Two unexpected examples concerning differentiability of Lipschitz functions on Banach spaces

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In this note we present two examples illustrating some surprising relations between the known existence theorems concerning two main concepts of differentiability of Lipschitz functions between Banach spaces. Recall that these concepts are: Gâteaux derivative of a mapping $\varphi : X \mapsto Y$ at $x \in X$, which is defined as a continuous linear map $\varphi'(x) : X \mapsto Y$ verifying

$$\langle \varphi'(x), u \rangle = \lim_{t \to 0} \frac{\varphi(x + tu) - \varphi(x)}{t}$$

for every $u \in X$, and Fréchet derivative which, in addition, requests that the above limit be uniform for $\|u\| \le 1$.

The examples we give point out that our understanding of differentiability properties of (real-valued) Lipschitz functions on Banach spaces is far from being complete. Since the motivation behind each of them is quite different, we will explain it in detail at the beginning of each of the two sections. Here we will confine ourselves to basic motivational remarks in the case of an infinitely dimensional separable Hilbert space H.

Our first example comes from the observation that the proof of existence of a point of Fréchet differentiability of a Lipschitz function $f : H \mapsto \mathbb{R}$ in [8] appears to be unnatural in the sense that it doesn't use the existing strong results about Gâteaux differentiability: Since we know that f is Gâteaux differentiable on a set $E \subset H$ whose complement is "negligible", the natural

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way would be to prove that any set with a negligible complement contains a point of Fréchet differentiability of f. We show, however, that the natural way has one disadvantage: It does not work, at least not with the presently known notions of negligibility.

The original motivation behind our second example arose from a technical point in the proof of the Fréchet differentiability result mentioned above; this will be explained later. The first examples, however, suggested that the real problem was how rich the set F of Fréchet derivatives of a Lipschitz mapping $f : H \mapsto \mathbb{R}^n$ has to be inside the set G of its Gâteaux derivatives. If $n = 1$, the mean value theorem for Fréchet derivative (see [8]) implies that the sets F and G have the same closed convex hull. For large n this completely fails: In the second part of this paper we construct a Lipschitz mapping $f : H \mapsto \mathbb{R}^3$ and three vectors $e_1, e_2, e_3 \in H$ such that

$$\langle f_1'(x), e_1\rangle + \langle f_2'(x), e_2\rangle + \langle f_3'(x), e_3\rangle = 0$$

at every point x of Fréchet differentiability of f, but such that f is Gâteaux differentiable at the origin and verifies

$$\langle f_1'(0), e_1\rangle + \langle f_2'(0), e_2\rangle + \langle f_3'(0), e_3\rangle = 1.$$

1. Incompatibility of Gâteaux and Fréchet differentiability results

We recall that the strongest present result about existence of Gâteaux derivative of Lipschitz functions due to Aronszajn [1] says that the set of points of Gâteaux non-differentiability of a Lipschitz mapping f of a separable Banach space X into a Banach space Y with the Radon-Nikodým Property is "small" in the following sense.

Definition 1. Let X be a separable Banach space. A set $E \subset X$ is negligible in the sense of Aronszajn if for every sequence e_i whose linear span $\operatorname{sp}\{e_1, e_2, \dots\}$ is dense in X one can find Borel sets E_i covering E (i.e., $E \subset \bigcup_{i=1}^{\infty} E_i$) such that the intersection of E_i with any line in the direction e_i has linear measure zero.

Remark. An easy way to see that $E = X$ is not negligible (and therefore that Aronszajn's differentiability result gives non-trivial information) is to observe that every set E negligible in the sense of Aronszajn is of measure zero for any non-degenerated Gaussian measure in X (see [7]).

The converse problem, i.e., whether Borel sets $E \subset X$ which are of measure zero for any non-degenerated Gaussian measure in X are necessarily negligible in the sense of Aronszajn, is still open. According to a result of Bogachev [3] the requirement that the sets E_i are Borel is not just a "formal" measurability requirement: For every sequence e_i whose linear span is dense in X one can decompose $X = \bigcup_{i=1}^{\infty} X_i$ such that the intersection of X_i with any line in the direction e_i has linear measure zero.

Bogachev uses his result to deduce that sets of measure zero for any non-degenerated Gaussian measure in X are negligible if one replaces the requirement that E_i be Borel by requiring only that they are measurable for any non-degenerated Gaussian measure in X. The main part of his argument is that any set $E \subset X$ is decomposed as $E = \bigcup_{i=1}^{\infty} E \cap X_i$. If now E is null for any non-degenerated Gaussian measure, then clearly each $E \cap X_i$ is measurable for any such measure, and we obtain the required decomposition of E.

This, however, seems to be far from solving the above mentioned problem: To see why, we may try to use the above argument for one fixed non-degenerated Gaussian measure γ in X. Replacing Borel measurability of E_i by their γ measurability, we conclude that the so defined negligible sets are precisely sets of γ measure zero. But this is obviously false for negligibility in the sense of Aronszajn.

The reason behind this is that the decomposition of X into X_i is badly based on the axiom of choice. To see its analogy in the plane, we recall that one of the tools Aronszajn uses is a simple corollary of Fubini's theorem saying that a Borel set E in the plane can be decomposed into two Borel sets E_1, E_2 such that E_1 is of linear measure zero on every line in the direction of the x-axis and E_2 is of linear measure zero on every line in the direction of y-axis. The above argument can be modeled, e.g., by first using the continuum hypothesis to decompose the plane into sets A and B such that A is countable on every line in the direction of the x-axis and B is countable on every line in the direction of y-axis and then considering the decomposition $E = (E \cap A) \cup (E \cap B)$. If E is of Lebesgue measure zero, this decomposes E into Lebesgue measurable sets which are not only of linear measure zero but even countable on required lines, but this decomposition is surely of a completely different nature than the Borel decomposition obtained by the use of Fubini's theorem.

After this side remark, we may return back to the real theme of this section. It is our intention to decompose every separable infinite dimensional Banach space X into two Borel sets U and V such that U is negligible in the sense of Aronszajn and V is negligible for whatever notion is appropriate to handle Fréchet differentiability. The latter is achieved by constructing a real-valued Lipschitz function f on X which is Fréchet non-differentiable at any point of V.

Our example is, indeed, much stronger than this: U is not only negligible in the sense of Aronszajn, but even of linear measure zero on every line and V is a countable union of closed porous sets.

Definition 2. A set $E \subset X$ is said to be porous if there is $c \in (0,1)$ such that for every $x \in E$ and every $\delta > 0$ there is $z \in X$ such that $0 < \|z-x\| < \delta$ and $E \cap B(z, c\|z-x\|) = \emptyset$.

We are really interested in countable unions of porous sets (usually termed σ-porous). These have been studied quite extensively in Real Analysis; an up-to-date survey can be found in [9]. Since we will not need any of the results, we just remark that σ-porous sets are of the first category and in case X is finite dimensional they are also of Lebesgue measure zero. Deeper results say, for example, that the family of σ-porous sets is much smaller than that of first category Lebesgue measure zero sets or that porosity could have been defined with one fixed value of c, say $c = 1/2$. (This would, of course, give a different notion of porosity, but it would lead to the same notion of σ-porosity.) We will need only the following simple reformulation of the definition of porosity for closed sets.

Lemma 1. *If E is a closed porous set, then there are $C > 1$ and a disjointed family $\mathcal{B}$ of closed balls of radii less then one whose union is disjoint from E and which has the property that for every $\delta > 0$ the family $\mathcal{B} \cup \{B(z, Cr);\ B(z,r) \in \mathcal{B}, r \in (0,\delta)\}$ covers X.*

Proof. Let $S_i \subset X \setminus E$ be maximal sets such that, whenever $x \in S_i$, then $B(x, 2^{-i}) \cap E = \emptyset$ and $\|y - x\| > 2^{-i+2}$ for every $y \in S_i$, $y \neq x$. By induction, we define $\mathcal{B}_0$ as the family $\{B(z,1);\ z \in S_0\}$ and $\mathcal{B}_i$ as the family of those balls from $\{B(z, 2^{-i});\ z \in S_i\}$ which are disjoint from all balls from the families $\mathcal{B}_j$ $(j < i)$. We prove that the statement holds with $\mathcal{B}$ being the union of the families $\mathcal{B}_0, \mathcal{B}_1, \dots$. Since for every $\delta > 0$ the family

$\mathcal{B} \cup \{B(z, Cr);\ B(z,r) \in \mathcal{B}, r \in (0,\delta)\}$ obviously covers the union of $\mathcal{B}$, let $x \in X$ be not in this union. Since the union of each family $\mathcal{B}_i$ is a closed set (every two different balls from $\mathcal{B}_i$ are at distance at least $2^{-(i+1)}$), we may start by finding $\sigma \in (0,\delta)$ such that every ball from $\mathcal{B}$ meeting $B(x,\sigma)$ has radius less than δ. If c is the number from the definition of porosity of E, we find $z \in X$ such that $0 < \|z-x\| < \sigma/4$ and $E \cap B(z, c\|z-x\|) = \emptyset$. (Since E is closed, we may find such z even if $x \notin E$.) Let j be the least natural number such that $2^{-j+2} < c\|z-x\|$. Then the maximality condition implies that there is a point $u \in S_j \cap B(z, c\|z-x\|) \subset B(x, \sigma/2)$. Hence we may find $0 \le i \le j$ and a ball $B(v, 2^{-i}) \in \mathcal{B}_i$ such that $B(v,2^{-i}) \cap B(u, 2^{-j}) \neq \emptyset$. Since $B(u,2^{-j}) \subset B(x,\sigma)$, we infer that $B(v,2^{-i}) \cap B(x,\sigma) \neq \emptyset$. Hence $2^{-i} < \delta$ and $x \in B(z, \|z-x\|) \subset B(z, 2^{-j+3}c^{-1}) \subset B(v, (18c^{-1})2^{-i})$. Consequently, the statement holds with $C = 18/c$.

The connection between porosity and differentiability is given in the following proposition. We do not know if its third statement holds for σ-porous sets and/or in non-separable spaces.

Proposition 1. (i) *If $E \subset X$ is porous, there is a real-valued Lipschitz function f on X which is Fréchet non-differentiable at any point of E.*

(ii) *Any σ-porous set belongs to the σ-ideal generated by sets of points of Fréchet non-differentiability of real-valued Lipschitz functions on X.*

(iii) *If X is separable and $E \subset X$ is a subset of a countable union of closed porous sets, then there is a real-valued Lipschitz function f on X which is Fréchet non-differentiable at any point of E.*

Proof. (i) Let $f(z)$ be defined as the distance of the point z to the set E. If $x \in E$ and h is such that $E \cap B(x+h, c\|h\|) = \emptyset$, then we have $f(x+h) + f(x-h) - 2f(x) \ge f(x+h) \ge c\|h\|$. Hence

$$\limsup_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{\|h\|} \ge c,$$

which easily implies that f is not Fréchet differentiable at x.

(ii) This follows immediately from (i).

(iii) If X is not an Asplund space, we may take for f any equivalent norm which is nowhere Fréchet differentiable. (See [6] or [4, Chapter I, Theorem 5.3].) If X is an Asplund space, we may, according to the Asplund's theorem (see [2] or [4, Chapter II, Theorem 2.6(ii)]), assume that the norm of X is differentiable away from the origin.

Let E_i be closed porous sets covering E and let $C_i > 1$ and families of balls $\mathcal{B}_i$ be as in Lemma 1. We define a real-valued function f_i on X by $f_i(x) = 0$ if x does not belong to any of the balls from $\mathcal{B}_i$ and $f_i(x) = (r^2 - \|z - x\|^2)/r$ if $x \in B(z, r) \in \mathcal{B}_i$. Then f_i is a Lipschitz function on X with Lipschitz constant at most two. It is clearly Fréchet differentiable at every point of the union of the interiors of the balls from $\mathcal{B}_i$. Whenever x does not belong to this union, we have, similarly as in the proof of (i),

$$\limsup_{h\to 0} \frac{f_i(x+h) + f_i(x-h) - 2f_i(x)}{\|h\|} \geq 1/C_i.$$

Let $0 < d_i < 2^{-i}$ be such that

$$\frac{d_j}{C_j} > 4 \sum_{i=j+1}^{\infty} d_i$$

for every j and let $f = \sum_i d_i f_i$. If $x \in E$, we find the least j for which x does not belong to the interior of any ball from $\mathcal{B}_j$ and use the differentiability of f_i $(i < j)$ at x to estimate

$$\begin{aligned}
&\limsup_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{\|h\|} \\
&\quad = \limsup_{h\to 0} \sum_{i=j}^{\infty} d_i \frac{f_i(x+h) + f_i(x-h) - 2f_i(x)}{\|h\|} \\
&\quad \geq \limsup_{h\to 0} d_j \frac{f_j(x+h) + f_j(x-h) - 2f_j(x)}{\|h\|} - 2 \sum_{i=j+1}^{\infty} d_i \operatorname{Lip}(f_i) \\
&\quad \geq d_j/C_j - 4 \sum_{i=j+1}^{\infty} d_i > 0.
\end{aligned}$$

Hence f is not Fréchet differentiable at x.

Lemma 2. *Every infinite dimensional separable Banach space X has a subset Z such that the balls $B(z, 6)$ $(z \in Z)$ cover X and*

$$\lim_{s\nearrow\infty} \sup \left\{ \frac{\mathcal{L}([x,y] \cap \bigcup_{z\in Z} B(z,1))}{\|y-x\|};\ x, y \in X, \|y-x\| \geq s \right\} = 0.$$

Proof. Let x_n $(n = 0, 1, \dots)$ be a sequence dense in X. We let $z_0 = x_0$ and choose, by induction, a point $z_k \in X$ such that $\|x_k - z_k\| \le 6$ and $\operatorname{dist}(z_k, \operatorname{sp}\{z_0, \dots, z_{k-1}\}) > 5$. (This is where we use that X is infinitely dimensional: Let $x^\star \in X^\star$ be such that $\|x^\star\| = 1$, $\langle x^\star, z_i\rangle = 0$ for $i < k$ and $\langle x^\star, x_k\rangle \ge 0$. Choosing $x \in X$ with $\|x\| = 1$ such that $\langle x^\star, x\rangle > 5/6$, we see that $z_k = x_k + 6x$ verifies $\operatorname{dist}(z_k, \operatorname{sp}\{z_0, \dots, z_{k-1}\}) \ge \langle x^\star, z_k\rangle \ge 6\langle x^\star, x\rangle > 5$.)

Let p be a line in X and let $n_1 < n_2 < \dots$ be all indices for which $\operatorname{dist}(z_{n_i}, p) \le 1$. We pick $w_i \in p$ such that $\|w_i - z_{n_i}\| \le 1$ and prove that, whenever $i < j < k$, then

$$\operatorname{dist}(w_k, [w_i, w_j]) > \|w_j - w_i\|. \tag{1}$$

Indeed, if $\operatorname{dist}(w_k, [w_i, w_j]) \le \|w_j - w_i\|$, we have $w_k = \alpha w_i + \beta w_j$ with $|\alpha| \le 2$ and $|\beta| \le 2$. Hence $\operatorname{dist}(z_{n_k}, \operatorname{sp}\{z_0, \dots, z_{n_k - 1}\}) \le \|z_{n_k} - (\alpha z_{n_i} + \beta z_{n_j})\| \le \|z_{n_k} - w_k\| + |\alpha|\|z_{n_i} - w_i\| + |\beta|\|z_{n_j} - w_j\| \le 5$, which contradicts the way in which z_{n_k} has been defined.

We infer from (1) that any subset of the sequence $w_1, w_2, \dots$ having at least $n \ge 2$ elements has diameter greater than 2^{n-1}. Indeed, this is obviously true for two element sets, since $\|w_i - w_j\| > \|z_{n_i} - z_{n_j}\| - 2 > 2$ if $i \ne j$. The estimate follows therefore by induction, since (1) shows that adding the element with the highest index multiplies the diameter by at least 2.

To finish the proof, assume for a while that $m \ge 2$, $x, y \in p$, $2^{m-2} < \|y - x\| \le 2^{m-1}$ and $\mathcal{L}([x, y] \cap \bigcup_{z \in Z} B(z, 1)) > 2m + 2$. Then we would infer from the estimate of the measure that the segment $[x, y]$ would contain a subset of the sequence $w_1, w_2, \dots$ having at least m elements. But this would imply that $\|y - x\|$ is at least the diameter of this set which is greater than 2^{m-1}. This contradiction shows that

$$\frac{\mathcal{L}([x, y] \cap \bigcup_{z \in Z} B(z, 1))}{\|x - y\|} \le (m + 1)2^{-m+3},$$

which proves the statement of the lemma.

Theorem 1. *Every infinite dimensional separable Banach space X may be decomposed into two sets U and V such that U is of linear measure zero on every line and V is a countable union of closed porous sets.*

In particular, U is negligible in the sense of Aronszajn and there is a Lipschitz real-valued function on X which is Fréchet non-differentiable at every point of V.

Proof. Let Z be the set from Lemma 2 and let G be the union of open balls with radius one centred at Z. For every $n = 0, 1, \dots$ let $s_n \in (1, \infty)$ be such that $\mathcal{L}([x, y] \cap G) < 2^{-n}\|y - x\|$ whenever $\|y - x\| \geq s_n$. Defining $G_n = \{z/(2^n s_n);\ z \in G\}$, we observe that $\mathcal{L}([x, y] \cap G_n) < 2^{-n}\|y - x\|$ whenever $\|y - x\| \geq 2^{-n}$. Let

$$U = \bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} G_n.$$

Whenever $\|y - x\| \geq 2^{-k}$, we may estimate

$$\mathcal{L}([x, y] \cap U) \leq \sum_{n=k}^{\infty} \mathcal{L}([x, y] \cap G_n) < \sum_{n=k}^{\infty} 2^{-n}\|y - x\| = 2^{-k+1}\|y - x\|.$$

Consequently, U is of linear measure zero on every segment. Moreover, the complement V of U is the union of the sets $X \setminus \bigcup_{n=k}^{\infty} G_n$ which are clearly closed and porous.

2. Strange difference between Fréchet differentiability of Lipschitz functions and of Lipschitz mappings

Even though we know that real-valued Lipschitz functions on Asplund spaces possess Fréchet derivatives at some points (see [8]), it is still an open problem if every finite (or countable) family of Lipschitz functions on such spaces (or even only on a separable Hilbert space) possesses a common point of Fréchet differentiability.

One of the facts behind the proof of the case of one function was the observation that, if f is a real-valued Lipschitz function on a separable Banach space X which is Gâteaux differentiable at x and if, in addition, $f'(x)$ is a weak* strongly exposed point of the set of all $f'(y)$ (where y runs through those points of X at which f is Gâteaux differentiable), then f is Fréchet differentiable at x. (Recall that $x^\star \in E^\star \subset X^\star$ is a weak* exposed point of $E^\star$ if there is $e \in X$ such that the diameters of the sets $\{y^\star \in E^\star;\ \langle y^\star, e\rangle > \langle x^\star, e\rangle - \delta\}$ tend to zero as $\delta \searrow 0$.) It was observed during a discussion of differentiability problems between the first named author and Joram Lindenstrauss that an analogy of this statement for more functions (i.e., for mappings into $\mathbb{R}^n$) is false. Here we strengthen this observation by showing that the following basic fact about differentiability of Lipschitz functions (proved in [8]) becomes false even for Lipschitz mappings of an infinitely dimensional Hilbert space into finitely dimensional spaces: Whenever f is a real-valued Lipschitz function on a separable Asplund space, then for every weak* slice S of the set of all Gâteaux derivatives of f (i.e., of the set $\{f'(x);\ f$ is Gâteaux differentiable at $x\}$) there is a point $x \in X$ at which f is Fréchet differentiable and $f'(x) \in S$. (Recall that a weak* slice of a set $E^\star \subset X^\star$ is any non-empty set of the form $\{x^\star \in E^\star;\ \langle x^\star, e\rangle > c\}$, where $e \in X$ and $c \in \mathbb{R}$.) In this section we prove

Theorem 2. *Let $1 < p < \infty$ and let n be a natural number greater than p. Then there is a Lipschitz mapping $f = (f_1, \dots, f_n)$ of ℓ_p to $\mathbb{R}^n$ such that*

$$\sum_{j=1}^{n} \langle f_j'(x), e_j\rangle = 0 \qquad (2)$$

at every point x at which f is Fréchet differentiable, but which, at the same time, is Gâteaux differentiable at the origin and verifies

$$\sum_{j=1}^{n} \langle f_j'(0), e_j\rangle = 1.$$

In addition, there are constants $0 < c, C < \infty$ such that f has, at every $x \in \ell_p$, the following properties.

(i) *For every linear mapping $T = (T_1, \dots, T_n)$ of ℓ_p to $\mathbb{R}^n$*

$$\limsup_{y \to x} \frac{\|f(y) - f(x) - \langle T, y-x\rangle\|}{\|y-x\|} \geq c \left|\sum_{j=1}^{n} \langle T_j, e_j\rangle\right|.$$

(ii) $\liminf_{t\to 0} \frac{1}{t} \sum_{j=1}^{n} (f_j(x + te_j) - f_j(x)) \geq 0,$ *and* $\limsup_{t\to 0} \frac{1}{t} \sum_{j=1}^{n} (f_j(x + te_j) - f_j(x)) \leq 1.$

(iii) $\|f'(x) - f'(0)\| \leq C \left(1 - \sum_{j=1}^{n} \langle f_j'(x), e_j\rangle\right)^{(p-1)/p}$ *whenever f is Gâteaux differentiable at x.*

We remark that, in particular, for the set S of all Gâteaux derivatives $f'(x)$ such that $\sum_{i=1}^{n}\langle f_i'(x), e_i\rangle > 0$ there is no point $z \in X$ at which f is Fréchet differentiable and $f'(z) \in S$. Since S is a (weak*) slice of the set of all Gâteaux derivatives of f (in the space of linear operators from ℓ_p to $\mathbb{R}^n$), this shows that the basic Fréchet differentiability result for real-valued functions does not have a simple analogy for mappings into finitely dimensional spaces. The difference between the higher dimensional and one dimensional ranges is stressed by the fact that f is not Fréchet differentiable at the origin, even though (ii) and (iii) imply that $f'(0)$ is a (weak*) exposed point of S.

Another way how to view this theorem is to consider it as a construction of a strange solution to the partial differential equation (2). Namely, an immediate consequence of the Gauss-Green Theorem ([5, 4.5.6]) is that, if we consider (2) as an equation for an unknown Lipschitz function f between finitely dimensional spaces, the solution set will be the same independently of whether we require its validity for almost all x or for all x at which f is differentiable. However, our mapping shows that even in the simplest infinite dimensional situation the notion of solution depends on whether we require (2) for points of Fréchet or Gâteaux differentiability.

We also remark that if $n = 1$ the validity of equation (2) at every point of Fréchet differentiability implies that the function is constant. (This is true in every Asplund space, see [8, Theorem 2.5].) So for $1 < p < 2$ the theorem gives an optimal result. However, in case of a Hilbert space we do not know if such a mapping exists with $n = 2$. As far as we know, an example with $n = 2$ could exist in every infinite dimensional Banach space.

We should also point out that the value of the main statement of this Theorem may depend on time. It is possible that there exist nowhere Fréchet differentiable Lipschitz mappings of ℓ_p into finite dimensional spaces. Once such functions are discovered, one may just observe that they may be easily modified to have the main properties stated in the theorem. However, according to a (so far unpublished) result of the first named author, for every Lipschitz mapping f of a space X with a uniformly rotund norm into a finite dimensional space Y there are points with an arbitrarily small error in Fréchet differentiability. In other words, for every $\varepsilon > 0$ there are $x \in X$ and a continuous linear mapping T of X to Y such that

$$\limsup_{z\to x} \frac{\|f(z) - f(x) - \langle T, z - x\rangle\|}{\|z - x\|} < \varepsilon. \tag{3}$$

Nevertheless, because of (i), the mapping f from our example has the property that even the closed convex hull of the set of continuous linear mappings T verifying (3) with $\varepsilon = c/2$ does not contain all Gâteaux derivatives of f.

We do not know if the construction of our mapping f can be modified to strengthen the above discrepancy between the set of ε-approximating linear mappings and Gâteaux derivatives to get, for a fixed $\varepsilon > 0$, that Gâteaux differentiability of f at x and (3) with $T = f'(x)$ imply (2). Another interesting open question is whether a mapping with the main properties from Theorem 2 can be everywhere Gâteaux differentiable.

2.1. Preliminaries

Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and let $n > p$ be an integer. We decompose the set $\mathbf{N}$ of all positive integers into infinitely many disjoint sets $\mathbf{N} = \bigcup_{i \geq 0} N_i$ such that $N_0 = \{1, 2, \ldots, n\}$ and all N_i with $i \geq 1$ are infinite. Then we put $X_i = \ell_p(N_i)$ and observe that $\ell_p = \oplus_{i=0}^{\infty} X_i$, where $\oplus$ means the ℓ_p sum. Further, let

$$\begin{array}{rcl} \pi_m & : & \ell_p \mapsto X_m, \\ \sigma_m & : & \ell_p \mapsto \oplus_{i=1}^{m-1} X_i, \\ \sigma^{(m)} & : & \ell_p \mapsto \oplus_{i=m+1}^{\infty} X_i \end{array}$$

denote the corresponding canonical projections. We also define $\sigma_1 = 0$. The symbol $\|\cdot\|$ is used for the norm in ℓ_p and $(e_j), j \geq 1$ stands for the usual basis of ℓ_p.

If $\varphi : \ell_p \mapsto X_0$ is a map and $v, z \in \ell_p$, then $\varphi'(z; v)$ denotes the derivative of φ at the point z in the direction v. In particular, the derivative of the norm at the point z in the direction v is $\|\cdot\|'(z; v)$. We will often use the simple fact that $-\|v\| \leq \|\cdot\|'(z; v) \leq \|v\|$. We also define

$$\operatorname{Tr} \varphi'(z) = \sum_{j=1}^{n} \langle \varphi'(z; e_j), e_j^{\star} \rangle,$$

where $(e_j^{\star})_{j=1}^{n}$ is the dual basis to $(e_j)_{j=1}^{n}$.

Let $h : [0, +\infty) \mapsto [0, +\infty)$ be the C^1-function defined by

$$\begin{array}{rcll} h(t) & = & 1 & \text{for} \quad t \in [0, 1] \\ & = & \frac{p+1}{t^p} - \frac{p}{t^{p+1}} & \text{for} \quad t \in (1, +\infty). \end{array}$$

Finally, let $g : [0, +\infty) \mapsto [0, +\infty)$ be another C^1-function given by

$$g(t) = \left(\int_t^{+\infty} \gamma(s)\, ds \right)^p,$$

where

$$\begin{array}{rcll} \gamma(t) & = & 0 & \text{for} \quad t \in [0,1) \cup [3, +\infty) \\ & = & t-1 & \text{for} \quad t \in [1,2) \\ & = & 3-t & \text{for} \quad t \in [2,3). \end{array}$$

For convenience, all (easy to show) facts about these functions which will be needed in the sequel are collected in the following lemma.

Lemma 3. *The functions g and h are continuously differentiable on $[0, \infty)$ and, for every $t \geq 0$, verify*

(i) $h'(t) \leq 0$, $0 < h(t) \leq 1$, $h(t) = 1$ *if* $t \leq 1$, *and* $h(t) \to 0$ *as* $t \to \infty$,

(ii) $h(t)\, t \leq (p+1) h^{\frac{1}{q}}(t)$,

(iii) $n\, h(t) + h'(t)\, t \geq (n-p) h(t)$,

(iv) $h(t) + |h'(t)|\, t \leq (p+1) h(t)$,

(v) $g'(t) \leq 0$, $0 \leq g(t) \leq 1$, $g(t) = 0$ *if* $t \geq 3$, *and* $g(t) = 1$ *if* $t \leq 1$, *and*

(vi) $|g'(t)| \leq p\, g^{\frac{1}{q}}(t)$.

Let r_m and s_m be positive reals and let us define $\varphi_m : \ell_p \mapsto X_0$ as

$$\varphi_m(z) = \frac{1}{n}\, g\!\left(\frac{\|\pi_m z\|}{r_m}\right) g\!\left(\frac{\|\sigma_m z\|}{s_m}\right) h\!\left(\frac{\|\pi_0 z\|}{r_m}\right) \pi_0 z.$$

Then we have the following lemma.

Lemma 4. *The mappings φ_m have the following properties.*

(i) $\|\varphi_m(z)\| \leq \frac{p+1}{n}\, r_m$.

(ii) $\varphi_m(z) = 0$ *provided that* $\|\pi_m z\| \geq 3r_m$ *or* $\|\sigma_m z\| \geq 3s_m$.

(iii) $\frac{n-p}{n}\, h\!\left(\frac{\|\pi_0 z\|}{r_m}\right) g\!\left(\frac{\|\pi_m z\|}{r_m}\right) g\!\left(\frac{\|\sigma_m z\|}{s_m}\right) \leq \operatorname{Tr} \varphi_m'(z) \leq h\!\left(\frac{\|\pi_0 z\|}{r_m}\right)$.

(iv) $\|\varphi_m'(z;v)\| \le \frac{n+1}{n-p} \operatorname{Tr} \varphi_m'(z) \|v\|$ *for* $v \in X_0$,

$\|\varphi_m'(z;v)\| \le \frac{p(p+1)}{n} \frac{r_m}{s_m} \left(\frac{n}{n-p} \operatorname{Tr} \varphi_m'(z) \right)^{\frac{1}{q}} \|v\|$ *for* $v \in \oplus_{i=1}^{m-1} X_i$, *and*

$\|\varphi_m'(z;v)\| \le \frac{p(p+1)}{n} \left(\frac{n}{n-p} \operatorname{Tr} \varphi_m'(z) \right)^{\frac{1}{q}} \|v\|$ *for* $v \in X_m$.

Proof. (i) Since $0 \le g \le 1$, we have

$$\|\varphi_m(z)\| \le h\left(\frac{\|\pi_0 z\|}{r_m}\right) \frac{\|\pi_0 z\|}{n} = \frac{r_m}{n} h\left(\frac{\|\pi_0 z\|}{r_m}\right) \frac{\|\pi_0 z\|}{r_m} \le \frac{p+1}{n} r_m,$$

where the last inequality follows from ((ii)) and ((i)) of Lemma 3.

(ii) Obvious since $g(t) = 0$ if $t \ge 3$.

(iii) A direct calculation gives

$$\operatorname{Tr} \varphi_m'(z) = \frac{1}{n} g\left(\frac{\|\pi_m z\|}{r_m}\right) g\left(\frac{\|\sigma_m z\|}{s_m}\right) \left[h'\left(\frac{\|\pi_0 z\|}{r_m}\right) \frac{\|\pi_0 z\|}{r_m} + nh\left(\frac{\|\pi_0 z\|}{r_m}\right) \right].$$

The upper estimate is now obvious, since $h' \le 0$ and $g \le 1$. The desired lower estimate follows directly from (iii) of Lemma 3.

(iv) Let $v \in X_0$ and $\|v\| = 1$. Then

$$\begin{aligned}
\|\varphi_m'(z;v)\| &= \frac{1}{n} g\left(\frac{\|\pi_m z\|}{r_m}\right) g\left(\frac{\|\sigma_m z\|}{s_m}\right) \\
&\quad \times \left\| h\left(\frac{\|\pi_0 z\|}{r_m}\right) \pi_0 v + h'\left(\frac{\|\pi_0 z\|}{r_m}\right) \frac{\|\cdot\|'(\pi_0 z; v)}{r_m} \pi_0 z \right\| \\
&\le \frac{1}{n} g\left(\frac{\|\pi_m z\|}{r_m}\right) g\left(\frac{\|\sigma_m z\|}{s_m}\right) \left[h\left(\frac{\|\pi_0 z\|}{r_m}\right) + \left| h'\left(\frac{\|\pi_0 z\|}{r_m}\right) \right| \frac{\|\pi_0 z\|}{r_m} \right].
\end{aligned}$$

Using (iv) of Lemma 3 and the already proven point (iii) of Lemma 4, this can be estimated by

$$\begin{aligned}
&\le \frac{p+1}{n} g\left(\frac{\|\pi_m z\|}{r_m}\right) g\left(\frac{\|\sigma_m z\|}{s_m}\right) h\left(\frac{\|\pi_0 z\|}{r_m}\right) \\
&\le \frac{p+1}{n-p} \operatorname{Tr} \varphi_m'(z).
\end{aligned}$$

Let $v \in \oplus_{1 \le i < m} X_i$ and $\|v\| = 1$. Then

$$\begin{aligned}
\|\varphi_m'(z;v)\| &= \frac{1}{n}\, g\left(\frac{\|\pi_m z\|}{r_m}\right) h\left(\frac{\|\pi_0 z\|}{r_m}\right) \|\pi_0 z\| \left| g'\left(\frac{\|\sigma_m z\|}{s_m}\right) \frac{\|\cdot\|'(\sigma_m z; v)}{s_m} \right| \\
&\le \frac{r_m}{n s_m}\, g\left(\frac{\|\pi_m z\|}{r_m}\right) h\left(\frac{\|\pi_0 z\|}{r_m}\right) \frac{\|\pi_0 z\|}{r_m} \left| g'\left(\frac{\|\sigma_m z\|}{s_m}\right) \right| .
\end{aligned}$$

Using that $0 \le g \le 1$ implies $g \le g^{1/q}$, applying Lemma 3(vi) and (ii), and using the estimate of $\operatorname{Tr}\varphi_m'(z)$ from (iii) of this Lemma, we infer that

$$\begin{aligned}
&\le \frac{r_m}{n s_m}\, g^{\frac{1}{q}}\left(\frac{\|\pi_m z\|}{r_m}\right) (p+1)\, h^{\frac{1}{q}}\left(\frac{\|\pi_0 z\|}{r_m}\right) p\, g^{\frac{1}{q}}\left(\frac{\|\sigma_m z\|}{s_m}\right) \\
&\le \frac{p(p+1)}{n} \frac{r_m}{s_m} \left(\frac{n}{n-p} \operatorname{Tr}\varphi_m'(z)\right)^{\frac{1}{q}} .
\end{aligned}$$

If $v \in X_m$ and $\|v\| = 1$, we use the same arguments as in the preceding case to estimate

$$\begin{aligned}
\|\varphi_m'(z;v)\| &= \frac{1}{n}\, g\left(\frac{\|\sigma_m z\|}{s_m}\right) h\left(\frac{\|\pi_0 z\|}{r_m}\right) \|\pi_0 z\| \left| g'\left(\frac{\|\pi_m z\|}{r_m}\right) \frac{\|\cdot\|'(\pi_m z; v)}{r_m} \right| \\
&\le \frac{1}{n}\, g\left(\frac{\|\sigma_m z\|}{s_m}\right) h\left(\frac{\|\pi_0 z\|}{r_m}\right) \frac{\|\pi_0 z\|}{r_m} \left| g'\left(\frac{\|\pi_m z\|}{r_m}\right) \right| \\
&\le \frac{1}{n}\, g^{\frac{1}{q}}\left(\frac{\|\sigma_m z\|}{s_m}\right) (p+1)\, h^{\frac{1}{q}}\left(\frac{\|\pi_0 z\|}{r_m}\right) p\, g^{\frac{1}{q}}\left(\frac{\|\pi_m z\|}{r_m}\right) \\
&\le \frac{p(p+1)}{n} \left(\frac{n}{n-p} \operatorname{Tr}\varphi_m'(z)\right)^{\frac{1}{q}} .
\end{aligned}$$

For each $m \ge 1$ we choose a maximal set $E_m \subset X_m$ containing the origin and such that $\|x_1 - x_2\| \ge 24 r_m$ whenever $x_1, x_2 \in E_m$ are different. Note that the maximality means that for any $z \in X_m$ there is $x \in E_m$ such that $\|x - z\| < 24 r_m$. To every point $x \in E_m$ we assign infinitely many points $y^{(m)}(x,i) \in X_m$, $i \ge 0$ such that

(α) $y^{(m)}(x,0) = x$,

(β) $\|y^{(m)}(x,i) - x\| = 8 r_m$, and

(γ) $\|y^{(m)}(x,i) - y^{(m)}(x,j)\| \ge 8 r_m$ whenever $i \ne j$.

We can obtain $y^{(m)}(x,i)$ by rearranging the points $x + 8 r_m e_j$ ($j \in N_m$) into a sequence. Let us denote, further, by $(d_i^{(m)})_{i \ge 0}$ a countable dense subset of $\oplus_{i=0}^{m-1} X_i$.

Lemma 5. *Let $m \geq 1$.*

(i) *Every open ball in ℓ_p of radius r_m meets at most one of the supports of the functions $z \mapsto \varphi_m(z - y^{(m)}(x,i) - d_i^{(m)})$ $(x \in E_m, i \geq 0)$.*

(ii) *For any choice of real coefficients $c(x,i)$ the function*

$$b(z) = \sum_{x \in E_m} \sum_{i=0}^{\infty} c(x,i)\varphi_m(z - y^{(m)}(x,i) - d_i^{(m)})$$

is well-defined and continuously differentiable on ℓ_p with derivative given by

$$b'(z) = \sum_{x \in E_m} \sum_{i=0}^{\infty} c(x,i)\varphi'_m(z - y^{(m)}(x,i) - d_i^{(m)}).$$

(iii) *If the $c(x,i)$ are bounded, then b is a bounded function with a bounded uniformly continuous derivative.*

(iv) *If $x \in E_m$ and $i \geq 0$, then $b(z) = c(x,i)\varphi_m(z - y^{(m)}(x,i) - d_i^{(m)})$ for all $z \in \ell_p$ such that $\|\pi_m z - y^{(m)}(x,i)\| < 4r_m$.*

(v) $b(z) = b((\pi_0 + \sigma_{m+1})(z))$.

(vi) $\|b(z)\| \leq \frac{p+1}{n}\, r_m \sup\{|c(x,i)|;\ x \in E_m, i \geq 0\}$.

(vii) *If $c(x,i) \geq 0$ for all $x \in E_m$ and $i \geq 0$, then $\operatorname{Tr} b'(z) \geq 0$ for all $z \in \ell_p$.*

(viii) *If $0 \leq c(x,i) \leq 1$ for all $x \in E_m$ and $i \geq 0$, then*
$\|b'(z;v)\| \leq \frac{n+1}{n-p} \operatorname{Tr} b'(z)\|v\|$ *for $v \in X_0$,*
$\|b'(z;v)\| \leq \frac{p(p+1)}{n} \frac{r_m}{s_m} \left(\frac{n}{n-p} \operatorname{Tr} b'(z)\right)^{\frac{1}{q}} \|v\|$ *for $v \in \oplus_{i=1}^{m-1} X_i$, and*
$\|b'(z;v)\| \leq \frac{p(p+1)}{n} \left(\frac{n}{n-p} \operatorname{Tr} b'(z)\right)^{\frac{1}{q}} \|v\|$ *for $v \in X_m$.*

Proof. Observing that

$$\begin{aligned}\|\pi_m(y^{(m)}(u,i) + d_i^{(m)}) - \pi_m(y^{(m)}(v,j) + d_j^{(m)})\| \\ = \|(y^{(m)}(u,i) - y^{(m)}(v,j)\| \geq 8r_m,\end{aligned}$$

whenever $(u,i) \neq (v,j)$, we deduce (i)–(iv) immediately from Lemma 4(ii).

The statement (v) follows from the definition of the mappings φ_m and (vi) from (i) and from Lemma 4(i).

Finally, using (i) once more, we deduce (vii) from Lemma 4(iii) and (viii) from Lemma 4(iv), where the only additional fact we need to observe is that $c(x,i) \le (c(x,i))^{1/q}$.

2.2. The construction

In the preliminary part we have worked for each $m \ge 1$ separately with so far free parameters r_m and s_m. Here we recursively choose their particular values; these will be defined together with mappings $G_m : \ell_p \mapsto X_0$ ($m \ge 0$) and $b_m : \ell_p \mapsto X_0$ ($m \ge 1$); to determine them, we will also need recursively defined real coefficients $c^{(m)}(x,i)$ ($m \ge 1$, $x \in E_m$, $i \ge 0$).

We start by choosing the starting values:

(R0) $G_0(z) = 0$ for all $z \in \ell_p$, $r_1 = s_1 = 1$, and $c^{(m)}(x,0) = 0$ for all $m \ge 1$ and $x \in E_m$.

If $r_1, \dots, r_{m-1}$, and $s_1, \dots, s_{m-1}$ have been already fixed, and if G_{m-1} has been defined in such a way that it has a uniformly continuous derivative, we choose the real parameter s_m to verify

(R1) $\|G'_{m-1}(z_1) - G'_{m-1}(z_2)\| \le 2^{-(m+1)} n^{-1}$ whenever $\|z_1 - z_2\| \le 6 s_m$.

Then we choose r_m small enough to satisfy the following two conditions:

(R2) $r_m \le 2^{-(m+2)} \min\{s_m, r_1, \dots, r_{m-1}\}$, and

(R3) $h\left(\frac{3 s_m}{r_m}\right) < 2^{-(m+1)}$.

Once r_m and s_m have been fixed, the mappings φ_m are fixed as well and we may use them in our final bunch of definitions:

(R4) $c^{(m)}(x,i) = 1 - 2^{-m} - \operatorname{Tr} G'_{m-1}(d_i^{(m)})$ for $x \in E_m$ and $i \ge 1$,

(R5) $b_m(z) = \sum_{x \in E_m} \sum_{i=0}^{\infty} c^{(m)}(x,i) \varphi_m(z - y^{(m)}(x,i) - d_i^{(m)})$, and

(R6) $G_m(z) = G_{m-1}(z) + b_m(z)$.

We observe that this construction does not stop, since, once we know that G_{m-1} has a bounded uniformly continuous derivative (which we surely know if $m = 1$), then we first use (R4) to infer that $c^{(m)}(x, i)$ are bounded, deduce from Lemma 5(iii) that b_m has a bounded uniformly continuous derivative, and finally use (R6) to conclude that G_m has a bounded uniformly continuous derivative as well.

2.3. Further properties

A simple combination of (R1) and Lemma 5(v) gives

Lemma 6. *Whenever* $\|(\pi_0 + \sigma_m)(z_1 - z_2)\| \le 6s_m$, *then*

(i) $\|G'_{m-1}(z_1) - G'_{m-1}(z_2)\| \le 2^{-(m+1)}n^{-1}$, *and*

(ii) $|\operatorname{Tr} G'_{m-1}(z_1) - \operatorname{Tr} G'_{m-1}(z_2)| \le 2^{-(m+1)}$.

Lemma 7. *For every* $z \in \ell_p$ *the sequence* $\operatorname{Tr} G'_m(z)$ *is non-decreasing and verifies* $0 \le \operatorname{Tr} G'_m(z) \le 1 - 2^{-(m+1)}$. *In particular,* $0 \le c^{(m)}(x, i) \le 1 - 2^{-m}$.

Proof. We prove the Lemma by induction with respect to m; the monotonicity statement is considered as the inequality $\operatorname{Tr} G'_m(z) \ge \operatorname{Tr} G'_{m-1}(z)$. Assume that $m \ge 1$ and that $0 \le \operatorname{Tr} G'_{m-1}(w) \le 1 - 2^{-m}$ for every $w \in \ell_p$. (This is clearly true if $m = 1$.) Since $c^{(m)}(x, i) = 1 - 2^{-m} - \operatorname{Tr} G'_{m-1}(d_i^{(m)})$ for $i \ge 1$ and $c^{(m)}(x, 0) = 0$, we immediately infer that

$$0 \le c^{(m)}(x, i) \le 1 - 2^{-m}.$$

Thus Lemma 5(vii) shows that $\operatorname{Tr} G'_m(z) \ge \operatorname{Tr} G'_{m-1}(z)$ and hence also that $\operatorname{Tr} G'_m(z) \ge 0$.

To finish the proof of the only remaining statement, namely of the inequality $\operatorname{Tr} G'_m(z) \le 1 - 2^{-(m+1)}$, we may assume that $\operatorname{Tr} G'_m(z) > \operatorname{Tr} G'_{m-1}(z)$. Then Lemma 5(i) and (ii) imply that there are $x \in E_m$ and $i \ge 0$ such that

$$\operatorname{Tr} G'_m(z) = \operatorname{Tr} G'_{m-1}(z) + c^{(m)}(x, i) \operatorname{Tr} \varphi'_m(z - y^{(m)}(x, i) - d_i^{(m)}),$$

and $c^{(m)}(x, i) \operatorname{Tr} \varphi'_m(z - y^{(m)}(x, i) - d_i^{(m)}) \ne 0$; because of Lemma 4(ii) the latter implies that $\|\sigma_m z - \sigma_m d_i^{(m)}\| \le 3s_m$.

If we now assume also that $\|\pi_0 z - \pi_0 d_i^{(m)}\| < 3s_m$, we get

$$\|(\pi_0 + \sigma_m)(z - d_i^{(m)})\| < 6s_m.$$

Hence Lemma 6(ii) gives

$$|\operatorname{Tr} G'_{m-1}(z) - \operatorname{Tr} G'_{m-1}(d_i^{(m)})| \le 2^{-(m+1)}.$$

Since $c^{(m)}(x,i) \ge 0$ and $\operatorname{Tr} \varphi'_m \le 1$ (see (iii) of Lemma 4), we have

$$\begin{array}{rcl} \operatorname{Tr} G'_m(z) & = & \operatorname{Tr} G'_{m-1}(z) + c^{(m)}(x,i) \operatorname{Tr} \varphi'_m(z - y^{(m)}(x,i) - d_i^{(m)}) \\ & \le & \operatorname{Tr} G'_{m-1}(z) + c^{(m)}(x,i) \\ & = & 1 - 2^{-m} + (\operatorname{Tr} G'_{m-1}(z) - \operatorname{Tr} G'_{m-1}(d_i^{(m)})) \\ & \le & 1 - 2^{-(m+1)}. \end{array}$$

In the remaining case when $\|\pi_0 z - \pi_0 d_i^{(m)}\| \ge 3s_m$ we use Lemma 4(iii) and the fact that $c^{(m)}(x,i) \le 1$ to infer that

$$\begin{array}{rcl} \operatorname{Tr} G'_m(z) & = & \operatorname{Tr} G'_{m-1}(z) + c^{(m)}(x,i) \operatorname{Tr} \varphi'_m(z - y^{(m)}(x,i) - d_i^{(m)}) \\ & \le & 1 - 2^{-m} + h\left(\dfrac{\|\pi_0 z - \pi_0 d_i^{(m)}\|}{r_m}\right) \\ & \le & 1 - 2^{-m} + h\left(\dfrac{3s_m}{r_m}\right) \le 1 - 2^{-(m+1)}, \end{array}$$

where we also used that h is non-increasing and (R3).

Now we are ready to define (almost) the mapping we need by putting

$$G(z) = \lim_{m\to\infty} G_m(z) = \sum_{m=1}^{\infty} b_m(z).$$

The limit exists because the estimate of $c^{(m)}(x,i)$ in Lemma 7 and Lemma 5(vi) imply that $\|b_m(z)\| \le \frac{p+1}{n} r_m$, and because (R2) shows that $\sum_{m=1}^{\infty} r_m$ converges.

Lemma 8. *Whenever $m = 1, 2, \ldots$ and $w \in \ell_p$, then*

(i) $\|(G - G_m)(w)\| \le 4r_{m+1}$,

(ii) $\|G_m(w+u) - G_m(w) - \langle G'_m(w), u\rangle\| \le 2^{-m}\|u\|$ *for* $\|u\| \le 6s_{m+1}$*, and*

(iii) $\|\frac{1}{2}(G_m(v) + G_m(w)) - G_m(\frac{1}{2}(v+w))\| \le 2^{-m}\|v-w\|$ *provided that* $\|v-w\| \le 6s_{m+1}$.

Proof. (i) The estimate $\|b_k\| \le \frac{p+1}{n} r_k$ mentioned above, the obvious inequality $\frac{p+1}{n} \le 2$, and (R2) imply

$$\|(G - G_m)(w)\| \le \sum_{k=m+1}^{\infty} \|b_k(w)\| \le \tfrac{p+1}{n} \sum_{k=m+1}^{\infty} r_k \le 4r_{m+1}.$$

(ii) Using the mean value estimate and (R1), we get

$$\begin{aligned} &\|G_m(w+u) - G_m(w) - \langle G'_m(w), u\rangle\| \\ &\quad \le \ \sup\{\|\langle G'_m(w+tu) - G'_m(w), u\rangle\|;\ 0 \le t \le 1\} \\ &\quad \le \ 2^{-m}\|u\|. \end{aligned}$$

(iii) Using (ii) with $u = v - w$ and with $u = \frac{1}{2}(v+w) - w = \frac{1}{2}(v-w)$, we get

$$\begin{aligned} &\|\tfrac{1}{2}(G_m(v) + G_m(w)) - G_m(\tfrac{1}{2}(v+w))\| \\ &\quad \le \ \|\tfrac{1}{2}(G_m(v) - G_m(w) - \langle G'_m(w), v-w\rangle)\| \\ &\qquad + \|G_m(\tfrac{1}{2}(v+w)) - G_m(w) - \langle G'_m(w), \tfrac{1}{2}(v-w)\rangle\| \\ &\quad \le \ 2^{-m}\|v-w\|. \end{aligned}$$

Lemma 9. *For every* $z \in \ell_p$

(i) $\lim_{m\to\infty} \sup\left\{\frac{\|G(z+u)-G(z)-\langle G'_m(z),u\rangle\|}{\|u\|};\ \|u\| = s_{m+1}\right\} = 0,$

(ii) $\lim_{m\to\infty} \left|\frac{G(z+s_{m+1}u)-G(z)}{s_{m+1}} - \langle G'_m(z), u\rangle\right| = 0$ *if* $u \in \ell_p$ *with* $\|u\| = 1$,

(iii) $\lim_{m\to\infty} \sum_{j=1}^{n} \left\langle \frac{G(z+s_m e_j)-G(z)}{s_m}, e_j^\star \right\rangle = \lim_{m\to\infty} \operatorname{Tr} G'_m(z),$

(iv) $\|G'(z)\| \le \liminf_{m\to\infty} \|G'_m(z)\|$ *if* G *is Gâteaux differentiable at* z*, and*

(v) $\lim_{m\to\infty} \|G'(z) - G'_m(z)\| = 0$ *if* G *is Fréchet differentiable at* z.

Proof. We use Lemma 8(ii) and (i) to estimate

$$\begin{aligned}\|G(z+u)-G(z)-\langle G'_m(z),u\rangle\| &\le \|G_m(z+u)-G_m(z)-\langle G'_m(z),u\rangle\| \\ &\quad+\|(G-G_m)(z+u)\| \\ &\quad+\|(G-G_m)(z)\| \\ &\le 2^{-m}\|u\|+8r_{m+1}.\end{aligned}$$

The first statement now follows by dividing by $\|u\| = s_{m+1}$ and observing that (R2) implies that $\lim_{m\to\infty} r_{m+1}/s_{m+1} = 0$.

The statement (ii) is just a special case of (i). To prove (iii), we use (ii) to infer that

$$\lim_{m\to\infty}\left|\sum_{j=1}^{n}\left\langle\frac{G(z+s_m e_j)-G(z)}{s_m}, e_j^\star\right\rangle - \operatorname{Tr} G'_m(z)\right| = 0,$$

and note that Lemma 7 implies that the limit $\lim_{m\to\infty} \operatorname{Tr} G'_m(z)$ exists.

To prove (iv), if suffices to note that if G is Gâteaux differentiable at z then (ii) implies that $\langle G'(z), u\rangle = \lim_{m\to\infty}\langle G'_m(z), u\rangle$ for every $u \in \ell_p$.

Finally, we observe that

$$\begin{aligned}\|G'(z)-G'_m(z)\| \le & \sup_{\|u\|=s_{m+1}}\left\{\frac{\|G(z+u)-G(z)-\langle G'_m(z),u\rangle\|}{\|u\|}\right\} \\ &+\sup_{\|u\|=s_{m+1}}\left\{\frac{\|G(z+u)-G(z)-\langle G'(z),u\rangle\|}{\|u\|}\right\}.\end{aligned}$$

Since we proved that the first supremum on the right hand side tends to zero as m tends to infinity, and since the second supremum tends to zero if G is Fréchet differentiable at z, this proves (v).

Lemma 10. *There is a constant $0 < c < \frac{1}{2n}$ such that, whenever $z \in \ell_p$ and T is a continuous linear mapping of ℓ_p to X_0, then*

$$\limsup_{y\to z}\frac{\|G(y)-G(z)-\langle T, y-z\rangle\|}{\|y-z\|} \ge 2c(1-\lim_{m\to\infty}\operatorname{Tr} G'_m(z)).$$

Consequently,

$$\limsup_{y\to z}\frac{\|G(y)-G(z)-\langle T, y-z\rangle\|}{\|y-z\|} \ge c\left|1-\sum_{j=1}^{\infty}\langle Te_j, e_j^\star\rangle\right|.$$

Proof. We prove that the statement holds with $c = \frac{1}{592n}$. To this end, let us assume first that for some z and T

$$\limsup_{u \to z} \frac{\|G(u) - G(z) - \langle T, u - z\rangle\|}{\|u - z\|} < c\theta,$$

where $\theta = 1 - \lim_{m\to\infty} \operatorname{Tr} G'_m(z)$.

Noting that clearly $\theta > 0$, we find m so large that $\theta > 2^{-m+7}(n+1)$, and

$$\|G(u) - G(z) - \langle T, u - z\rangle\| \le c\theta\|u - z\|$$

whenever $\|u - z\| \le 37r_m$.

We choose $x \in E_m$ and $i \ge 1$ having the properties that $\|\pi_m z - x\| < 24r_m$ and $\|(\pi_0 + \sigma_m)z - d_i^{(m)}\| < r_m$. We intend to estimate the contribution of the summand

$$c^{(m)}(x, i)\varphi_m(z - y^{(m)}(x, i) - d_i^{(m)})$$

to the value of the function G. Since the sequence $\operatorname{Tr} G'_m(z)$ is non-decreasing, we have $\operatorname{Tr} G'_{m-1}(z) \le 1 - \theta$. Since

$$\|(\pi_0 + \sigma_m)(z - d_i^{(m)})\| = \|(\pi_0 + \sigma_m)z - d_i^{(m)}\| < r_m \le 6s_m$$

(where the last inequality comes from (R2)), we infer from Lemma 6(ii) that

$$|\operatorname{Tr} G'_{m-1}(z) - \operatorname{Tr} G'_{m-1}(d_i^{(m)})| \le 2^{-(m+1)}.$$

Hence the coefficient $c^{(m)}(x, i)$ can be estimated from below

$$\begin{aligned} c^{(m)}(x, i) &= 1 - 2^{-m} - \operatorname{Tr} G'_{m-1}(d_i^{(m)}) \ge 1 - 2^{-m} - \operatorname{Tr} G'_{m-1}(z) - 2^{-(m+1)} \\ &\ge 1 - 2^{-m} - 2^{-(m+1)} - 1 + \theta = \theta - 2^{-m} - 2^{-(m+1)} \ge \theta/2. \end{aligned}$$

Let $y_1, y_2 \in X_m$ be such that

$$\tfrac{1}{2}(y_1 + y_2) = y^{(m)}(x, i) \qquad \text{and} \qquad \|y_1 - y_2\| = 6r_m,$$

and let $d \in X_0$ be any vector with $\|d\| = r_m$. Then we put, for $j = 1, 2$,

$$u_j = d + d_i^{(m)} + y_j + \sigma^{(m)} z,$$

and estimate

$$\begin{aligned}\|u_j - z\| &= \|d + d_i^{(m)} + y_j - (\pi_0 + \sigma_m + \pi_m)z\| \\ &\leq \|d\| + \|d_i^{(m)} - (\pi_0 + \sigma_m)z\| + \|y_j - y^{(m)}(x,i)\| \\ &\quad + \|y^{(m)}(x,i) - x\| + \|x - \pi_m z\| \\ &\leq 37r_m.\end{aligned}$$

Since $\|\pi_m u_j - y^{(m)}(x,i)\| = \|y_j - y^{(m)}(x,i)\| = 3r_m$, we have

$$\varphi_m(u_j - y^{(m)}(x,i) - d_i^{(m)}) = 0.$$

Using 5(iv), we infer that

$$b_m(u_j) = c^{(m)}(x,i)\varphi_m(u_j - y^{(m)}(x,i) - d_i^{(m)}) = 0.$$

Using 5(iv) once more, we get

$$\begin{aligned}\|b_m(\tfrac{1}{2}(u_1+u_2))\| &= \|c^{(m)}(x,i)\varphi_m(\tfrac{1}{2}(u_m+v_m) - y^{(m)}(x,i) - d_i^{(m)})\| \\ &= c^{(m)}(x,i)\|\varphi_m(d)\| = c^{(m)}(x,i)\tfrac{1}{n}\|\pi_0(d)\| \\ &\geq \tfrac{\theta}{2n}r_m.\end{aligned}$$

Noting that Lemma 8(iii) gives

$$\begin{aligned}&\|\tfrac{1}{2}(G_{m-1}(u_1) + G_{m-1}(u_2)) - G_{m-1}(\tfrac{1}{2}(u_1+u_2))\| \\ &\leq 2^{-m+1}\|u_1 - u_2\| \leq 2^{-m+4}r_m < \tfrac{\theta}{8n}r_m,\end{aligned}$$

and that Lemma 8(i) and (R2) imply

$$\|(G - G_m)(w)\| \leq 4r_{m+1} \leq 2^{-m}r_m < \tfrac{\theta}{16n}r_m,$$

we conclude that

$$\begin{aligned}&\|\tfrac{1}{2}(G(u_1) + G(u_2)) - G(\tfrac{1}{2}(u_1+u_2))\| \\ &\geq \|\tfrac{1}{2}(b_m(u_1) + b_m(u_2)) - b_m(\tfrac{1}{2}(u_1+u_2))\| \\ &\quad - \|\tfrac{1}{2}(G_{m-1}(u_1) + G_{m-1}(u_2)) - G_{m-1}(\tfrac{1}{2}(u_1+u_2))\| \\ &\quad - \tfrac{1}{2}\|(G-G_m)(u_1)\| - \tfrac{1}{2}\|(G-G_m)(u_2)\| - \|(G-G_m)(\tfrac{1}{2}(u_1+u_2))\| \\ &> \tfrac{\theta}{2n}r_m - \tfrac{\theta}{8n}r_m - 2\tfrac{\theta}{16n}r_m = \tfrac{\theta}{4n}r_m.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
&\|\tfrac{1}{2}(G(u_1)+G(u_2))-G(\tfrac{1}{2}(u_1+u_2))\| \\
&\quad\le\ \tfrac{1}{2}\|G(u_1)-G(z)-\langle T,u_1-z\rangle\|+\tfrac{1}{2}\|G(u_2)-G(z)-\langle T,u_2-z\rangle\| \\
&\qquad\ +\|G(\tfrac{1}{2}(u_1+u_2))-G(z)-\langle T,\tfrac{1}{2}(u_1+u_2)-z\rangle\| \\
&\quad<\ \tfrac{1}{2}c\theta\|u_1-z\|+\tfrac{1}{2}c\theta\|u_2-z\|+c\theta\|\tfrac{1}{2}(u_1+u_2)-z\|\le 74c\theta r_m \\
&\quad=\ \tfrac{\theta}{4n}r_m,
\end{aligned}$$

which is a contradiction proving the first statement of the Lemma.

To prove the second statement, we observe that the inequality

$$\limsup_{y\to z}\frac{\|G(y)-G(z)-\langle T,y-z\rangle\|}{\|y-z\|}<c\Big|1-\sum_{j=1}^{\infty}\langle Te_j,e_j^\star\rangle\Big|$$

and Lemma 9(iii) together with $nc<\frac{1}{2}$ would imply that

$$\Big|\sum_{j=1}^{\infty}\langle Te_j,e_j^\star\rangle-\lim_{m\to\infty}\operatorname{Tr}G_m'(z)\Big|\le\tfrac{1}{2}\Big|1-\sum_{j=1}^{\infty}\langle Te_j,e_j^\star\rangle\Big|.$$

But then

$$\begin{aligned}
\Big|1-\sum_{j=1}^{\infty}\langle Te_j,e_j^\star\rangle\Big| &\le \Big|1-\lim_{m\to\infty}\operatorname{Tr}G_m'(z)\Big|+\Big|\sum_{j=1}^{\infty}\langle Te_j,e_j^\star\rangle-\lim_{m\to\infty}\operatorname{Tr}G_m'(z)\Big| \\
&\le \Big|1-\lim_{m\to\infty}\operatorname{Tr}G_m'(z)\Big|+\tfrac{1}{2}\Big|1-\sum_{j=1}^{\infty}\langle Te_j,e_j^\star\rangle\Big|,
\end{aligned}$$

which would give

$$\limsup_{y\to z}\frac{\|G(y)-G(z)-\langle T,y-z\rangle\|}{\|y-z\|}<2c(1-\lim_{m\to\infty}\operatorname{Tr}G_m'(z)),$$

contradicting thus the already established first part of the Lemma.

Lemma 11. *There is a constant $0<C<\infty$ such that for all $z,v\in\ell_p$*

$$\|G_m'(z;v)\|\le C\,(\operatorname{Tr}G_m'(z))^{\frac{1}{q}}\,\|v\|.$$

In particular, the function G is Lipschitz.

Proof. From Lemma 7 we see that $0 \le \operatorname{Tr} G'_m(z) \le 1$ and that the assumptions of Lemma 5(viii) are satisfied with the constants $c^{(k)}(x,i)$. Hence

$$
\begin{aligned}
\|b'_k(z;v)\| &\le \frac{n+1}{n-p} \operatorname{Tr} b'_k(z)\|v\| && \text{for } v \in X_0,\\
\|b'_k(z;v)\| &\le \frac{p(p+1)}{n}\frac{r_m}{s_m}\left(\frac{n}{n-p}\operatorname{Tr} b'_k(z)\right)^{\frac{1}{q}}\|v\| && \text{for } v \in \oplus_{i=1}^{k-1} X_i, \text{ and}\\
\|b'_k(z;v)\| &\le \frac{p(p+1)}{n}\left(\frac{n}{n-p}\operatorname{Tr} b'_k(z)\right)^{\frac{1}{q}}\|v\| && \text{for } v \in X_k.
\end{aligned}
$$

Using also that $b'_k(z;\sigma^{(k)}v) = 0$ (see Lemma 5(v)) and the inequalities $\operatorname{Tr} b'_k(z) \le \operatorname{Tr} G'_m(z)$ if $k \le m$ and $\operatorname{Tr} G'_m(z) \le (\operatorname{Tr} G'_m(z)^{1/q}$, we estimate

$$
\begin{aligned}
\|G'_m(z;v)\| \;&\le\; \sum_{k=1}^m \|b'_k(z;v)\| = \sum_{k=1}^m \|b'_k(z;\pi_0 v + \sigma_k v + \pi_k v)\|\\
&\le\; \sum_{k=1}^m \|b'_k(z;\pi_0 v)\| + \sum_{k=1}^m \|b'_k(z;\sigma_k v)\| + \sum_{k=1}^m \|b'_k(z;\pi_k v)\|\\
&\le\; \sum_{k=1}^m \frac{n+1}{n-p}\operatorname{Tr} b'_k(z)\|\pi_0 v\|\\
&\quad + \sum_{k=1}^m \frac{p(p+1)}{n}\frac{r_m}{s_m}\left(\frac{n}{n-p}\operatorname{Tr} b'_k(z)\right)^{\frac{1}{q}}\|\sigma_k v\|\\
&\quad + \sum_{k=1}^m \frac{p(p+1)}{n}\left(\frac{n}{n-p}\operatorname{Tr} b'_k(z)\right)^{\frac{1}{q}}\|\pi_k v\|\\
&\le\; \frac{n+1}{n-p}\operatorname{Tr} G'_m(z)\|v\|\\
&\quad + \frac{p(p+1)}{n}\left(\frac{n}{n-p}\operatorname{Tr} G'_m(z)\right)^{\frac{1}{q}}\|v\|\sum_{k=1}^m \frac{r_m}{s_m}\\
&\quad + \frac{p(p+1)}{n}\left(\sum_{k=1}^m \frac{n}{n-p}\operatorname{Tr} b'_k(z)\right)^{\frac{1}{q}}\left(\sum_{k=1}^m \|\pi_k v\|^p\right)^{\frac{1}{p}}\\
&\le\; C(\operatorname{Tr} G'_m(z))^{\frac{1}{q}}\|v\|,
\end{aligned}
$$

where

$$C = \frac{n+1}{n-p} + \frac{p(p+1)}{n}\left(\frac{n}{n-p}\right)^{\frac{1}{q}}\left(1+\sum_{k=1}^{m}\frac{r_m}{s_m}\right),$$

which is finite because of (R2).

Finally, to prove that G is Lipschitz, we note that our estimate of the derivative implies that $\|G_m(u) - G_m(v)\| \leq C\|u-v\|$ $(u, v \in \ell_p)$, and take the limit as m tends to infinity .

Lemma 12. $G'(0) = 0$.

Proof. Recalling that, for each $k \geq 1$, the origin belongs to E_k, and that $y^{(k)}(0,0) = 0$ and $c^k(0,0) = 0$, we infer from 5(iv) that $b_k(z) = 0$ whenever $\|\pi_k z\| < 4r_k$. In particular, $G_m(z) = \sum_{k=1}^{m} b_k(z) = 0$ if $\|z\| < 4r_m$ and $b_k(z) = 0$ if $z \in \oplus_{i=0}^{k-1} X_i$.

Let $v \in \oplus_{i=0}^{m} X_i$. Then the above discussion shows that

$$\frac{G(tv) - G(0)}{t} = \frac{G_m(tv) - G_m(0)}{t} + \sum_{k=m+1}^{\infty} \frac{b_k(tv) - b_k(0)}{t} = 0$$

if $|t|$ is so small that $\|tv\| < 4r_m$. So, taking the limit $t \to 0$, we get that $G'(0; v) = 0$ for every $v \in \oplus_{i=1}^{m} X_i$ and every $m \geq 0$. Since the vectors v of this type are dense in ℓ_p and since the function G is Lipschitz we conclude that $G'(0) = 0$.

2.4. Proof of Theorem 2

We prove that the statement of the Theorem holds with

$$f = \tfrac{1}{n}\pi_0 - G.$$

Because of Lemma 11, the mapping f is Lipschitz. Obviously, the points of (any kind of) differentiability of f and G coincide. In particular, because $G'(0) = 0$ according to Lemma 12, f is Gâteaux differentiable at the origin and $\operatorname{Tr} f'(0) = 1$. However, if z is a point of Fréchet differentiability of the function f, we deduce from Lemma 10 that $\lim_{m\to\infty} \operatorname{Tr} G'_m(z) = 1$ and from Lemma 9(iii) that $\operatorname{Tr} G'(z) = 1$; this shows that $\operatorname{Tr} f'(z) = 0$.

The stronger statements of the Theorem follow similarly easily: (i) from the second statement of Lemma 10, (ii) from Lemma 7 and Lemma 9(iii), and (iii) from Lemma 11 and Lemma 9(iii) and (iv).

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