# Positivity Principle for more concentrated measures

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**Abstract**

The inequality $\mu \geq \nu$ between two finite measures $\mu$ and $\nu$ on a metric space $X$ is deduced from the inequality $\mu B \geq \nu B$ for balls $B$ with radius less than a given constant and from some restrictions on the support of $\nu$.

## 1. Introduction.

For the purpose of this paper, the *Positivity Principle* will mean the following statement:

> If $\mu$ is a bounded signed measure on a metric space $X$ for which there is $r > 0$ such that $\mu B \geq 0$ for every ball $B$ with the radius less than $r$, then $\mu$ is a non-negative measure.

This may be equivalently stated using instead of one signed measure a pair of finite measures (by measure we always mean a non-negative measure, if we want to allow also negative values, we use the term signed measure): For such a pair $\mu$ and $\nu$ the inequality $\nu B \leq \mu B$ for all "small" balls should imply that $\nu \leq \mu$. In [6] it was proved that the validity of the Positivity Principle for all finite Borel measures in a separable Hilbert space $H$ is equivalent to

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$\dim H < +\infty$. It follows that, in order for statements like the Positivity Principle to hold in infinitely dimensional spaces, additional assumptions either on the space or on the measures involved are needed.

The main result of this note (Theorem 7) gives a new simple criterion for the validity of such restricted Positivity Principle based on the notion of "essential support" which may well turn out to be of independent interest, in particular because for Gaussian measures it consists precisely from the points belonging to the reproducing kernel Hilbert space (as may be easily seen from the deduction of the third application of our theorem).

Several seemingly quite different results concerning Positivity Principles which appeared during the last two decades are now covered by our main theorem. The first such results were discovered by J.P.R. Christensen and extended by M. Studený. To describe them, we need the notion of (almost) uniformly distributed measures.

A locally finite Borel measure $\mu$ on a metric space $X$ is called *uniformly distributed* if $\mu B(x,r) = \mu B(y,r) > 0$ for all $x, y \in X$ and $r > 0$, where $B(x,r)$ denotes an open ball with the center $x$ and radius $r$. If there is a function $h : (0,+\infty) \longrightarrow (0,+\infty)$ and a constant $c \in (0,1)$ such that

$$ch(r) \leq \mu B(x,r) \leq h(r)$$

for all $x \in X$ and $r > 0$ we say that $\mu$ is *almost uniformly distributed.*

**Theorem 1.** (Christensen, [3]) *Let $X$ be a metric space on which there exists a finite uniformly distributed measure. Then the Positivity Principle holds true in $X$.*

A generalization for almost uniformly distributed measures is due to Studený.

**Theorem 2.** (Studený, [8]) *Let $X$ be a metric space with a finite almost uniformly distributed measure. Then the Positivity Principle holds true in $X$.*

The next result for Hilbert spaces seems to be of a completely different nature. It imposes a very special approximation condition upon one of the measures.

**Theorem 3.** (Preiss, Tišer, [7]) *Let $H$ be a separable Hilbert space and let $(H_k)$ be a sequence of its finite dimensional subspaces. Assume that $\nu$ is a finite Borel measure such that*

$$\operatorname{dist}(x, H_k) = o\left(\frac{1}{\sqrt{\dim H_k}}\right)$$

*for $\nu$-a.e. $x \in H$. If $\mu$ is a finite Borel measure and $r_0 > 0$ such that $\mu B(x,r) \geq \nu B(x,r)$ for all $x \in H$ and $r_0 \geq r > 0$, then $\mu \geq \nu$.*

The last result we want to mention is due to U. Dinger. It restricts the support of one of the measures in, it appears, a completely different way. (Less importantly, our result also improves its statement to the full Positivity Principle. As it stands, it gives what could be called a "Zero Principle", which is implied by, but does not imply, the Positivity Principle.)

**Theorem 4.** (Dinger, [5]) *Let $X$ be a separable Banach space and let the finite measure $\nu$ have the support in a reproducing kernel $H_\gamma$ for some Gaussian measure $\gamma$. If $r_0 > 0$ and $\mu$ is a finite measure with $\mu B(x,r) = \nu B(x,r)$ for all $x \in X$ and $0 < r < r_0$, then $\mu = \nu$.*

## 2. Main Theorem.

We start with the general form of our result.

**Theorem 5.** *Let $\nu$ be a finite Borel measure on a metric space $X$ and let $\gamma_n$ be a sequence of $\sigma$-finite Borel measures on $X$ and $r_n$ a sequence of positive numbers tending to zero such that the numbers*

$$h_n = \sup\{\gamma_n B(x, r_n);\ x \in X\}$$

*are positive and finite and*

$$\liminf_{n\to\infty} \frac{\gamma_n B(x, r_n)}{h_n} > 0$$

*for $\nu$ almost all $x \in X$. If $\mu$ is a finite measure with $\mu B(x, r_n) \geq \nu B(x, r_n)$ for all $x \in X$ and all $n$, then $\mu \geq \nu$.*

**Proof.** Consider the Jordan decomposition of the bounded signed measure $\mu - \nu$ into positive and negative parts,

$$\mu - \nu = u_+ - u_-.$$

Then $u_+$ and $u_-$ are finite non-negative mutually singular measures. Also, since $(\mu - \nu)B(x, r_n) \geq 0$, then

$$u_+ B(x, r_n) \geq u_- B(x, r_n) \tag{1}$$

for all $x \in X$ and $r_0 \geq r > 0$. The proof will be completed if we show that $u_-$ is identically zero. For that, it is sufficient to prove that $u_-$ is absolutely continuous with respect to $u_+$.

Let

$$\Theta(x) = \liminf_{n\to\infty} \frac{\gamma_n B(x, r_n)}{h_n}.$$

Obviously, $0 \leq \Theta \leq 1$ and, by the assumption, $\Theta(x) > 0$ for $\nu$-a.e. $x \in X$. Since $u_- \leq \nu$, we infer that $\Theta(x) > 0$ for $u_-$-a.e. $x \in X$ as well.

We denote by $\chi_n$ the characteristic function of the set

$$K_n = \{x \in X \ ; \ 0 < \Theta(x) h_n < 2\gamma_n B(x, r_n)\},$$

and infer from the definition of $\Theta$ that the sequence $\chi_n$ converges to the characteristic function of the set $\{x;\ \Theta(x) > 0\}$. Thus

$$\lim_{n\to\infty} \chi_n \Theta = \Theta. \tag{2}$$

Let $\varphi : X \longrightarrow [0, +\infty)$ be a fixed bounded continuous function and let

$$\varphi_n(x) = \begin{cases} \dfrac{1}{\gamma_n B(x, r_n)} \displaystyle\int_{B(x,r_n)} \varphi(y)\, d\gamma_n(y) & \text{if } \gamma_n B(x, r_n) > 0 \\ \\ \varphi(x) & \text{otherwise.} \end{cases}$$

The continuity of $\varphi$ implies that

$$\lim_{n\to\infty} \varphi_n = \varphi. \tag{3}$$

We note that the Borel measurability of $\Theta$, $K_n$, $\chi_n$ and $\varphi_n$ follows immediately from the lower semi-continuity of the functions $x \mapsto \gamma_n B(x, r_n)$. Using Fubini's Theorem, the inequality

$$\frac{\Theta(x)}{\gamma_n B(x, r_n)} \leq \frac{2}{h_n} \text{ for } x \in K_n,$$

which follows from the definition of the set $K_n$, and the inequalities (1) and $h_n \geq \gamma_n B(x, r_n)$ for $x \in X$, we estimate that

$$
\begin{aligned}
\int \varphi_n \chi_n \Theta \, du_- &= \int \varphi(y) \int_{B(y,r_n)\cap K_n} \frac{\Theta(x)}{\gamma_n B(x,r_n)} \, du_-(x) \; d\gamma_n(y) \\
&\leq \int \varphi(y) \int_{B(y,r_n)\cap K_n} \frac{2}{h_n} \, du_-(x) \; d\gamma_n(y) \\
&\leq \frac{2}{h_n} \int \varphi(y) \; u_- B(y,r_n) \; d\gamma_n(y) \\
&\leq \frac{2}{h_n} \int \varphi(y) \; u_+ B(y,r_n) \; d\gamma_n(y) \\
&= 2 \int \frac{\gamma_n B(x,r_n)}{h_n} \; \varphi_n(x) \; du_+(x) \\
&\leq 2 \int \varphi_n(x) \; du_+(x).
\end{aligned}
$$

Since $\lim_{n\to\infty} \varphi_n = \varphi$ according to (3) and $\lim_{n\to\infty} \varphi_n \chi_n \Theta = \varphi\Theta$ by combining (3) and (2), and since all the functions involved are uniformly bounded, we conclude from Lebesgue Theorem that

$$
\int \varphi\Theta \; du_- \leq 2 \int \varphi \; du_+ .
$$

This holds true for any bounded non-negative continuous function $\varphi$, so

$$
\Theta u_- \leq 2 u_+ .
$$

Since $\Theta$ is a $u_-$ almost everywhere positive function, we see that $u_-$ is absolutely continuous with respect to $u_+$, which finishes the proof.

An easy trick provides us with the following generalization. Its main corollary is the relaxation of the requirement that the measures in Theorem 5 be finite.

**Theorem 6.** *Let $\nu$ be a locally finite $\sigma$-finite Borel measure on a metric space $X$ and let $\gamma_{n,i}$ be a family of $\sigma$-finite Borel measures on $X$ and $r_{n,i}$ $(n, i = 1, 2, \dots)$ a family of positive numbers such that* $\lim_{n\to\infty} r_{n,i} = 0$ *for each $i$, the numbers*

$$
h_{n,i} = \sup\{\gamma_{n,i} B(x, r_{n,i}); \; x \in X\}
$$

*are positive and finite, and the sets*

$$E_i = \{x \in X;\ \liminf_{n\to\infty} \frac{\gamma_{n,i} B(x, r_{n,i})}{h_{n,i}} > 0\}$$

*cover $\nu$ almost all of $X$. If $\mu$ is a locally finite $\sigma$-finite Borel measure with $\mu B(x, r_{n,i}) \ge \nu B(x, r_{n,i})$ for all $x \in E_i$, then $\mu \ge \nu$.*

**Proof.** If $\mu$ and $\nu$ are finite, we use, for each $i = 1, 2, \dots$, Theorem 5 with $\nu$ replaced by the restriction $\nu_i$ of $\nu$ to $E_i$ to infer that $\mu \ge \nu_i$. Hence for every Borel measurable set $E$,

$$\mu(E) = \sum_i \mu(E \cap (E_i \setminus \bigcup_{j<i} E_j)) \ge \sum_i \nu_i(E \cap (E_i \setminus \bigcup_{j<i} E_j)) = \nu(E).$$

If $\mu$ and $\nu$ are locally finite and $\sigma$-finite, there is a non-decreasing sequence of open sets $G_k$ of $\mu + \nu$ finite measure covering $\mu + \nu$ almost all of $X$. For every $k$ and every $m$ we observe that the restrictions

$$\nu_{k,m} = \nu \llcorner \{x \in G_k;\ \mathrm{dist}(x, X \setminus G_k) > 1/m\} \text{ and } \mu_k = \mu \llcorner G_k$$

verify the assumptions of the already proved finite case of the theorem. Hence $\mu_k \ge \nu_{k,m}$, and the limit as $m \to \infty$ gives that $\mu \llcorner G_k \ge \nu \llcorner G_k$ for each $k$. Taking the limit as $k \to \infty$ finishes the proof.

It is useful to formulate these results as saying that the Positivity Principle is verified once the measure $\nu$ is supported by a certain set. To this end we introduce the following definition.

**Definition 1.** Let $\Gamma = (\gamma_n)$ be a sequence of $\sigma$-finite Borel measures on a metric space $X$. We put

$$h_n(r) = \sup\{\gamma_n B(x, r);\ x \in X\}.$$

The set

$$\mathrm{Ess}\,(\Gamma) = \{x \in X;\ \liminf_{r\to 0} \liminf_{n\to\infty} \frac{\gamma_n B(x, r)}{h_n(r)} > 0\}$$

is called the *essential support* of $\Gamma$.

If $\gamma_n = \gamma$ for each $n$, we speak about the essential support of $\gamma$ and write $\mathrm{Ess}\,(\gamma)$ instead of $\mathrm{Ess}\,(\Gamma)$.

In this definition we may encounter expressions $\frac{0}{0}$ and $\frac{\infty}{\infty}$ which are both let to be zero. This means that in order for $x$ to belong to $\mathrm{Ess}\,(\Gamma)$ it is necessary that for sufficiently small $r > 0$ one has $0 < \gamma_n B(x,r) \leq h_n(r) < \infty$ if $n > n_r$.

We are now ready for the main result.

**Theorem 7.** *Let $\nu$ be a finite Borel measure on a metric space $X$ and let $\Gamma = (\gamma_n)$ be a sequence of $\sigma$-finite Borel measures such that $\mathrm{Ess}\,(\Gamma)$ covers $\nu$ almost all of $X$. If $r_0 > 0$ and $\mu$ is a finite measure with $\mu B(x,r) \geq \nu B(x,r)$ for all $x \in X$ and $r_0 \geq r > 0$, then $\mu \geq \nu$.*

**Proof.** We fix a sequence $(r_k)_{k=1}^{\infty}$ such that $r_0 \geq r_1 > r_2 > \dots$ and $\lim r_k = 0$, and we put

$$\Theta_k(x) = \liminf_{n\to\infty} \frac{\gamma_n B(x,r_k)}{h_n(r_k)} \quad \text{and} \quad \Theta(x) = \liminf_{k\to\infty} \Theta_k(x)$$

for $x \in X$; our conventions imply that $\Theta(x) > 0$ for $x \in \mathrm{Ess}\,(\Gamma)$. Since the functions $x \mapsto \gamma_n B(x,r)$ are lower semi-continuous, $\Theta$ is Borel measurable. Obviously, $0 \leq \Theta \leq 1$, and, by the assumption, $\Theta(x) > 0$ for $\nu$-a.e. $x \in X$.

Since

$$1 \geq (\Theta(x) - \Theta_k(x))_+ \to 0 \text{ as } k \to \infty$$

(where $c_+ = \max\{c, 0\}$), the Lebesgue Theorem provides us with a sequence $k_j$ such that

$$\int (\Theta(x) - \Theta_{k_j}(x))_+ d\nu(x) < 2^{-j-1}.$$

Similarly, since

$$1 \geq \left(\Theta_{k_j}(x) - \frac{\gamma_n B(x,r_{k_j})}{h_n(r_{k_j})}\right)_+ \to 0 \text{ as } n \to \infty,$$

there are $n_j$ such that

$$\int \left(\Theta_{k_j}(x) - \frac{\gamma_{n_j} B(x,r_{k_j})}{h_{n_j}(r_{k_j})}\right)_+ d\nu(x) < 2^{-j-1}.$$

Thus

$$\int \left(\Theta(x) - \frac{\gamma_{n_j} B(x,r_{k_j})}{h_{n_j}(r_{k_j})}\right)_+ d\nu(x) < 2^{-j},$$

and Borel-Cantelli Lemma implies that

$$\liminf_{j\to\infty} \frac{\gamma_{n_j} B(x, r_{k_j})}{h_{n_j}(r_{k_j})} \geq \Theta(x) > 0$$

for $\nu$ a.e. $x$. So the assumptions of Theorem 5 are verified, and the statement follows by its application.

Using Theorem 6 instead of Theorem 5, or repeating the same trick as in the proof of Theorem 6, we immediately get

**Theorem 8.** *Let $\nu$ be a locally finite $\sigma$-finite measure on a metric space $X$ and let $(\Gamma_i)$ be a countable family of the sequences $\Gamma_i = (\gamma_n^{(i)})$ of $\sigma$-finite measures such that the union $\bigcup_i \mathrm{Ess}\,(\Gamma_i)$ covers $\nu$ almost all of $X$. If $r_0 > 0$ and $\mu$ is a locally finite $\sigma$-finite measure on $X$ such that $\mu B(x, r) \geq \nu B(x, r)$ for all $x \in X$ and $r_0 \geq r > 0$, then $\mu \geq \nu$.*

**Remark.** Essential support may be defined, instead for sequences, for more general nets of measures. The proofs of Theorems 7 and 8 remain valid as long as the nets are of countable type.

# 3. Special cases.

1. Let $X$ be a metric space with a $\sigma$-finite uniformly distributed measure $\gamma$. Since clearly $\mathrm{Ess}\,(\gamma) = X$, Theorem 7 gives the result of Christensen (Theorem 1 above). The same choice works also in case of almost uniformly distributed measures and Theorem 2 follows as well.

2. Let $H$ be a separable Hilbert space and let $(H_k)$ be an increasing sequence of finite dimensional subspaces, $n_k = \dim H_k$. We put $\gamma_k = \mathcal{L}_{n_k} \llcorner H_k$ and $\Gamma = (\gamma_k)$. Let $x \in H$ and $r > 0$. An easy calculation reveals that

$$\gamma_k B(x, r) = \mathcal{L}_{n_k} B(0, 1) \left(r^2 - \mathrm{dist}^2(x, H_k)\right)_+^{n_k/2}.$$

So $h_k(r) = \mathcal{L}_{n_k} B(0, 1) r^{n_k}$ and the assumption in Theorem 3 implies that

$$\liminf_{r\to 0} \liminf_{k\to\infty} \frac{\gamma_k B(x, r)}{h_k(r)} = 1$$

for $\nu$-a.e. $x \in H$. Hence Theorem 7 applies.

3. Let $X$ be a separable Banach space and let a measure $\nu$ have the support in the reproducing kernel $H_\gamma$ of some (centered) Gaussian measure $\gamma$. The results of C. Borell [1] say that

$$h(r) = \sup\{\gamma B(x,r);\ x \in X\} = \gamma B(0,r)$$

and [2] that

$$\lim_{r \to 0} \frac{\gamma B(x,r)}{\gamma B(0,r)} = e^{-\|x\|_\gamma},$$

where the norm $\|.\|_\gamma$ arises from the covariance operator of $\gamma$. For us, it is only important to know that

$$H_\gamma = \{x \in X \ ;\ \|x\|_\gamma < +\infty\}.$$

Hence, if the support of $\nu$ is contained in $H_\gamma = \mathrm{Ess}\,(\gamma)$, then Theorem 7 implies that $\mu \geq \nu$ provided $\mu B \geq \nu B$ for all balls $B$ of small radius. If we moreover assume the equality $\mu B = \nu B$ for all small balls, then $\mathrm{spt}\,\mu = \mathrm{spt}\,\nu$ and the requirement for $\mathrm{spt}\,\nu$ is verified also for $\mathrm{spt}\,\mu$. It follows that we can obtain $\mu \leq \nu$ as well, and consequently $\mu = \nu$. This is exactly the content of Theorem 4.

4. Theorems 1 and 2 were particular cases of the situation when the Positivity Principle holds because there is a measure whose essential support is the whole space. The existence of such a measure imposes strong conditions upon the space. Indeed, we have

**Observation 1.** *If $\gamma$ measures a complete space $X$ and* $\mathrm{Ess}\,(\gamma) = X$, *then $X$ contains a compact set with non-empty interior.*

**Proof.** The sets

$$E_k = \{x \in X;\ \frac{\gamma B(x,r)}{h(r)} \geq \frac{1}{k} \ \text{ for } \ 0 < r < \frac{1}{k}\}$$

are closed and cover $X$. Hence by Baire Category Theorem one of them, say $E_m$, has an interior point, say $z$. Let $0 < s < \frac{1}{2m}$ be such that $B(z, 2s) \subset E_m$. If the closed ball $B$ of centre $z$ and radius $s$ were not compact, we would be able to find $0 < \delta < s$ and an infinite set $S \subset B$ such that $\mathrm{dist}(x,y) > 2\delta$ for any pair $x, y$ of different points of $S$. But then

$$\gamma B(z, 2s) \geq \sum_{x \in S} \gamma B(z, \delta) \geq \sum_{x \in S} \frac{h(\delta)}{m} = \infty,$$

which contradicts the definition of $E_m$.

In particular, no infinitely dimensional Banach space can be the essential support of a single measure. However, our next result shows that it can be the essential support of a sequence of measures. (Of course, because of the result of [6] mentioned in the introduction, we know that this is not true for all Banach spaces. Indeed, the Positivity Principle does not hold in an infinite dimensional separable Hilbert space $H$, so Theorem 7 implies that $H$ cannot be the essential support of any sequence of measures.)

**Theorem 9.** *The space* $c_1$ *of all convergent sequences is the essential support of a sequence of measures. In particular, the Positivity Principle holds true in* $c_1$.

**Proof.** We put

$$M_n = \{(x_k) \ : \ x_k = x_{k+1} \text{ for all } k \geq n\}.$$

Since $\dim M_n = n$ we may define $\gamma_n = \mathcal{L}_n \llcorner M_n$, and finally, $\Gamma = (\gamma_n)$.

Let $x \in c_1$ and $r > 0$. Let $\varepsilon > 0$ and let $n_0$ be such that

$$|x_n - x_\infty| \leq \varepsilon \quad \text{for all } n \geq n_0,$$

where $x_\infty = \lim x_n$. If $n \geq n_0$, then

$$B(x,r) \cap M_n \supset \{y \in M_n \ ; \ |y_k - x_k| \leq r, \ 1 \leq k \leq n-1, \text{and } |y_n - x_\infty| \leq r - \varepsilon\}.$$

Hence

$$\gamma_n B(x,r) = \mathcal{L}_n(B(x,r) \cap M_n) \geq 2^n \ r^{n-1}(r - \varepsilon).$$

It follows that

$$\begin{aligned} \liminf_{n\to\infty} \frac{\gamma_n B(x,r)}{h_n(r)} &= \liminf_{n\to\infty} \frac{\gamma_n B(x,r)}{\gamma_n B(0,r)} \\ &\geq \liminf_{n\to\infty} \frac{2^n \ r^{n-1}(r-\varepsilon)}{2^n \ r^n} = 1 - \frac{\varepsilon}{r}. \end{aligned}$$

This is true for all $\varepsilon > 0$, so we obtain

$$\liminf_{n\to\infty} \frac{\gamma_n B(x,r)}{\gamma_n B(0,r)} = 1$$

for all $x \in c_1$. Consequently, $\mathrm{Ess}\,(\Gamma) = c_1$.

Another example of a Banach space which is the essential support of a sequence of measures is given by the space $c_0$ of sequences converging to zero. The proof may be obtained by replacing in the above argument the sets $M_n$ by

$$M_n = \{(x_k) \ : \ x_k = 0 \text{ for all } k > n\}$$

and observing that for any $x \in c_0$ and $r > 0$ one has $\gamma_n B(x,r) = 2^n r^n$ if $n$ is large enough. (However, not every subset of $c_1$ is the essential support of a sequence of measures, indeed, not every subset of $c_1$ verifies the Positivity Principle. See [8] for a modification of Davies's [4] construction of a compact metric space admitting different Borel probability measures agreeing on all balls which can be embedded into $c_0$.)

5. Finally, we show how our result gives a natural generalization of Theorem 3 to all Banach spaces. Of course, we pay for the generality by stronger assumptions on the support of the measure.

**Corollary 1.** *Let $X$ be a separable Banach space and $(X_k)$ a sequence of its finite dimensional subspaces. Let $\nu$ be a finite measure such that*

$$\operatorname{dist}(x, X_k) = o\left(\frac{1}{\dim X_k}\right)$$

*for $\nu$-a.e. $x \in X$. If $r_0 > 0$ and $\mu$ is a finite measure with $\mu B \geq \nu B$ for all $x \in X$ and $r_0 \geq r > 0$, then $\mu \geq \nu$.*

**Proof.** We define

$$\gamma_k = \mathcal{L}_{n_k} \llcorner X_k,$$

where $n_k = \dim X_k$. Let $x \in X$ and $r > 0$. We denote by $x_k$ the closest point from $X_k$ to the point $x$, $\operatorname{dist}(x, X_k) = \|x - x_k\|$. Then a simple geometrical observation reveals that

$$\begin{aligned} \gamma_k B(x,r) &= \mathcal{L}_{n_k}\left(B(x,r) \cap X_k\right) \geq \mathcal{L}_{n_k} B\left(x_k, r - \operatorname{dist}(x, X_k)\right)) \\ &= \mathcal{L}_{n_k} B(0,1) \ \left(r - \operatorname{dist}(x, X_k)\right)^{\dim X_k}. \end{aligned}$$

Hence

$$\liminf_{k\to\infty} \frac{\gamma_k B(x,r)}{h_k(r)} = \liminf_{k\to\infty} \frac{\gamma_k B(x,r)}{\gamma_k B(0,r)} \geq \lim_{k\to\infty} \left(1 - \frac{\operatorname{dist}(x, X_k)}{r}\right)^{\dim X_k} = 1$$

for $\nu$-a.e. $x \in X$ and Theorem 7 applies.

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