipschitz maps and porous sets in Banach spaces

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This paper is an outline of the main results obtained by the authors on this subject in recent years. The proofs of many of those results are very technical and involved. The complete proofs will be presented in a research monograph the authors are preparing at present. The content of this article is closely related to talks presented by the first named author at conferences: Banach spaces and their applications in analysis, Oxford, Ohio, 2006 and New trends in Banach space theory, Caceres, Spain, 2006.

1 Preliminaries

We start with the definition of porous sets and related notions.

Definition 1.1. A set E in a metric space (X, d) is said to be c-porous, $0 < c < 1$, if for every $x \in E$ and $0 < \varepsilon < c$ there is a $z \in X \setminus E$ such that $d(x, z) \leq \varepsilon$ and $B(z, cd(x, z)) \cap E = \emptyset$. A set which is c-porous for some $0 < c < 1$ is called porous.

If X is a Banach space and Y a subspace of X we can talk of a porous set in the direction of Y. A set $E \subset X$ is c-porous in the direction of Y if for every $x \in E$ and $0 < \varepsilon < c$ there is a $y \in Y$, $0 < \|y\| \leq \varepsilon$, such that $B(x + y, c\|y\|) \cap E = \emptyset$. A set which is porous in the direction of a 1-dimensional subspace of X will be called directionally porous.

A countable union of porous sets is called σ-porous. Similarly we define the notion of σ-directionally porous sets in a Banach space.

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We will use in the sequel several notions of null sets in separable Banach spaces X. The most simple notion is that of a Haar null set. A Borel set $E \subset X$ is said to be *Haar null* if there is a Borel probability measure μ on X such that $\mu(x+E) = 0$ for all $x \in X$. If $\dim X < \infty$ a set is Haar null iff it is Borel and its Lebesgue measure is 0. This fact will be true also for the other notions of null sets we shall encounter in the future except for the class $\widetilde{\mathcal{A}}$ mentioned below, where it is still open at present. For more informations on Haar null sets as well as most of the notions which appear in this section, and for references to original papers we refer to the book [1].

A map f defined on an open set in a Banach space X into a Banach space Y is said to be *Gâteaux differentiable* at a point x_0 if

$$\lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t} = Tu \tag{1}$$

exists for every $u \in X$ and T is a bounded linear operator from X to Y. The operator T is called the Gâteaux derivative and is denoted by $Df(x_0)$.

If for some fixed $u \in X$ the limit

$$f'(x_0; u) := \lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t}$$

exists we call it the directional derivative of f at x_0 in the direction u. Thus f is Gâteaux differentiable at x_0 if and only if $f'(x_0; u)$ exists for every u and forms a bounded linear operator as a function of u. In this case we have

$$f'(x_0; u) = Df(x_0)u.$$

A Banach space is said to have the *Radon Nikodým property*, *RNP* for short, if every Lipschitz map $f\colon \mathbb{R} \longrightarrow X$ has a derivative almost everywhere. This is known to be equivalent to the requirement that every such f has at least one point of differentiability.

It is well known that every Lipschitz map f from a separable space X into a space Y with *RNP* has many points of Gâteaux differentiability. More precisely,

Theorem 1.2. *Every Lipschitz map $f\colon X \longrightarrow Y$ with X separable and Y having the* RNP *is Gâteaux differentiable almost everywhere.*

This theorem has many versions which depend only on the nature of the notion “almost everywhere”, i.e. on the nature of the exceptional sets. One can take as the exceptional sets Haar null sets. A stronger result is obtained by taking a smaller class of exceptional sets, namely Gauss null sets. A Borel set E in a separable Banach space X is said to be *Gauss null* if for every non degenerate (i.e. not supported on a closed hyperplane of X) Gaussian measure μ on X we have $\mu(E) = 0$. At present the strongest known result of Theorem 1.2 (i.e. the smallest class of exceptional sets) is due to Preiss and Zajíček [10]. They define a class $\widetilde{\mathcal{A}}$ of exceptional sets. The definition of this class is somewhat involved and since we are not going to use it we do not reproduce the definition here.

We now pass to the definition of the main notion considered in this paper.

Definition 1.3. A function $f\colon G \longrightarrow Y$ defined on an open set G in a separable Banach space X into a Banach space Y is said to be Fréchet differentiable at a point $x_0 \in G$ if there is a bounded linear operator $T\colon X \longrightarrow Y$ so that

$$f(x_0 + u) = f(x_0) + Tu + o(\|u\|), \ u \to 0.$$

Of course, if f is Fréchet differentiable at x_0 then f is Gâteaux differentiable at x_0 and the operator T above is necessarily $Df(x_0)$. Stated otherwise: f is Fréchet differentiable at x_0 iff it is Gâteaux differentiable there and the limit in (1) exists uniformly on the unit sphere $\{u \mid \|u\| = 1\}$. The notion of Fréchet derivative is the most natural and useful notion of a derivative. If $\dim X < \infty$ and f is Lipschitz then the notions of Fréchet derivative and Gâteaux derivative coincide. This fails if $\dim X = \infty$.

To get a feeling of the difference between these two notions consider the simple example of the function $f\colon L^2[0,1] \longrightarrow L^2[0,1]$ defined by $f(u) = \sin u$. This function is everywhere Gâteaux differentiable with $Df(u) = \cos u$ (i.e. $Df(u)x = x\cos u$), but as it can be easily checked nowhere Fréchet differentiable.

It is an unfortunate fact that existence theorems for Fréchet derivatives are quite rare and usually quite hard to prove.

In the context of Fréchet differentiability we have to assume in existence results that the dual space X^* is separable, i.e. X is what is called an Asplund space. It is known, for example, that if X is separable with X^* non separable, there is an equivalent norm $\|\cdot\|$ on X so that the convex continuous function $f\colon X \longrightarrow \mathbb{R}$ defined by $f(u) = \|u\|$ is nowhere Fréchet differentiable. On the other hand if X is Asplund every convex $f\colon X \to \mathbb{R}$ is Fréchet differentiable on a dense G_δ subset of X, actually even outside a σ-porous set. We should point out in this connection that there is a continuous convex $f\colon \ell^2 \longrightarrow \mathbb{R}$ which is Fréchet differentiable only on a Gauss null set. (For these results and references to original papers we refer as before to [1].)

In view of the difficulty in proving existence results for Fréchet derivatives it is of interest to use some related weaker notions where existence proofs are more accessible (but often also not simple).

Definition 1.4. The function $f\colon X \longrightarrow Y$ is said to be ε-Fréchet differentiable at x_0, $\varepsilon > 0$, if there is a $\delta > 0$ and a bounded linear operator $T\colon X \longrightarrow Y$ so that

$$\|f(x_0 + u) - f(x_0) - Tu\| \leq \varepsilon \|u\|$$

if $\|u\| \leq \delta$.

If f is ε-Fréchet differentiable at some x_0 for every $\varepsilon > 0$ then clearly f is Fréchet differentiable at x_0. However, if we know only that for every $\varepsilon > 0$ there is a point x_ε at which f is ε-Fréchet differentiable then it is not clear that f is Fréchet differentiable at some point because x_ε depend on ε and in general there is no way to control how x_ε changes with ε.

Another notion which we find convenient to introduce in this paper is *asymptotic Fréchet differentiability.*

Definition 1.5. A function $f\colon X \longrightarrow Y$ is said to be asymptotically Fréchet differentiable at x_0 if there is a bounded linear operator $T\colon X \longrightarrow Y$ so that for every $\varepsilon > 0$ there is $\delta > 0$ and a subspace $Z \subset X$ of finite codimension so that

$$\|f(x_0 + z) - f(x_0) - Tz\| \leq \varepsilon \|z\|$$

whenever $z \in Z$ and $\|z\| < \delta$.

We would like to point out a simple connection between the notions of differentiability and the notions of porous set introduced in the beginning of this section. If $E \subset X$ is a porous set then the Lipschitz function $f\colon X \longrightarrow \mathbb{R}$ defined by

$$f(x) := \operatorname{dist}(x, E)$$

is nowhere Fréchet differentiable on E. Indeed, since f attains its minimum on E the only possible derivative of f at a point $x \in E$ is 0. However, by the definition of c-porosity there are points z in X arbitrarily close to x with

$$\|f(z) - f(x)\| = \operatorname{dist}(z, E) \geq c\|z - x\|.$$

Similarly if E is directionally porous then $\mathrm{dist}(x, E)$ is not Gâteaux differentiable at any point of E and thus E is negligible in any sense in which Theorem 1.2 is valid.

Our results below depend on some smoothness assumptions. Let us recall here the basic notions. A Banach space X is uniformly smooth if

$$\rho_X(t) := \sup_{\|x\|=1} \sup_{\|y\|\le t} \big(\|x+y\| + \|x-y\| - 2\big) = o(t),\ t \searrow 0.$$

For example, let $X = L^p$ and $t \in [0,1]$. If $1 < p \le 2$ then $\rho_{L^p}(t) \le C_p t^p$. If $p = 2$ then $\rho_{L^2}(t) = 2t^2$ and for $2 < p < \infty$ we have $\rho_{L^p}(t) \le C_p t^2$ and none of these estimates can be improved.

A variant of this notion which was introduced in [3] is asymptotic uniform smoothness. The modulus of asymptotic uniform smoothness of a space X is defined by

$$\bar\rho_X(t) := \sup_{\|x\|=1} \inf_{\dim X/Y<\infty} \sup_{\substack{y\in Y\\ \|y\|\le t}} \big(\|x+y\| - 1\big).$$

Clearly a uniformly smooth space is asymptotically uniform smooth but the converse is false since e.g. for $X = c_0$ the modulus $\bar\rho_X(t) \equiv 0$. There are duals to these notions (uniform convexity and asymptotic uniform convexity). We shall not need these notions so we do not define them here. On the other hand we shall need the extension of these notions for general convex functions $\Theta\colon X \longrightarrow \mathbb{R}$. The modulus of smoothness of Θ at the point $x \in X$ is defined by

$$\rho_\Theta(x,t) := \sup_{\|y\|\le t} \big(\Theta(x+y) + \Theta(x-y) - 2\Theta(x)\big).$$

The modulus of asymptotic smoothness of Θ at the point x is defined by

$$\bar\rho_\Theta(x,t) := \inf_{\substack{Y\subset X\\ \dim X/Y<\infty}} \sup_{\substack{y\in Y\\ \|y\|\le t}} \big(\Theta(x+y) + \Theta(x-y) - 2\Theta(x)\big).$$

Up to a constant factor this extends the previous notions defined for norms, for example $\bar\rho_X(t) \le \sup_{\|x\|=1} \bar\rho_{\|\cdot\|}(x,t) \le 2\bar\rho_X(t)$.

An important existence theorem for points of Fréchet differentiability is known for scalar valued Lipschitz functions.

Theorem 1.6. *Let X be an Asplund space and $f\colon X \longrightarrow \mathbb{R}$ a Lipschitz function. Then f is Fréchet differentiable on a dense set in X.*

This theorem is proved in [8]. A somewhat simplified proof (but not simple either) is presented in [6]. This is not an almost everywhere result. The proof in both papers is achieved by a (complicated) iterative construction of a sequence of points x_n which are shown to converge to a point x of Fréchet differentiability. Because the nature of the set of points of Fréchet differentiability is not known we do not know for example if two Lipschitz functions from X to $\mathbb{R}$ have a common point of Fréchet differentiability. Stated otherwise, it is not known if a Lipschitz map $f\colon X \longrightarrow \mathbb{R}^2$ must have a point of Fréchet differentiability. This is the source of difficultly for the arguments in Section 5 where the difficulty is overcome in the most important case, namely for X being a Hilbert space.

Both proofs of Theorem 1.6 yield also a mean value theorem for Fréchet derivatives.

Theorem 1.7. *Let f be a Lipschitz function on an open set G in an Asplund space X into the real line. Then for every pair of points $u, v \in G$ such that the line segment connecting them is in G we have*

$$\begin{aligned}\inf\bigl\{Df(x)(v-u) \bigm| x \in G,\ f \text{ is Fréchet differentiable at } x\bigr\}\\ \le f(v) - f(u)\\ \le \sup\bigl\{Df(x)(v-u) \bigm| x \in G,\ f \text{ is Fréchet differentiable at } x\bigr\}.\end{aligned}$$

The mean value theorem will have an important role in the coming sections.

In the paper [5] an investigation was carried out on the existence of Fréchet derivatives for functions with a more general range space. In this connection a new class of null sets in Banach spaces was defined as a notion combining measure and category.

Let $T = [0,1]^{\mathbb{N}}$ be endowed with the product topology and the product Lebesgue measure μ. Let $\Gamma(X)$ be the space of all continuous mappings $\gamma\colon T \longrightarrow X$ having continuous partial derivatives $\frac{\partial\gamma}{\partial t_i}$ with respect to all the variables (with one-sided derivatives at the points where the i-th coordinate is 0 or 1). The elements of $\Gamma(X)$ will be called (∞-dimensional) surfaces in X. We equip $\Gamma(X)$ with the topology generated by the semi-norms

$$\|\gamma\|_0 = \sup_{t\in T}\|\gamma(t)\|,\ \|\gamma\|_i = \sup_{t\in T}\left\|\frac{\partial\gamma}{\partial t_i}(t)\right\|,\ 1 \le i < \infty.$$

The space $\Gamma(X)$ with this topology is a Fréchet space (in particular it is a complete separable metric space).

A Borel set E in X will be called Γ-*null* if

$$\mu\{t \in T \mid \gamma(t) \in E\} = 0$$

for residually many $\gamma \in \Gamma(X)$.

This class can be easily shown to have the requirements we put on null sets. In particular, Theorem 1.2 is valid Γ almost everywhere (actually any set belonging to the class $\widetilde{\mathcal{A}}$ in sense of [10] is Γ null).

For the statement of the main result of [5] we need also the notion of a regular point of a function. Let f be a map from an open set in X to a Banach space Y. We say that $x \in X$ is a *regular point* of f if for every $\in X$ for which $f'(x;v)$ exists

$$\lim_{t \to 0} \frac{f(x+tu+tv) - f(x+tu)}{t} = f'(x;v)$$

uniformly for $\|u\| \leq 1$.

We can now state the main result of [5].

Theorem 1.8. *Suppose that G is an open set in a separable Banach space X. Let $\mathcal{L}$ be a norm separable subspace in the space of all bounded linear operators $L(X,Y)$. Let $f\colon G \longrightarrow Y$ be a Lipschitz function. Then f is Fréchet differentiable at Γ almost every point $x \in X$ at which it is regular, Gâteaux differentiable and $Df(x) \in \mathcal{L}$.*

It is not hard to show that for convex continuous $f\colon X \longrightarrow \mathbb{R}$ every point x is regular. For a general Lipschitz function $f\colon X \longrightarrow Y$ the set of irregular points of f is σ-porous. As a consequence we have

Corollary 1.9. *If X is an Asplund space then every convex continuous $f\colon X \longrightarrow \mathbb{R}$ is Γ almost everywhere Fréchet differentiable.*

Comparing this result with the example of a convex function $f\colon \ell^2 \longrightarrow \mathbb{R}$ which was mentioned above (being Fréchet differentiable only on a Gauss null set) we get that Γ null sets may be "orthogonal" to Gauss null sets in the sense that the subset E of ℓ^2 on which f is Fréchet differentiable is Gauss null while $\ell^2 \setminus E$ is Γ null.

Corollary 1.10. *Let X be an Asplund space. Then every Lipschitz function from X to $\mathbb{R}$ is Γ almost everywhere Fréchet differentiable if and only if every porous set in X is Γ null.*

The "only if" part follows from the observation made above concerning the function $f(x) = \operatorname{dist}(x, E)$ with E porous. It was proved in [5] that the class of spaces X in which every porous set is Γ null includes $C(K)$ spaces with K scattered, subspaces of c_0 and even some reflexive infinite dimensional Banach spaces (but not ℓ^p for $1 < p < \infty$, in particular not the Hilbert space). For spaces satisfying the condition in Corollary 1.10 we can restate Theorem 1.8 as follows

Theorem 1.11. *Let G be an open set in a separable Banach space X in which every porous set is Γ null, $\mathcal{L}$ a norm separable subspace of $L(X,Y)$ and $f\colon G \longrightarrow Y$ a Lipschitz map. Then f is Fréchet differentiable at Γ almost every point in which f is Gâteaux differentiable and $Df(x) \in \mathcal{L}$.*

There is also a formulation of Theorem 1.8 connected to the mean value theorem. This formulation involves slices of the set of Gâteaux derivatives. Let Υ be a subset of $L(X,Y)$. By a *slice* S of Υ we mean a set of the form

$$S(\Upsilon, v_1, \dots, v_m, y_1^*, \dots, y_m^*, \delta) = \Big\{ T \in \Upsilon \;\Big|\; \sum_{i=1}^{m} y_i^*(Tv_i) > \alpha - \delta \Big\},$$

where $m \in \mathbb{N}$, $v_1, \dots, v_m \in X$, $y_1^*, \dots y_m^* \in Y^*$, $\delta > 0$ and

$$\alpha = \sup_{T \in \Upsilon} \sum_{i=1}^{m} y_i^*(Tv_i).$$

The following is proved in [5].

Theorem 1.12. *Let $f\colon G \longrightarrow Y$ be a Lipschitz map which is Fréchet differentiable at Γ almost every point of an open set G of X. Then for every slice S of the set of Gâteaux derivatives of f the set of points x at which f is Fréchet differentiable and $Df(x) \in S$ is not Γ null.*

To clarify that this is a mean value theorem note that if the range is 1-dimensional then slices are the sets of the form $\{T \in \Upsilon \mid Tx > \alpha - \delta\}$ where $\alpha = \sup\{Tx \mid T \in \Upsilon\}$.

As already pointed out, porous sets are not necessarily Γ-null in spaces such as ℓ_p, $1 < p < \infty$. In such spaces, before the work announced here, there were no results asserting Fréchet differentiability of Lipschitz maps into even two dimensional spaces. However, existence of points of ε-Fréchet differentiability of Lipschitz maps from super-reflexive spaces into finite dimensional spaces was shown in [4] and this result was extended to asymptotically uniformly smooth spaces in [3].

Theorem 1.13. *Let $f\colon X \longrightarrow Y$ be a Lipschitz map, where X is asymptotically uniformly smooth and Y is finite dimensional. Then for every $\varepsilon > 0$ there is $x \in X$ at which f is both Gâteaux differentiable and ε-Fréchet differentiable.*

In the coming sections we shall announce some of the main results of our recent investigation.

2 Γ_n-null sets and a variational principle

We first introduce the notions of Γ_n-null sets, $n \geq 1$. These are similar to the notion of Γ-null sets from [5] mentioned above in which the infinite dimensional surfaces in X are replaced by the n-dimensional ones.

Let $T_n = [0,1]^n$ be endowed with the product topology and product Lebesgue measure μ_n. We denote by $\Gamma_n(X)$ the space of n-dimensional surfaces in X, i.e., functions $\gamma : [0,1]^n \to X$ which are continuous and continuously differentiable (except for the boundary points where we require only the appropriate one sided derivatives). We norm $\Gamma_n(X)$ by the norm

$$\sup_{t\in T_n} |\gamma(t)| + \sum_{j=1}^{n} \sup_{t\in T_n} \left| \frac{\partial\gamma}{\partial t_j}(t) \right|.$$

With this norm $\Gamma_n(X)$ becomes a Banach space. A Borel set E in X will be called Γ_n-null if

$$\mu_n\{t \in T_n \mid \gamma(t) \in E\} = 0$$

for residually many $\gamma \in \Gamma_n(X)$.

One dimensional surfaces are called, as usual, curves.

There is a natural connection between Γ_n-null sets and Γ-null sets as the following simple Proposition shows.

Proposition 2.1. *If a G_δ set E in X is Γ_n-null for infinitely many values of n then it is Γ-null.*

Our aim is to show that under suitable assumptions, given a (porous) set $E \subset X$ we can modify a surface $\gamma \in \Gamma_n(X)$ a little so that it cuts E only by a set of measure zero. A natural way of doing this is to construct a sequence $(\gamma_j)_{j=1}^{\infty}$ so that the measure of $\{t \in T_n \mid \gamma_j(t) \in E\}$ decreases to zero while keeping control of some parameter ("energy") associated with γ_j (e.g., in the case of curves we may use their length) and in the limit to get a surface $\tilde{\gamma}$ which intersect E in a set of measure zero and, thanks to the control of the "energy," is close to γ.

Instead of working with sequences, we found it useful to work with a variational principle. The main difference between this principle and the previous ones is that we need to work in an incomplete metric space (the set of surfaces that meet a given G_δ set in a set of large measure) and we need to minimize a function f that is not lower semi-continuous. Fortunately, this set S of surfaces is a G_δ subset of $\Gamma_n(X)$ even if $\Gamma_n(X)$ is equipped just with the norm $\sup_{t\in T_n} |\gamma(t)|$. So in S, Cauchy sequences satisfying a mild additional assumption converge. A similar observation handles the problem of semi-continuity of f.

Proposition 2.2. *Suppose that X is a set equipped with two metrics $d_0 \leq d$, X is d-complete, $S \subset X$ is G_δ in (X, d_0) and $f : S \to \mathbb{R}$ is d_0-upper semi-continuous. Then there are functions $\delta_j(x_1, \dots, x_j) : S^j \to (0, \infty)$ such that every d-Cauchy sequence $(x_j)_{j=1}^\infty$ in S with*

$$d_0(x_j, x_{j+1}) \leq \delta_j(x_1, \dots, x_j)$$

converges in the metric d to some $x \in S$ and

$$f(x) \leq \liminf_{j\to\infty} f(x_j).$$

This leads us to a variational principle for spaces equipped with two metrics, which contains the variational principle from [2] as a special case (when $d_0 = 0$).

Theorem 2.3. *Let S be a set equipped with two metrics $d_0 \leq d$ and let $f : S \to \mathbb{R}$ be a function bounded from below. Suppose further that there are functions $\delta_j(x_1, \dots, x_j) : S^j \to (0, \infty)$ with the following property: every d-Cauchy sequence $(x_j)_{j=1}^\infty$ in S such that*

$$d_0(x_j, x_{j+1}) \leq \delta_j(x_1, \dots, x_j),$$

converges in the d metric to some $x \in S$ and

$$f(x) \leq \liminf_{j\to\infty} f(x_j).$$

Let $F : S \times S \to [0,\infty]$ be d lower semi-continuous in the second variable with $F(x,x) = 0$ for all $x \in S$ and $\inf_{d(x,y)>s} \max(F(x,y), d_0(x,y)) > 0$ for each $s > 0$.

Then for every sequence of positive numbers $(\lambda_j)_{j=1}^{\infty}$ there is a sequence $(x_j)_{j=1}^{\infty}$ in S such that for some d_0 continuous $\varphi : S \to [0,\infty)$, the function

$$h(x) := f(x) + \varphi(x) + \sum_{j=1}^{\infty} \lambda_j F(x_j, x)$$

attains its minimum on S.

Using a somewhat more detailed version of this principle we get the following results.

Theorem 2.4. *Suppose that X admits a convex function Θ which is smooth with modulus of smoothness $o(t^n \log^{n-1}(1/t))$ (as $t \searrow 0$) at every point and satisfies $\Theta(0) = 0$ and $\inf_{\|x\|>s} \Theta(x) > 0$ for every $s > 0$. Then every porous set E in X is Γ_n-null.*

Theorem 2.5. *In an Asplund space every porous set is Γ_1-null.*

Theorem 2.6. *Every σ-directionally porous subset of any Banach space is Γ_1-null as well as Γ_2-null.*

Theorem 2.7. *Let X be a separable Banach space with*

$$\bar{\rho}_X(t) = o(t^n \log^{n-1}(1/t)) \text{ as } t \to 0.$$

Then every porous set in X is contained in a union of a σ-directionally porous set and a Γ_n-null G_δ set.

By combining these results with Proposition 2.1 we get

Theorem 2.8. *Every porous set in a separable Banach space X with $\bar{\rho}_X(t) = o(t^n)$ for each n is Γ-null.*

This gives a somewhat more general class of examples of spaces for which Theorem 1.11 holds.

3 Criteria of ε differentiability

To state the main result of this section, we use the following notation for slices of the set of Gâteaux derivatives of $\mathbb{R}^n$ valued Lipschitz maps. For a Lipschitz map $f = (f_1, \ldots, f_n)$ of a non-empty open set $G \subset X$ of a separable Banach space X into $\mathbb{R}^n$ we denote by $D(f)$ the set of points in G at which f is Gâteaux differentiable. Then slices of the set of Gâteaux derivatives of f are sets of the form

$$S(f, (u_i)_{i=1}^n, \delta) = \Big\{ Df(x) \;\Big|\; x \in D(f),\ \sum_{i=1}^n f_i'(x; u_i) > \alpha - \delta \Big\},$$

where $(u_i)_{i=1}^n$ are points in X and

$$\alpha = \sup_{x \in D(f)} \sum_{i=1}^n f_i'(x; u_i).$$

To understand the connection between smallness of porous sets and differentiability, we recall that Corollary 1.10 says that once we know that porous sets in X are Γ-null, Fréchet differentiability of Lipschitz maps into finite dimensional spaces follows. It is therefore natural to hope that in spaces in which porous sets are Γ_n-null, Fréchet differentiability of Lipschitz maps into at least spaces of small finite dimension can be shown. In view of the mean value estimates, we may even hope that in these spaces, all slices of the set of Gâteaux derivatives of any Lipschitz map into a space with dimension not exceeding n contain Fréchet derivatives. (See the following Section for examples showing that in this context the dimension of the target space cannot exceed n.)

These statements remain open, but we were able to prove them after replacing Fréchet derivatives with ε-Fréchet derivatives.

Theorem 3.1. *Suppose that the Asplund space X has the property that every porous set in X can be covered by a union of a Haar null set and a Γ_n-null G_δ set. Let f be a Lipschitz map from a non-empty open set $G \subset X$ to a Banach space of dimension not exceeding n. Then for every $\varepsilon, \delta > 0$ and $(u_i)_{i=1}^n \in X$, the slice*

$$S(f; u_1, \ldots, u_n; \delta)$$

contains a point of ε-Fréchet differentiability of f.

4 Examples of big porous sets and not too often Fréchet differentiable maps

In this section we announce the existence of big porous sets and Lipschitz maps into finite dimensional spaces which have no ε-Fréchet derivatives in some slices of their sets of Gâteaux derivatives. This shows that the results of Sections 2 and 3 are quite sharp. We use here the word "quite" firstly to indicate that instead of working in spaces that are not asymptotically smooth with modulus of smoothness $o(t^n \log^{n-1}(1/t))$ we work in spaces that are asymptotically convex with a slightly bigger modulus, and secondly to indicate that instead of uniform convexity in the space we work with uniform convexity in the dual with the dual modulus $o(t^{n/(n-1)} \log^{\beta}(1/t))$, $\beta > 1$, (which may be a stronger requirement when the space is not reflexive).

The starting point of these examples is the following observation.

Example 4.1. *In ℓ_1 there is a σ-porous set whose complement meets every curve in a set of* 1*-dimensional Hausdorff measure zero. In particular, this set has Γ_1-null complement and is not Haar null.*

Its construction is based on a connection between size of porous sets on curves and existence of functions that are close to being non-differentiable at points where some directional derivative is positive.

Example 4.2. *For every $\varepsilon > 0$ there is a porous set E in ℓ_1 and a function $f : \ell_1 \to [0, \varepsilon]$ with* $\mathrm{Lip}(f) \le 1$ *such that f is increasing in the first coordinate direction e_1 and $f'(x; e_1) = 1$ whenever $x \notin E$.*

It follows that for suitable $\eta > 0$, E has small measure on every curve γ with $\|\gamma'(t) - e_1\| < \eta$, from which it is easy to finish the argument showing Example 4.1. Also, it would be easy to use Example 4.2 to find a Lipschitz function $f : \ell_1 \to \mathbb{R}$ such that f is increasing but not constant in the direction e_1 but, however small $\varepsilon > 0$ may be, there is no point of ε-Fréchet differentiability with $f'(x; e_1) > \varepsilon$. Of course, since ℓ_1 is not Asplund, such an example may be found in an easier way by starting from a nowhere ε-Fréchet differentiable function. However, an analogous argument for maps into $\mathbb{R}^n$ gives similar, but considerably more interesting, examples in, e.g., ℓ_p spaces for $1 < p < \infty$. (See Theorem 4.5.)

Returning to Example 4.1 we remark that the method indicated above can be used to show that any space containing ℓ_1 has a σ-porous subset with

Γ_1-null complement. By a very different approach, namely by a surprising use of the theory of Lipschitz quotients, this result was proved in [7]. In spite of these results, the question whether the converse to Theorem 2.5 holds, i.e, whether in any non-Asplund space there is a σ-porous subset with Γ_1-null complement, remains open.

Let w be an Orlicz function and use in the Orlicz space associated to w the norm

$$\|\{c_i\}_{i=1}^{\infty}\|_w = \inf\{\lambda > 0 \mid \sum_{i=1}^{\infty} w(|c_i|/\lambda) \leq 1\}.$$

A sequence of elements $\{x_i\}_{i=1}^{\infty}$ in a Banach space X is said to satisfy the upper w estimate if there is a constant A so that for every sequence $\{c_i\}_{i=1}^{\infty}$ of real numbers and $k \in N$

$$\|\sum_{i=1}^{k} c_i x_i\| \leq A\|\{c_i\}_{i=1}^{k}\|_w.$$

In case $w(t) \approx t^q$ as $t \to 0$ we speak about the upper q estimate.

The following is a rather simple proposition

Proposition 4.3. *A Banach space which is asymptotically uniformly smooth with modulus w contains a normalized sequence $\{x_i\}_{i=1}^{\infty}$ which satisfies the upper w estimate.*

Note that the converse to Proposition 4.1 is false since $C[0,1]$ contains for any w a normalized sequence satisfying the upper w estimate while it is not isomorphic to an asymptotically smooth space (e.g. since it is not an Asplund space).

The following are the main results in this section.

Theorem 4.4. *Let X be a separable Banach space and let $n > 1$. Suppose that for some $\beta > 1$, X^* contains a normalized sequence satisfying the upper $t^{n/(n-1)}/\log^{\beta}(1/t)$ estimate. Then X contains a σ-porous set whose complement meets every n-dimensional surface in a set of n-dimensional Hausdorff measure zero.*

In particular, the σ-porous set from this Theorem has Γ_n-null complement and is neither Haar null nor Γ_n null.

The construction of such a σ-porous set resembles a construction in [9]. In the same paper there is also a much simpler construction in every separable infinite dimensional space of a σ-porous set whose complement has measure zero on every line (see also [1], Chapter 6). Such a set obviously cannot be Haar null.

Theorem 4.5. *Let X be a separable Banach space and let $n > 1$. Suppose that for some $q > n/(n-1)$, X^* contains a normalized sequence satisfying the upper q estimate. Then there is a Lipschitz map $f : X \to \mathbb{R}^n$ and a continuous linear functional y^* on $L(X, \mathbb{R}^n)$ so that $y^*(Df(x)) \geq 0$ whenever f is Gâteaux differentiable at x and for every small $\varepsilon > 0$ the set*

$$\{Df(x) \mid f \text{ is Gâteaux differentiable at } x \text{ with } y^*(Df(x)) > \varepsilon\}$$

is a nonempty slice of the set of all Gâteaux derivatives of f which contains no ε-Fréchet derivative of f.

5 Asymptotic Fréchet differentiability for Lipschitz maps into $\mathbb{R}^n$

The main result in this section actually is concerned with proper Fréchet differentiability.

Theorem 5.1. *Assume that the space X admits a continuous convex function Θ such that for some $n \geq 1$*

(i) $\inf\{\Theta(x); \|x\| > s\} > \Theta(0)$ *for all $s > 0$,*

(ii) $\sup_{x \in X} \rho_\Theta(x,t) = O(t^n \log^{n-1}(1/t))$ *as $t \searrow 0$,*

(iii) *for every $x \in X$, $\rho_\Theta(x,t) = o(t^n \log^{n-1}(1/t))$ as $t \searrow 0$.*

Let f be a Lipschitz map from a nonempty open set G of X to a space of dimension not exceeding n. Then every slice of the set of Gâteaux derivatives of f contains a Fréchet derivative.

For $n = 1$, this theorem is just a restatement of Theorems 1.7. Condition (iii) above can only hold for $n \leq 2$ since affine maps are the only convex functions which satisfy $\rho_\Theta(x,t) = o(t^2)$ at every $x \in X$ even if $X = \mathbb{R}$ (clearly affine maps fail to satisfy (i)). The proof of Theorem 5.1 resembles the proof in [8]. It involves a delicate iterative construction of a sequence

$\{x_n\}_{n=1}^{\infty}$ in X which is shown to converge to a point x at which f is Fréchet differentiable.

The main special case of Theorem 5.1 is

Corollary 5.2. *Every pair (f,g) of real valued Lipschitz functions defined on an open set G of a Hilbert space possesses a common point of Fréchet differentiability. Moreover, for any $u, v \in X$ and $c \in \mathbb{R}$ the existence of a common point x of Gâteaux differentiability of f and g such that*

$$f'(x,u) + g'(x,v) > c$$

implies the existence of such a point of Fréchet differentiability.

Using a similar argument to the proof of Theorem 5.1 we obtain a result which is meaningful for every finite n.

Theorem 5.3. *Assume that the space X admits a continuous convex function Θ such that for some $n \geq 1$*

(i) $\inf\{\Theta(x); \|x\| > s\} > \Theta(0)$ *for all* $s > 0$,

(ii) $\sup_{x \in X} \overline{\rho}_{\Theta}(x,t) = O(t^n \log^n(1/t))$ *as* $t \searrow 0$,

(iii) *For every* $x \in X$, $\overline{\rho}_{\Theta}(x,t) = o(t^n \log^{n-1}(1/t))$ *as* $t \searrow 0$.

Then every Lipschitz map f from a nonempty open set G in X to a space of dimension not exceeding n is asymptotically Fréchet differentiable at some points in G.

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