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How small are σ porous sets and why are we interested in it

Abstract

We describe some of the results of our ongoing investigation of differentiability of Lipschitz functions on infinite dimensional Banach spaces with particular emphasis on the role of porous sets played both in positive results and in key open problems.

Introduction

These notes are an expanded version of the talk given by the second named author at the Real Analysis Symposium. Their main purpose is briefly to inform about some of the results of our study of differentiability problems for Lipschitz functions and to sketch some of the key arguments without going into difficult technical details. The key notions needed to understand the material are explained as well, although the definitions may be somewhat simplified (and sometimes not equivalent to original ones) and appear only after the results have been stated and/or discussed. Background notions and much other relevant material may be found in [2].

The questions we are interested in have, a priory, nothing in common with σ porous sets. We intend to convince the reader that there is a deep connection which can be used to obtain at least some partial answers. However, many basic questions about differentiability of Lipschitz functions in Banach spaces remain unanswered, for example

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Problem 1. *Is is true that any three real-valued Lipschitz functions on an (infinite dimensional separable) Hilbert space have a common point of Fréchet differentiability?*

This is a special case of a number of more general problems, one of which, in spite of weakness of the evidence, we state as

Conjecture 2. *Any finite (or even countable) collection of real-valued Lipschitz functions on a Banach space with separable dual has a common point of Fréchet differentiability.*

Our goal here is mainly to comment on some of the connections between these problems and the questions about size of σ porous sets. Details of the arguments and more detailed results will be contained in a text that is currently being written.

Basic notions

Throughout this talk, *functions will be real-valued*, otherwise they will be called *maps*. X will be a separable Banach space. A function $f\colon X \to \mathbb{R}$ is *Lipschitz* if there is $C < \infty$ so that

$$|f(x) - f(y)| \le C\|x - y\| \text{ all } x, y \in X$$

Definition. A function $f\colon X \to \mathbb{R}$ is said to be *Fréchet differentiable* at a point x_0 if there is $x^* \in X^*$ so that

$$f(x_0 + u) = f(x_0) + x^*(u) + o(\|u\|),\ u \to 0.$$

Definition. The *directional derivative of f at x_0 in direction $u \in X$* is

$$f'(x_0; u) := \lim_{t\to 0} \frac{f(x_0 + tu) - f(x_0)}{t} \qquad (\star)$$

provided that the limit exists.

Observation. *The function f is Fréchet differentiable at x_0 if and only if*

(1) its directional derivative $f'(x_0; u)$ exists for every u and

(2) forms a bounded linear operator as a function of u and

(3) the limit in $(\star)$ is uniform for $\|u\| \le 1$.

Definition. If (1) holds, we say that f is *directionally differentiable* at x_0, if (1) and (2) hold, we say that f is *Gâteaux differentiable* at x_0.

Two remarks are in order: first, and rather easy to see, that if f is Lipschitz and $\dim(X) < \infty$, then (1) $\iff$ (3); and second, that we will not explain here, that in the problems we are interested in, condition (2) is rather easy to achieve.

Existence of Gâteaux derivative

Lebesgue's result (usually quoted as) that monotone functions $\mathbb{R} \to \mathbb{R}$ are differentiable almost everywhere implies that Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ are differentiable a.e. as well: just consider the function $x \to f(x) + Cx$ for C large enough or use the standard extension of Lebesgue's theorem to functions of bounded variation.

So for any Lipschitz $f : X \to \mathbb{R}$ and any $u \in X$, the set of points where f is non-differentiable in direction u belongs to the family $\mathcal{A}(u)$ of Borel sets that are null on every line parallel to u. It follows that every Lipschitz $f : X \to \mathbb{R}$ is directionally (and also Gâteaux) differentiable except points of a set from the σ-ideal $\mathcal{A}$ generated by sets from $\mathcal{A}(u)$, $u \in X$.

One can prove that $\mathcal{A}$ is non-trivial, ie, $X \notin \mathcal{A}$. Strengthening (or weakening) of these observations led in the beginning of 70's a number of authors to the following result, with various notions of "almost everywhere." (The case when X is finite dimensional was proved, with "almost everywhere with respect to the Lebesgue measure," by Rademacher in 1919.)

Theorem 1 (Mankiewicz; Christensen; Aronszajn; Phelps)**.** *Every real-valued Lipschitz function on a separable Banach space is Gâteaux differentiable almost everywhere.*

For example, Aronszajn's notion of null sets is: A set $A \subset X$ is Aronszajn null if for every sequence $u_i \in X$ whose linear span is dense in X, $A = \bigcup_i A_i$ where $A_i \in \mathcal{A}(u_i)$.

Current strongest results on Gâteaux differentiability

Following Aronszajn's idea, we observe that in his definition we do not quite need the full strength of the requirement $A_i \in \mathcal{A}(u_i)$.

Definition. For $u \in X$ and $\eta > 0$ let $\widetilde{\mathcal{A}}(u, \eta)$ be the class of Borel sets $E \subset X$ having the property that $|\gamma^{-1}(E)| = 0$ for every Lipschitz curve γ with $\|\gamma'(t) - u\| \le \eta$.

A set $A \subset X$ is $\widetilde{\mathcal{A}}$-null if for every sequence $u_i \in X$ whose linear span is dense in X there are $\eta_i > 0$ such that $A = \bigcup_i A_i$ where $A_i \in \widetilde{\mathcal{A}}(u_i, \eta_i)$.

Theorem 2 (Zajíček, Preiss 2001)**.** *Every real-valued Lipschitz function on a separable Banach space is Gâteaux differentiable $\widetilde{\mathcal{A}}$-almost everywhere.*

Based on the same idea, a number of formally weaker or formally stronger results may be (rather easily) generated. The use of the word "formally" indicates that it is open whether or not the corresponding variants of Theorem 2 are equivalent.

An interesting formally stronger result is obtained by replacing $\widetilde{\mathcal{A}}(u, \eta)$ by the class $\widetilde{\mathcal{A}}_r(u, \eta)$ of Borel sets $E \subset X$ having the property that for every $\varepsilon > 0$ there is an open set $G \supset E$ such that $|\gamma^{-1}(G)| < \varepsilon$ for every Lipschitz curve γ with $\|\gamma'(t) - u\| \leq \eta$.

Another interesting, this time formally weaker, version of this result replaces $\widetilde{\mathcal{A}}$-sets by the σ-ideal generated by the sets $\widetilde{\mathcal{A}}(u, \eta)$.

We also may notice that, if X has the Radon-Nikodým property, the definition of $\widetilde{\mathcal{A}}(u, \eta)$ may equivalently require that $|\gamma^{-1}(E)| = 0$ for every Lipschitz curve γ such that $t \to \gamma(t) - ut$ has Lipschitz constant $\leq \eta$. Without the Radon-Nikodým property it is not clear if these two definitions lead to the same notion of "almost everywhere," yet Theorem 2 remains valid.

Weaker differentiability, stronger assumptions

Although we are unable to Fréchet differentiate three Lipschitz functions on a Hilbert spaces at the same point, we can at least show that for any prescribed $\varepsilon > 0$ there are points at which the increment of all of them is ε close to linear. More precisely, this property is defined as follows.

Definition. A function $f : X \to \mathbb{R}$ is called almost Fréchet differentiable if for every $\varepsilon > 0$ there is $x \in X$ at which it is ε-Fréchet differentiable, i.e., there are $x^* \in X^*$ and $\delta > 0$ so that

$$\|f(x + u) - f(x) - x^*(u)\| \leq \varepsilon \|u\|$$

if $\|u\| \leq \delta$.

Since Fréchet differentiability of f at x implies (and is in fact equivalent to) its ε-Fréchet differentiability at x for every $\varepsilon > 0$, the validity of Conjecture 2 for countable collections is equivalent to showing that in every Banach space with separable dual and for any $\varepsilon > 0$, any countable collection of Lipschitz functions has a common point of ε-Fréchet differentiability. This is unknown even for finite collections, but for such collections it was proved under the assumption that the norm is smooth in a suitable uniform sense. The notion used to define this is given in

Definition. The modulus of asymptotic uniform smoothness of a space X is defined by

$$\bar{\rho}_X(t) := \sup_{\|x\|=1} \inf_{\dim X/Y<\infty} \sup_{\substack{y\in Y \\ \|y\|\le t}} \bigl(\|x+y\|-1\bigr).$$

Example 3. $X = \ell_p$, $\bar{\rho}_X(t) \asymp t^p$ as $t \searrow 0$.

Current status of the conjecture

Theorem 4 (Preiss 1990; a simpler proof is in Lindenstrauss, Preiss (2000))**.** *Every Lipschitz function on a Banach space with separable dual is Fréchet differentiable at least at one point.*

Theorem 5 (Lindenstrauss, Preiss (1996) with smoothness; Johnson, Lindenstrauss, Schechtman, Preiss (2002) with asymptotic smoothness)**.** *Every finite collection of Lipschitz functions on a separable Banach space that admits a norm with modulus of (asymptotic) smoothness $o(t)$ has, for each $\varepsilon > 0$, a common point of ε-Fréchet differentiability.*

Theorem 6 (Lindenstrauss, Preiss 2003)**.** *On some separable Banach spaces (such as c_0) every Lipschitz function is differentiable Γ-almost everywhere.*

Theorem 7 (Lindenstrauss, Tišer, Preiss 2008)**.** *Every pair of Lipschitz functions on a separable Banach space that admits a norm with modulus of asymptotic smoothness $o\bigl(t^2\log(1/t)\bigr)$ has a common point of Fréchet differentiability.*

Definition of Γ_n- and Γ-null sets

Denote by $\Gamma_n(X)$ the space of continuously differentiable mappings from $[0,1]^n$ to X.

Definition. A Borel set $E \subset X$ is Γ_n*-null* if

$$\bigl\{\gamma \in \Gamma_n(X) : |\gamma^{-1}(E)| > 0\bigr\}$$

is a first category subset of $\Gamma_n(X)$.

Notice that the Baire category theorem shows that Γ_n-null sets form a nontrivial σ-ideal of Borel subsets of X.

The definition makes sense also for $n = \infty$, in which case we leave out the index ∞. So $\Gamma(X)$ is the space of continuous mappings from $[0,1]^{\mathbb{N}}$ to X having continuous partial derivatives. Then $\Gamma(X)$ is not a Banach space but a Fréchet space. So the Baire category theorem is still applicable and the Γ-null sets also form a nontrivial σ-ideal of Borel subsets of X.

Non-differentiability in $\mathbb{R}$

The situation on the real line is well known, in fact, so well known that we couldn't find out who was the first to observe it. A more precise result is due to Zahorski [19]: A subset of $\mathbb{R}$ is the set of points of non-differentiability of a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ if and only if it is a $G_{\delta\sigma}$-set of measure zero.

Observation. *For every Lebesgue null set $E \subset \mathbb{R}$ there is a Lipschitz $f : \mathbb{R} \to \mathbb{R}$ which is non-differentiable at any point of E.*

Proof. Recursively find open $\mathbb{R} = G_0 \supset G_1 \supset G_2 \supset \cdots \supset E$ so small that G_{k+1} is small in every component of G_k. For example, require that $|G_{k+1} \cap C| < 2^{-k-1}|C|$ for any component C of G_k.

Let $\psi(x) = (-1)^k$ where k is the least index such that $x \in G_k$, observe that ψ is well defined almost everywhere and define $f(x) = \int \psi(x)\, dx$. If $x \in E$ and (a, b) is a component of G_k containing x, then

$$\left| \frac{f(b) - f(a)}{b - a} - (-1)^k \right| \le 2^{-k},$$

so f is not differentiable at x. □

Porosity, directional porosity and σ-(directional) porosity

Definition. A set E in X is said to be *c-porous at $x \in E$*, $0 < c < 1$, if for every $\varepsilon > 0$ there is a $z \in X \setminus E$ such that $\|x - z\| < \varepsilon$ and $B(z, c\|x - z\|) \cap E = \emptyset$.

Definition. A set E in X is said to be *directionally c-porous at $x \in E$*, $0 < c < 1$, if there is a line L passing through x such that for every $\varepsilon > 0$ there is a $z \in (X \setminus E) \cap L$ such that $\|x - z\| < \varepsilon$ and $B(z, c\|x - z\|) \cap E = \emptyset$.

Notice that porosity notions that play an important role in differentiability questions are "upper porosities;" the holes do not occur in all radii, but only in arbitrarily small radii.

Definition. A set which is (directionally) c-porous at x for some $0 < c < 1$ is called *(directionally) porous at x*. A set is *(directionally) porous* if it is (directionally) porous at each of its points. A countable union of (directionally) porous sets is called *σ-(directionally) porous.*

Observation. *E is (directionally) porous iff the function $x \to \operatorname{dist}(x, E)$ is not Fréchet (directionally, Gâteaux) differentiable at any point of E.*

While in finite dimensional spaces every σ-porous set is σ-directionally porous, this is false in all infinite dimensional spaces:

Example 8 (Tišer, Preiss 1995)**.** Every infinite dimensional separable Banach space is the union of a σ-porous set and of a set that is null on every line.

The σ-porous set from this example cannot be σ-directionally porous: If it were, it would belong to $\mathcal{A}$, since a directionally porous set is null on all lines in the porosity direction. Hence X would belong to $\mathcal{A}$, and we know that it doesn't.

To construct the example, recursively find points $x_1, x_2, \ldots \in X$ that are 500-dense in X and x_{k+1} has distance 100 from the linear span of $x_1, \ldots, x_k$. The key point is that the set $\bigcup_{k=1}^{\infty} B(x_k, 1)$ is small on sufficiently long segments. Hence it suffices to define the required σ-porous set as the complement of

$$\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} B(r_j x_k, r_j),$$

where $r_j \searrow 0$ sufficiently fast.

In uniformly smooth spaces, a considerable strengthening of this example is due to E. Matoušková [13].

Non-differentiability in infinite dimensions

At the present time, we know three differently behaved classes of sets in infinite dimensional spaces for which we can construct non-differentiable functions.

(a) Preimages of Lebesgue null sets from $\mathbb{R}$ under linear projections (or under non-linear projections satisfying rather obvious requirements).

(b) σ-directionally porous sets.

(c) σ-porous, but not σ-directionally porous sets.

Notice that (a), (b) lead to Gâteaux non-differentiability while (c) leads to Fréchet non-differentiability.

By recent results of Alberti, Csörnyei and Preiss, we may replace $\mathbb{R}$ by $\mathbb{R}^2$ in (a). Or we may use any $\mathbb{R}^n$ and replace Lebesgue null sets by their description of sets of non-differentiability of Lipschitz function on $\mathbb{R}^n$.

The mysterious role of porous sets

Theorem 9 (Lindenstrauss, Preiss 2003)**.** *Every real-valued Lipschitz function on a Banach space X with separable dual is Fréchet differentiable Γ-almost everywhere provided that every porous set in X is Γ-null.*

Notice that already when $X = \mathbb{R}$ there are many more non-differentiability sets than σ-porous sets: any null set can be the former, but the latter are necessarily of the first category.

Notice also that there are infinite dimensional spaces satisfying the conditions of this Theorem. (For example c_0.) But in the most interesting spaces, the infinite dimensional Hilbert spaces, there are σ-porous sets that are not Γ-null; one may even find σ-porous subsets whose complement is Γ-null.

Remark. The previous theorem becomes perhaps less mysterious if we observe that for every Lipschitz $f : X \to \mathbb{R}$, where X has separable dual, there is a σ-porous set of "irregular" points and, if this set is Γ-null then f is Fréchet differentiable Γ-almost everywhere.

The "mean value" estimates

When a Fréchet differentiability result holds for a real-valued Lipschitz function on X, we may expect that the corresponding monotonicity (or mean value) estimate holds:

(♣) If $u \in X$ is such that $f'(x; u) \leq 0$ for every x at which f is Fréchet differentiable, then f decreases in direction u.

For functions on spaces with separable dual this is actually true; both proofs of Theorem 4 give this additional information.

Similarly, assuming that the Conjecture is true, we may expect that the mean value estimate (which is now a corollary of the divergence theorem) holds also for several functions. This mean value estimate is somewhat technical to state, but we may simplify our life by observing that (♣) holds for Gâteaux differentiability and so may be equivalently stated as:

(♠) For every $y \in X$ at which f is Gâteaux differentiable, $u \in X$ and $\varepsilon > 0$ there is $x \in X$ at which f is Fréchet differentiable and $f'(x; u) > f'(y; u) - \varepsilon$.

A similar simplification for several functions leads to the following "mean value" variant of our Conjecture.

Question 3. *Given a Banach space with separable dual and an integer n, is it true that for any $f_1, \dots, f_n : X \to \mathbb{R}$, $u_1, \dots, u_n \in X$ and $\varepsilon > 0$ and for every $y \in X$ at which all f_i are Gâteaux differentiable there is $x \in X$ at which all f_i are Fréchet differentiable and*

$$\textstyle\sum_{i=1}^n f_i'(x; u_i) > \sum_{i=1}^n f_i'(y; u_i) - \varepsilon?$$

There is however a big difference between Problem 1 and Conjecture 2 on one side and Question 3 on the other side: Question 3 has been already answered. In fact, it was answered before it was stated.

Answer (Tišer, Preiss 1995). *If $X = \ell_p$ and $n > p$, the above fails even with almost Fréchet derivative. In particular, it fails when X is an infinitely dimensional Hilbert space and $n \geq 3$.*

This answer suggests that the correct "mean value problem" for Fréchet derivatives is for what X and n is the statement from Question 3 true. Our recent work gives a reasonably satisfactory answer to this problem for almost Fréchet derivatives and even for full Fréchet derivatives if $n = 2$. One direction is a considerably refined version of the above example. The key to the other direction is

Theorem 10. *Suppose that the Banach space X with separable dual has the property that every c-porous set in X can be covered by a union of a σ-directionally porous set and a Γ_n-null G_δ set.*

Then for any Lipschitz $f_1, \dots, f_n : X \to \mathbb{R}$, $u_1, \dots, u_n \in X$ and $\varepsilon > 0$ and for any $y \in X$ at which all f_i are Gâteaux differentiable there is $x \in X$ at which all f_i are Gâteaux differentiable, ε-Fréchet differentiable and

$$\textstyle\sum_{i=1}^n f_i'(x; u_i) > \sum_{i=1}^n f_i'(y; u_i) - \varepsilon$$

So we are again left with the question of smallness of porous sets, this time not in the sense of Γ-nullness but in the sense of Γ_n-nullness.

Before coming to the description of the use of this result, we should comment on its strange assumption. Since every c-porous set is contained in a c-porous G_δ set, it looks to be almost the same as requiring that every c-porous set in X is Γ_n-null. Indeed, this would be the case if all directionally porous sets were Γ_n-null. Funnily enough, we do not know whether this is the case, except when $n \leq 2$.

Theorem 11. *Every σ-directionally porous subset of any Banach space is Γ_1-null as well as Γ_2-null.*

Porous sets and Γ_n-null sets

Theorem 12. *Let X be a separable Banach space with*

$$\bar{\rho}_X(t) = o(t^n \log^{n-1}(1/t)) \text{ as } t \to 0.$$

Then every porous set in X is contained in a union of a σ-directionally porous set and a Γ_n-null G_δ set.

Combined with the previous theorem, we get an answer to the "mean value" version of our problems for almost Fréchet derivatives.

Corollary 13. *Let X be a separable Banach space with*

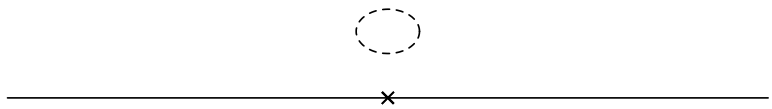
$$\bar{\rho}_X(t) = o(t^n \log^{n-1}(1/t)) \text{ as } t \to 0.$$

Then for any Lipschitz $f_1, \dots, f_n : X \to \mathbb{R}$, $u_1, \dots, u_n \in X$ and $\varepsilon > 0$ and for any $y \in X$ at which all f_i are Gâteaux differentiable there is $x \in X$ at which all f_i are Gâteaux differentiable, ε-Fréchet differentiable and

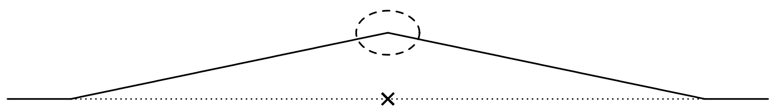
$$\textstyle\sum_{i=1}^n f_i'(x; u_i) > \sum_{i=1}^n f_i'(y; u_i) - \varepsilon$$

The main argument

Starting with a given curve (surface) we want to modify it so that it passes through a hole in the porous set E. Choose a density point t of $\gamma^{-1}(E)$ and close to $x = \gamma(t)$ find a hole.



Modify γ so that it runs through the hole and notice that, at least in the Hilbert space case, the preimage of the hole has much bigger measure than the prolongation of the curve.



Repeating this process should lead to a curve with length almost equal to the length of the original curve (so close to it) but almost never passing through the porous set.

In the Hilbert space, and for curves, this sketch can be completed to a precise argument. However, for more general spaces and higher dimensional surfaces the argument would get rather involved. We therefore observe that another way of stating the above is that we are trying to find curves minimizing a suitable perturbation of the function $\gamma \to |\gamma^{-1}(\overline{E})|$; the perturbation being related to length. To show existence of such minima, we follow the route taken in [3] to obtain a smooth variant of Ekeland's variational principle [6]. (See also [5].) These principles, however, work only in complete spaces and for lower semi-continuous functions. These assumptions do not hold in our situation and so we push the principles further to allow for "conditionally" complete spaces and "conditionally" lower semi-continuous function. It turns out that these notions match the properties that the function $f(\gamma) := |\gamma^{-1}(\overline{E})|$ defined on an appropriate domain satisfies.

Conditional completeness and lower semi-continuity

We consider $\Gamma_n(X)$ with two metrics: d induced by the C^1-norm and d_0 induced by the maximum norm.

The first observation is that, arguing by contradiction, we manage to find a somewhere d-dense, d_0-G_δ subset M of $\Gamma_n(X)$ on which $|\gamma^{-1}(E)| > c > 0$.

We wish to find a minimum of perturbed f on M, where the perturbation should be such that the "running around construction" leads to a contradiction. It is easy to see that such a contradiction may be found for any d_0 continuous perturbation. So everything would be trivial were M complete when equipped with the metric d_0. It is not, but the following weaker property, which can be deduced from it being d_0-G_δ, suffices.

Definition. A space (M, d, d_0), where d_0 is a pseudometric continuous with respect to the metric d, is (d, d_0)-*complete* if there are functions $\delta_j(x_0, \dots, x_j) : M^{j+1} \to (0, \infty)$ such that every d-Cauchy sequence $(x_j)_{j=0}^\infty$ converges provided that

$$d_0(x_j, x_{j+1}) \le \delta_j(x_0, \dots, x_j) \text{ for each } j = 0, 1, \dots.$$

The second important point is that the function $f(\gamma) := |\gamma^{-1}(\overline{E})|$ that we wish to "minimise" is Baire-1 in the metric d_0 (in fact it is upper semi-continuous). We observe that this implies a "conditional lower semi-continuity," which is exactly the property we will need in our variational principle.

Definition. Suppose that (M, d) is a metric space and d_0 a continuous pseudometric on M. We say that a function $f : M \to \mathbb{R}$ is (d, d_0)-lower semi-continuous if there are functions $\delta_j(x_0, \dots, x_j) : M^{j+1} \to (0, \infty)$ such that

$$f(x) \le \liminf_{j \to \infty} f(x_j)$$

whenever $x_j \in M$ d-converge to x and

$$d_0(x_j, x_{j+1}) \le \delta_j(x_0, \dots, x_j) \text{ for each } j = 0, 1, \dots.$$

We use the smoothness assumption on X to define suitable perturbation functions (similar to length) and are ready to deduce the statement of Theorem 12 from a variational principle.

A variational principle

Let (M,d) be a metric space and $F_j : M \times M \to [0,\infty]$ be d-lower semi-continuous in the second variable and such that $F_j(x,x) = 0$ for all $x \in M$ and, for some $r_j \searrow 0$,

$$\inf_{d(x,y)>r_j} F_j(x,y) > 0$$

Suppose further that d_0 is a continuous pseudometric on M, M is (d,d_0)-complete and $f : M \to \mathbb{R}$ is (d,d_0)-lower semi-continuous function and bounded from below.

Then, given any $x_0 \in M$ and any sequence of positive numbers $(\varepsilon_j)_{j=0}^{\infty}$ such that $f(x_0) \le \varepsilon_0 + \inf_{x\in M} f(x)$, one may find a sequence $(x_j)_{j=1}^{\infty}$ in M converging in the metric d to some $x_\infty \in M$ and a d_0 continuous function $\phi : M \to \mathbb{R}$ such that the function

$$h(x) := f(x) + \phi(x) + \textstyle\sum_{j=0}^{\infty} F_j(x_j, x)$$

attains its minimum on M at $x = x_\infty$.

What about full Fréchet differentiability?

It seems reasonable to conjecture that the statement of Corollary 13 holds with full Fréchet differentiability instead of ε-Fréchet differentiability. We can however prove it only when $n \le 2$. For $n = 1$ it is the mean value inequality which we have already mentioned and for $n = 2$ it is the following stronger version of Theorem 7. This Theorem together with Example 8 explain the enormous difference between the size of sets of common points of differentiability of two and three Lipschitz functions.

Theorem 14. *Let X be a separable Banach space with*

$$\bar\rho_X(t) = o(t^2 \log(1/t)) \text{ as } t \to 0.$$

Then for any Lipschitz $f, g : X \to \mathbb{R}$, $u, v \in X$ and $\varepsilon > 0$ and for any $y \in X$ at which both f, g are Gâteaux differentiable there is $x \in X$ at which both f, g are Fréchet differentiable and

$$f'(x;u) + g'(x;v) > f'(y;u) + g'(y;v) - \varepsilon$$

The proof of this result is rather involved and we will not try to explain it here. To a certain extent it follows the almost Fréchet differentiability arguments without the simplifications achieved by the use of the σ-porous sets and the variational principle.

Nevertheless, a (non-trivial) special case of this theorem when the functions f, g are everywhere Gâteaux differentiable, follows naturally from the variational principle. We explain it in the case of one function on a Hilbert space. Observe that, if f is (continuous and) everywhere Gâteaux differentiable then the function $(x,u) \to f'(x,u)$ is (d, d_0)-continuous on $X \times X$, where d is given, say, by the maximum norm in $X \times X$ and d_0 measures just the distance of the first projections. Hence our variational principle provides us with a suitable perturbation of f that attains its maximum and a variant of the "running around construction" shows that f is Fréchet differentiable at any point at which this maximum is attained.

Further open problems

Problem 4. *Is there a real-valued Lipschitz function on a separable Hilbert space (or at least on some Banach space with separable dual) whose set of points of Gâteaux differentiability is σ-porous?*

Problem 5. *Can an infinite dimensional separable Hilbert space (or just some Banach space with separable dual) be decomposed as a union of a σ-porous set and an $\widetilde{\mathcal{A}}$ or $\widetilde{\mathcal{A}}_r$ null set?*

Problem 6. *Are porous subsets of $\mathbb{R}^{n+1}$ Γ_n-null ($n \geq 3$)?*

Problem 7. *Is every set belonging to some $\widetilde{\mathcal{A}}(u, \varepsilon)$ necessarily $\widetilde{\mathcal{A}}$-null?*

Problem 8. *In $\mathbb{R}^n$, $n \geq 3$, do $\widetilde{\mathcal{A}}$ null sets coincide with Lebesgue null sets?*

Problems 7 and 8 are open also for $\widetilde{\mathcal{A}}_r$ null sets. This is hardly surprising, since we do not know whether these classes are different or not.

Problem 9. *Do $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{A}}_r$ null sets coincide?*

There is, however, one difference between what we know about $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{A}}_r$ null sets. To explain it, recall that already the very first Gâteaux differentiability results were proved for more general target spaces, namely those Banach spaces that have the Radon-Nikodým Property. (See [2].) Similarly Theorem 2 holds, with $\widetilde{\mathcal{A}}$-null sets, also for maps into such Banach spaces. However, the argument replacing $\widetilde{\mathcal{A}}$ with $\widetilde{\mathcal{A}}_r$ uses that the functions in question are real valued, thus leaving the following problem open.

Problem 10. *Is every Lipschitz map of a separable Banach space to a Banach space with the Radon-Nikodým property Gâteaux differentiable* $\widetilde{\mathcal{A}}_r$ *almost everywhere?*

References

[1] N. Aronszajn, Differentiability of Lipschitzian mappings between Banach spaces, *Studia Math* **57**(1976), 147 – 190

[2] Y. Benyamini, J. Lindenstrauss, Geometric Nonlinear Functional Analysis, Vol. 1, *Colloquium Publications* Vol 48, AMS 2000

[3] J. Borwein, D. Preiss, A smooth variational principle with applications to subdifferentiability of convex functions, *Trans. Amer. Math. Soc.* **303**(1987), 517 – 527

[4] J. P. R. Christensen, Measure theoretic zero sets in infinite dimensional spaces and applications to differentiability of Lipschitz mappings, Actes du Deuxième Colloque d'Analyse Fonctionnelle de Bordeaux (Univ. Bordeaux, 1973), I, pp. 29–39. *Publ. Dép. Math. (Lyon)* **10**(1973), no. 2, 29–39

[5] R. Deville, G. Godefroy, V. Zizler *Smoothness and renormings in Banach spaces, Pitman Monographs and Surveys* Vol. 64, Longman 1993

[6] I. Ekeland, Nonconvex minimization problem, *Bull. Amer. Math. Soc.* **1**(1979), 443–474

[7] W.B. Johnson, J. Lindenstrauss, D. Preiss, G. Schechtman, Almost Fréchet differentiability of Lipschitz mapping between infinite dimensional Banach spaces, *Proc. London Math. Soc.* (3) **84** (2002), 711–146

[8] J. Lindenstrauss, D. Preiss, Almost Fréchet differentiability of finitely many Lipschitz functions, *Mathematika*, **86**(1996), 393–412

[9] J. Lindenstrauss, D. Preiss, A new proof of Fréchet differentiability of Lipschitz functions, *J. Eur. Math. Soc.* **2**, No.3 (2000), 199–216

[10] J. Lindenstrauss, D. Preiss, On Fréchet differentiability of Lipschitz maps between Banach spaces, *Annals of Math.* **157**(2003), 257–288

[11] J. Lindenstrauss, D. Preiss, J. Tišer, in preparation; expected to be finished in 2008

[12] P. Mankiewicz, On the differentiability of Lipschitz mappings in Fréchet spaces, *Studia Math.* **45**(1973), 15–29

[13] E. Matoušková, An almost nowhere Fréchet smooth norm on superreflexive spaces, *Studia Math.* **133**(1999), no. 1, 93–99

[14] R.R. Phelps, Gaussian null sets and differentiability of Lipschitz map on Banach spaces, *Pac. J. Math.* **77**(1978), 523-531

[15] D. Preiss, Fréchet differentiability of Lipschitz functions, *J. Functional Anal.* **91**(1990), 312–345

[16] D. Preiss, L. Zajíček, Directional derivatives of Lipschitz functions, *Israel J. Math* **125** (2001), 1–27

[17] D. Preiss, J. Tišer, Two unexpected examples concerning differentiability of Lipschitz functions on Banach spaces, *GAFA Israel Seminar* 92–94, *Birkhäuser* 1995, 219–238

[18] H. Rademacher, Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variablen und über die Transformation der Doppelintegrale, *Math. Ann.* 79(1919), 340-359

[19] Z. Zahorski, Sur l'esemble des points de non-dérivabilité d'une fonction continue, *Bull. Soc. Math. France* **74**(1946), 147–178