Vitali Covering Theorem in Hilbert Space

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Abstract

It is shown that the statement of Vitali Covering theorem does not hold for a certain class of measures in a Hilbert space. This class contains all infinite dimensional Gaussian measures.

1. Introduction

We start with recalling the statement of the classical covering theorem due to G. Vitali, [9].

Theorem. *Let $A \subset \mathbb{R}^n$ be a set. Assume that for every $x \in A$ there is a sequence $(B[x, r_k(x)])_k$ of closed balls centred at x and radii $r_k(x)$ such that $\lim_{k\to\infty} r_k(x) = 0$. Then there is an at most countable family of disjoint balls, $\{B[x_i, r_{k_i}(x_i)] \mid i \in \mathbb{N}\}$, such that*

$$\mathscr{L}_n\Big(A \setminus \bigcup_{i\in\mathbb{N}} B[x_i, r_{k_i}(x_i)]\Big) = 0.$$

The balls in the original paper were considered with respect to the norm $\|.\|_\infty$. In fact, the statement of the Theorem above holds true for balls in any equivalent norm in $\mathbb{R}^n$.

Since the time of Vitali there appeared many generalizations of the statement in various directions. To mention at least one of them, now already classical, we have to point out the version based on the Besicovitch Covering

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theorem. It extends the statement from Lebesgue measure to any σ-finite measure on $\mathbb{R}^n$, see e.g. de Guzmán [1].

Our aim is to study what happens if we replace the n-dimensional Euclidean space $\mathbb{R}^n$ by an infinite dimensional Hilbert space. The first result of this type is due to D. Preiss, [4]. He gave an example of a Gaussian measure on a separable Hilbert space for which the covering theorem fails to hold.

One of the most important consequences of Vitali Covering theorem is the so-called Differentiation theorem. The original version goes back to H. Lebesgue. Employing the above mentioned generalization of the covering theorem one has the following form of the Differentiation theorem. Here, and also in the sequel, $B[x,r]$ denotes the closed ball with the center x and radius $r > 0$.

Differentiation Theorem. 1 *Let μ be a locally finite measure on $\mathbb{R}^n$ and let $f \in L^1_{loc}(\mu)$. Then*

$$\lim_{r\to 0} \frac{1}{\mu B[x,r]} \int_{B[x,r]} f\, d\mu = f(x) \quad \mu - a.e. \tag{1}$$

The negative result of D. Preiss [4] was later strengthen in [5] by constructing a bounded function and a Gaussian measure on a Hilbert space such that (1) does not hold. Moreover, in [6] the same author obtained a Gaussian measure γ together with the integrable function $f \in L^1(\gamma)$ such that the means of f over the balls in (1) tend to infinity uniformly with respect to x.

On the other hand, J. Tišer [8] has shown the validity of (1) for some class of Gaussian measures on a Hilbert space and for all L^p functions, $1 < p < \infty$. This result could indicate that there is a chance for having the Vitali Covering theorem at least for some infinite dimensional Gaussian measures. However, Theorem 1 below makes clear that it is not the case, and that the Preiss' example [4] was not accidental from this point of view.

Before stating Theorem 1 we recall the concept of Vitali system.

Definition. *Let $A \subset X$ be a subset of a metric space X. A family*

$$\mathcal{V} \subset \{B[x,r] \mid x \in A,\ r > 0\}$$

is called the Vitali system on A if for every $x \in A$ and for every $\varepsilon > 0$ the system $\mathcal{V}$ contains a ball $B[x,r]$ with $r \le \varepsilon$.

Theorem 1. *Let H be a separable Hilbert space and let γ be a Gaussian measure with* $\dim \operatorname{spt}\gamma = \infty$*. Then for every $\varepsilon > 0$ there exists a Vitali system $\mathcal{V}$ on* $\operatorname{spt}\gamma$ *such that any disjoint subfamily $\mathcal{S} \subset \mathcal{V}$ satisfies*

$$\gamma\Big(\bigcup \mathcal{S}\Big) \leq \varepsilon, \quad \textit{i.e.} \quad \gamma\Big(\operatorname{spt}\gamma \setminus \bigcup \mathcal{S}\Big) \geq 1 - \varepsilon.$$

The Theorem 1 is an easy consequence of the following Proposition 1, which is formulated for more general measures than the Gaussian ones. We make some comments on the other consequences of the Proposition 1 at the end of this section. First, however, we shall introduce some notions and notations.

Symbol $\operatorname{spt}\mu$ will denote the support of a measure μ. The projection μ_U of the measure μ onto a closed subspace U of the Hilbert space H is defined by the formula

$$\mu_U A = \mu\, \pi_U^{-1}(A),$$

where $\pi_U \colon H \to U$ denotes the projection and $A \subset U$ is any Borel set in U. If $U \subset H$ is a finite dimensional subspace, then we shall denote by $\mathscr{L}_U$ the corresponding $\dim U$- dimensional Lebesgue measure.

We shall also mention some basic facts concerning Gaussian measures.

Definition. *A probability measure ν on the real line $\mathbb{R}$ is called a Gaussian measure, if either ν is the Dirac measure supported at* 0*, or it has the Radon-Nikodým derivative with respect to the Lebesgue measure of the form*

$$\frac{d\nu}{d\mathscr{L}_1} = \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{x^2}{2\sigma^2}\Big)$$

for some $\sigma > 0$. A Borel probability γ on a separable Hilbert space H is called a Gaussian measure, if every projection of γ onto one-dimensional subspace is a Gaussian measure.

We consider the Dirac measure to be a Gaussian measure only for convenience. It enables to include among the Gaussian measures also the measures which are supported by a proper subspace of the Hilbert space H.

Let γ be a Gaussian measure on H. The covariance operator $S_\gamma \colon H \to H$ is defined by

$$\langle S_\gamma x, y \rangle = \int_H \langle x, h \rangle \, \langle y, h \rangle \, d\gamma(h), \quad x, y \in H.$$

The operator S_γ is always non-negative ($\langle S_\gamma x, x\rangle \geq 0$), self-adjoint and nuclear, see e.g. [3]. If $\mathrm{spt}\gamma = H$, the covariance operator is even positive definite. In that case the eigenvectors of S_γ form an orthonormal basis (e_n) of H with the following nice property: If γ_n is the projection of γ onto the line spanned by e_n, then

$$\gamma = \prod_n \gamma_n. \tag{2}$$

Such representation of γ as a countable product will be useful.

Definition. *Let $r > 0$. The symbol $\mathfrak{B}(r)$ denotes the set of all disjoint families of closed balls in H of radius $r > 0$,*

$$\mathfrak{B}(r) = \{\mathfrak{B} \mid \mathfrak{B} \textit{ is a disjoint family of balls of radius } r\}.$$

Proposition 1. *Let H be a separable Hilbert space and let μ be a finite Borel measure on H with the following property: For every $n \in \mathbb{N}$ there is a finite dimensional subspace $U \subset H$ such that*

(i) $\dim U \geq n$,

(ii) *μ_U is absolutely continuous with respect to the Lebesgue measure $\mathscr{L}_U$ on U,*

(iii) $\mu \leq \mu_U \times \mu_{U^\perp}$.

Then

$$\lim_{r\to 0} \sup\Big\{\mu \bigcup \mathfrak{B} \;\Big|\; \mathfrak{B} \in \mathfrak{B}(r)\Big\} = 0.$$

Proof of Theorem 1. Without loss of generality we may obviously assume that $\mathrm{spt}\gamma = H$. If we recall the representation (2) of a Gaussian measure as a countable product of one-dimensional Gaussian measures, then we see that the conditions (i) – (iii) of Proposition are satisfied. Indeed, let (e_n) be the orthonormal basis of H consisting of the eigenvectors of the covariance operator S_γ. Then for any $n \in \mathbb{N}$ we put $U = \mathrm{span}\{e_1, \dots, e_n\}$. The conditions (i) and (ii) are obviously true and in the condition (iii) we even obtain equality.

Let $\varepsilon > 0$ be given. By Proposition 1, there is a decreasing sequence of numbers $r_k \searrow 0$ such that

$$\sup\Big\{\gamma \bigcup \mathfrak{B} \mid \mathfrak{B} \in \mathfrak{B}(r_k)\Big\} \leq \frac{\varepsilon}{2^k}. \tag{3}$$

We define the following Vitali system

$$\mathcal{V} = \{B[x, r_k] \mid x \in H,\ k \in \mathbb{N}\}.$$

Let $\mathcal{S} \subset \mathcal{V}$ be any disjoint subfamily. Then

$$\mathcal{S} = \bigcup_{k \in \mathbb{N}} \mathcal{S}_k, \quad \mathcal{S}_k = \{B \in \mathcal{S} \mid \text{radius}\,(B) = r_k\}.$$

Now, by using (3),

$$\gamma \bigcup \mathcal{S} = \sum_{k=1}^{\infty} \gamma \bigcup \mathcal{S}_k \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

$\square$

Remark. Note that the finite dimensional subspaces $U \subset H$ from the Proposition 1 need not be nested. Also, if we choose for any $n \in \mathbb{N}$ the corresponding subspace U_n with the properties (i)–(iii), then the linear span of $\{U_n \mid n \in \mathbb{N}\}$ need not be dense in H. Hence the conclusion of Proposition 1, and consequently non-validity of Vitali Covering theorem can be obtained e.g. for the following type of measures: Let

$$H = H_0 \oplus H_0^{\perp}$$

be an orthogonal decomposition of H such that $\dim H_0 = \infty$. Let (μ_n) be any sequence of absolutely continuous probability measures on $\mathbb{R}$. We consider the measure

$$\mu = \prod_n \mu_n$$

on the space $\mathbb{R}^{\mathbb{N}}$. Since $H_0 \approx \ell^2 \subset \mathbb{R}^{\mathbb{N}}$, by the $0-1$ law there are only two possibilities: either $\mu H_0 = 0$ or $\mu H_0 = 1$. Assume the latter. In that case for arbitrary finite measure ν on $H_0^{\perp}$ the product $\mu \times \nu$ on H is an example of a measure satisfying the assumptions of Proposition 1.

2. Lemmata

Let $U \subset H$ be a closed subspace of the Hilbert space H and let $\mathfrak{B}$ be a family of disjoint closed balls in H of radius r, $\mathfrak{B} \in \mathfrak{B}(r)$. We denote by $\mathfrak{B}_U$ the family

$$\mathfrak{B}_U = \{U \cap B \mid B \in \mathfrak{B}\}.$$

Obviously, $\mathfrak{B}_U$ is a disjoint family of closed balls in U of the radii at most r.

The first Lemma establishes one simple geometrical relationship among the balls in $\mathfrak{B}_U$.

Lemma 1. *Let $U \subset H$ be a subspace of a separable Hilbert space H and let $\mathfrak{B} \in \mathfrak{B}(1)$. Let $B[u_1, r_1]$ and $B[u_2, r_2]$ be two different balls from $\mathfrak{B}_U$. If either $2r_1 \le r_2$ or $2r_2 \le r_1$ then*

$$\|u_1 - u_2\| \ge r_1 + r_2 + \frac{1}{2}(\sqrt{10} - 3)\max\{r_1, r_2\}.$$

Proof. Since the balls $B[u_1, r_1]$ and $B[u_2, r_2]$ belong to $\mathfrak{B}_U$, there are two unit balls $B[x_1, 1]$ and $B[x_2, 1] \in \mathfrak{B}$ such that

$$B[u_1, r_1] = U \cap B[x_1, 1] \quad \text{and} \quad B[u_2, r_2] = U \cap B[x_2, 1].$$

Also, since $H = U \oplus U^\perp$, one has

$$x_1 = u_1 + v_1 \quad \text{and} \quad x_2 = u_2 + v_2,$$

where u_1, $u_2 \in U$ and v_1, $v_2 \in U^\perp$. By the disjointness of the balls in $\mathfrak{B}$ it is readily seen that

$$\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2 = \|x_1 - x_2\|^2 \ge 4. \tag{4}$$

Note that $r_1^2 = 1 - \|v_1\|^2$ and $r_2^2 = 1 - \|v_2\|^2$. Using this in the estimate (4) we obtain

$$\begin{aligned} \|u_1 - u_2\|^2 &\ge 2 + r_1^2 + r_2^2 + \|v_1\|^2 + \|v_2\|^2 - \|v_1 - v_2\|^2 \\ &= 2 + r_1^2 + r_2^2 + 2\langle v_1, v_2\rangle \ge 2 + r_1^2 + r_2^2 - 2\|v_1\|\,\|v_2\| \\ &= 2 + r_1^2 + r_2^2 - 2\sqrt{(1 - r_1^2)(1 - r_2^2)}. \end{aligned} \tag{5}$$

Without loss of generality we may assume that $r_2 \le r_1$. Then the assumption in Lemma implies that even $2r_2 \le r_1$. Let $\delta = \frac{1}{2}(\sqrt{10} - 3)$. In order to prove

$$\|u_1 - u_2\| \ge r_1(1 + \delta) + r_2$$

we are going to show that

$$\|u_1 - u_2\|^2 - \big(r_1(1 + \delta) + r_2\big)^2 \ge 0.$$

To this end, we use the estimate (5):

$$
\begin{aligned}
\|u_1-u_2\|^2 &- \big(r_1(1+\delta)+r_2\big)^2 \\
&\geq 2+r_1^2+r_2^2-2\sqrt{(1-r_1^2)(1-r_2^2)}-\big(r_1(1+\delta)+r_2\big)^2 \\
&= 2-2\sqrt{(1-r_1^2)(1-r_2^2)}-r_1^2\big((1+\delta)^2-1\big)-2r_1r_2(1+\delta) \\
&= g(r_1,r_2).
\end{aligned}
$$

We shall have to find the minimal value of the function $g(r_1,r_2)$ on the set $\{(r_1,r_2)\mid 0\leq 2r_2\leq r_1\leq 1\}$. Some elementary calculation reveals that the function $r_2\mapsto g(r_1,r_2)$ is nonincreasing on $[0,\frac{1}{2}r_1]$. One more calculation gives that the function $r_1\mapsto g(r_1,\frac{1}{2}r_1)$ is nondecreasing on $[0,1]$ provided

$$(1+\delta)^2+(1+\delta)-1\leq\frac{5}{4}.$$

This condition is guaranteed by our choice of δ. Hence the minimal value of $g(r_1,r_2)$ is attained at the point $(0,0)$ and is equal to 0. This completes the proof. $\square$

The next Lemma estimates the Lebesgue measure of the intersection of two balls in a special position. The symbol $\alpha(n)$ denotes the volume of the unit Euclidean ball in $\mathbb{R}^n$,

$$\alpha(n)=\mathscr{L}_nB[0,1]=\frac{\pi^{n/2}}{\Gamma(1+n/2)}.$$

Lemma 2. *There is $\Delta_0>0$ such that for any $x\in\mathbb{R}^n$ with $\|x\|=3$ and $0<\delta\leq\Delta_0$ we have the following estimate*

$$\mathscr{L}_n\Big(B[0,1+\delta]\cap B[x,2(1+3\delta)]\Big)\leq\alpha(n-1)\ 10^{\frac{n+1}{2}}\ (1+\delta)^n\ \delta^{\frac{n+1}{2}}.$$

Proof. Let $x=(3,0,\dots,0)\in\mathbb{R}^n$. If we write a point $z\in\mathbb{R}^n$ in the form $z=(z_1,z_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$, then the following equations determine the intersection of the spheres $\{y\mid\|y\|=1+\delta\}\cap\{y\mid\|y-x\|=2(1+3\delta)\}$:

$$
\begin{aligned}
z_1^2+\|z_2\|^2&=(1+\delta)^2\\
(z_1-3)^2+\|z_2\|^2&=4(1+3\delta)^2.
\end{aligned}
$$

Eliminating $\|z_2\|$, we get $z_1 = \frac{1}{6}(9 + (1+\delta)^2 - 4(1+3\delta)^2)$. Then a simple geometrical observation reveals that

$$\begin{aligned}\mathscr{L}_n\Big(B[0, 1+\delta] \cap B[x, 2(1+\delta)]\Big)& \\ &\leq 2\ \alpha(n-1) \int_{z_1}^{1+\delta} \Big((1+\delta)^2 - t^2\Big)^{\frac{n-1}{2}} dt \\ &= 2\ \alpha(n-1)\ (1+\delta)^n \int_\theta^1 (1-u^2)^{\frac{n-1}{2}} du,\end{aligned}$$

where $\theta = \frac{z_1}{(1+\delta)}$. It is clear that $\theta > 0$ for δ small enough. The explicit condition for δ is $\delta \leq \frac{\sqrt{330}-11}{35}$. We estimate the function $(1-u^2)^{\frac{n-1}{2}}$ by its maximal value on the interval $[\theta, 1]$ and we obtain

$$\begin{aligned}&\leq 2\ \alpha(n-1)\ (1+\delta)^n (1-\theta^2)^{\frac{n-1}{2}} (1-\theta) \\ &\leq 2^{\frac{n+1}{2}}\ \alpha(n-1)\ (1+\delta)^n (1-\theta)^{\frac{n+1}{2}}.\end{aligned}$$

Since a short calculation gives that $1 - \theta \leq 5\delta$ again for small δ ($\delta \leq 2/35$), we get the desired estimate. Finally, we finish the proof by putting $\Delta_0 = \min\{\frac{\sqrt{330}-11}{35}, \frac{2}{35}\} = \frac{2}{35}$. ☐

We introduce the following notation. Let $B = B[x,r]$ be a ball. The symbol

$$(1+\delta)B = B[x, (1+\delta)r]$$

denotes the enlarged ball with the *same* center and $(1+\delta)$ times bigger radius. We shall be using both notations $(1+\delta)B$ and $B[x, (1+\delta)r]$.

The next Lemma contains the key estimate needed in proof of Proposition 1.

Lemma 3. *There is a number* $\delta_0 > 0$ *such that for every* $r > 0$*, every family* $\mathfrak{B} \in \mathfrak{B}(r)$ *of disjoint balls of radius* r*, and every finite dimensional subspace* $U \subset H$ *the following estimate holds*

$$\mathscr{L}_U\Big((1+\delta)B_0 \setminus \bigcup\{(1+\delta)B \mid B \in \mathfrak{B}_U,\ B \neq B_0\}\Big) \geq \frac{1}{2}(1+\delta)^{\dim U} \mathscr{L}_U B_0$$

provided $0 < \delta \leq \delta_0$ *and* $B_0 \in \mathfrak{B}_U$*.*

Proof. Let $B_0 \in \mathfrak{B}_U$ be fixed. Without loss of generality we assume that B_0 has center at the origin, $B_0 = B[0, r_0]$, say. Let $\delta > 0$ be such that $2\delta < \frac{1}{2}(\sqrt{10} - 3)$. Then, by Lemma 1, we see that the ball $(1+\delta)B_0$ is disjoint with

$$\bigcup \{B[x, (1+\delta)r_x] \mid B[x, r_x] \in \mathfrak{B}_U,\ 2r_x \le r_0 \ \text{ or }\ 2r_0 \le r_x\}.$$

Accordingly, the only relevant balls in $\mathfrak{B}_U$ which may interfere with the $(1+\delta)B_0$ are those of radii comparable to r_0. We denote the centres of such balls by

$$C = \Big\{x \in U \setminus \{0\} \mid B[x, r_x] \in \mathfrak{B}_U,\ (1+\delta)B_0 \cap B[x, (1+\delta)r_x] \neq \emptyset\Big\}.$$

Note that for the ball $B[x, r_x] \in \mathfrak{B}_U$ with $x \in C$ we have $\frac{1}{2}r_0 \le r_x \le 2r_0$.

We have to estimate the measure of $(1+\delta)B_0 \cap \bigcup_{x\in C} B[x, (1+\delta)r_x]$. Since

$$\mathscr{L}_U\Big((1+\delta)B_0 \cap \bigcup_{x\in C} B[x, (1+\delta)r_x]\Big) \le \sum_{x\in C} \mathscr{L}_U\Big((1+\delta)B_0 \cap B[x, (1+\delta)r_x]\Big),$$

we shall look closer at each intersection $(1+\delta)B_0 \cap B[x, (1+\delta)r_x]$.

Let $x \in C$. First note that

$$r_0 + r_x \le \|x\| \le (1+\delta)(r_0 + r_x).$$

Also, $r_x \le 2r_0$. We show that

$$B[x, (1+\delta)r_x] \subset B\Big[3r_0\frac{x}{\|x\|}, 2(1+3\delta)r_0\Big]. \tag{6}$$

To see this, let $y \in B[x, (1+\delta)r_x]$, i.e. $\|y - x\| \le (1+\delta)r_x$. Then

$$\begin{aligned}\Big\|y - 3r_0\frac{x}{\|x\|}\Big\| &= \Big\|y - x + x\Big(1 - \frac{3r_0}{\|x\|}\Big)\Big\| \\ &\le \|y - x\| + \|x\|\Big|1 - \frac{3r_0}{\|x\|}\Big| \\ &\le (1+\delta)r_x + \big|\|x\| - 3r_0\big|.\end{aligned}$$

If $3r_0 \geq \|x\|$, then the calculation finishes in

$$\begin{aligned}&\leq (1+\delta)r_x + 3r_0 - (r_0 + r_x) = 2r_0 + \delta r_x \\ &\leq 2(1+\delta)r_0 < 2(1+3\delta)r_0.\end{aligned}$$

If, on the other hand, $3r_0 \leq \|x\|$, then we proceed as

$$\begin{aligned}&\leq (1+\delta)r_x + (1+\delta)(r_0 + r_x) - 3r_0 \\ &\leq 5(1+\delta)r_0 - 3r_0 < 2(1+3\delta)r_0.\end{aligned}$$

It follows immediately from (6)

$$(7) \qquad (1+\delta)B_0 \cap B[x,(1+\delta)r_x] \subset (1+\delta)B_0 \cap B\Big[3r_0\frac{x}{\|x\|}, 2(1+3\delta)r_0\Big].$$

Now let $n = \dim U$ for short. If, moreover, $\delta \leq \Delta_0$ from Lemma 2 we obtain the estimate of the intersection on the right hand side of (7):

$$\begin{aligned}&\mathscr{L}_n\Big((1+\delta)B_0 \cap B[x,(1+\delta)r_x]\Big) \\ &\qquad \leq \mathscr{L}_n\Big(B[0,(1+\delta)r_0] \cap B\Big[3r_0\frac{x}{\|x\|}, 2(1+3\delta)r_0\Big]\Big) \\ &\qquad = r_0^n \mathscr{L}_n\Big(B[0,(1+\delta)] \cap B\Big[3\frac{x}{\|x\|}, 2(1+3\delta)\Big]\Big) \\ &\qquad \leq r_0^n\ \alpha(n-1)\ 10^{\frac{n+1}{2}}\ (1+\delta)^n \delta^{\frac{n+1}{2}} \\ &\qquad = \frac{\alpha(n-1)}{\alpha(n)}\ 10^{\frac{n+1}{2}}\ (1+\delta)^n\ \delta^{\frac{n+1}{2}} \mathscr{L}_n B[0,r_0].\end{aligned}$$

Hence

$$\begin{aligned}(8)\quad &\mathscr{L}_n\Big((1+\delta)B_0 \cap \bigcup_{x\in C} B[x,(1+\delta)r_x]\Big) \\ &\qquad \leq \frac{\alpha(n-1)}{\alpha(n)}\ 10^{\frac{n+1}{2}}\ (1+\delta)^n\ \delta^{\frac{n+1}{2}}\ \mathscr{L}_n B[0,r_0] \cdot \operatorname{card} C.\end{aligned}$$

What is missing now is some control over the cardinality of the set C. Fortunately, for our purpose we shall not need any hard estimate. The sufficient upper bound for $\operatorname{card} C$ follows from the comparison of certain volumes. To this end, recall that for all $x \in C$

$$\|x\| \leq (1+\delta)(r_x + r_0) \quad \text{and} \quad \frac{1}{2}r_0 \leq r_x \leq 2r_0.$$

Hence

$$\bigcup_{x\in C} B[x, r_x] \subset B[0, (5+3\delta)r_0]. \tag{9}$$

Also, $B[x, r_x] \supset B\big[x, \frac{1}{2}r_0\big]$ for $x \in C$. Combining it with (9) we get

$$\mathscr{L}_n B\Big[x, \frac{1}{2}r_0\Big] \operatorname{card} C \le \mathscr{L}_n B[0, (5+3\delta)r_0].$$

Thus

$$\operatorname{card} C \le 10^n \Big(1 + \frac{3}{5}\delta\Big)^n \le 10^n (1+\delta)^n.$$

Using this estimate in (8) we have

$$\begin{aligned}
&\mathscr{L}_n\Big((1+\delta)B_0 \cap \bigcup_{x\in C} B[x, (1+\delta)r_x]\Big)\\
&\quad\le \frac{\alpha(n-1)}{\alpha(n)}\ 10^{\frac{n+1}{2}}\ (1+\delta)^n\ \delta^{\frac{n+1}{2}}\ \mathscr{L}_n B[0, r_0]\ 10^n (1+\delta)^n\\
&\quad= \frac{\alpha(n-1)}{\alpha(n)}\ 10^{\frac{3n+1}{2}}\ (1+\delta)^{2n}\ \delta^{\frac{n+1}{2}}\ \mathscr{L}_n B[0, r_0].
\end{aligned}$$

Since $\frac{\alpha(n-1)}{\alpha(n)} \approx \sqrt{n}$ for $n \to \infty$, there is $\delta_1 > 0$ such that

$$\frac{\alpha(n-1)}{\alpha(n)}\ 10^{\frac{3n+1}{2}}\ (1+\delta)^n\ \delta^{\frac{n+1}{2}} \le \frac{1}{2}$$

for all $n \in \mathbb{N}$ and $0 < \delta \le \delta_1$. With this choice of δ one has

$$\mathscr{L}_n\Big((1+\delta)B_0 \cap \bigcup_{x\in C} B[x, (1+\delta)r_x]\Big) \le \frac{1}{2}(1+\delta)^n \mathscr{L}_n B_0. \tag{10}$$

To complete the proof, we put $\delta_0 = \min\{\Delta_0, \delta_1, \frac{1}{4}(\sqrt{10}-3)\}$. If now $0 < \delta \le \delta_0$, then by (10)

$$\begin{aligned}
&\mathscr{L}_n\Big((1+\delta)B_0 \setminus \bigcup\{(1+\delta)B \mid B \in \mathfrak{B}_U,\ B \ne B_0\}\Big)\\
&\quad= \mathscr{L}_n(1+\delta)B_0 - \mathscr{L}_n\Big((1+\delta)B_0 \cap \bigcup_{x\in C} B[x, (1+\delta)r_x]\Big)\\
&\quad\ge (1+\delta)^n \mathscr{L}_n B_0 - \frac{1}{2}(1+\delta)^n \mathscr{L}_n B_0 = \frac{1}{2}(1+\delta)^n \mathscr{L}_n B_0
\end{aligned}$$

and the proof is finished. $\square$

We associate with every $B_0 \in \mathfrak{B}_U$ the set

$$D_{B_0} = (1+\delta)B_0 \setminus \Big(B_0 \cup \bigcup \{(1+\delta)B \mid B \in \mathfrak{B}_U,\ B \neq B_0\}\Big).$$

Then, obviously, $\{D_B \mid B \in \mathfrak{B}_U\}$ is the disjoint system of subsets in U. One consequence of Lemma 3 is the following estimate of the measure of D_B.

Corollary 1. *Let $\delta_0 > 0$ be as in Lemma 3 and $U \subset H$ a finite dimensional subspace. Then*

$$\mathscr{L}_U D_{B_0} \geq \Big(\frac{1}{2}(1+\delta)^{\dim U} - 1\Big)\ \mathscr{L}_U B_0$$

for every $0 < \delta \leq \delta_0$ and every $B_0 \in \mathfrak{B}_U$.

Proof. Since

$$D_{B_0} \cup B_0 = (1+\delta)B_0 \setminus \bigcup \{(1+\delta)B \mid B \in \mathfrak{B},\ B \neq B_0\},$$

we obtain by using Lemma 3,

$$\mathscr{L}_U(D_{B_0} \cup B_0) \geq \frac{1}{2}(1+\delta)^{\dim U}\ \mathscr{L}_U B_0.$$

The sets D_{B_0} and B_0 are disjoint, so $\mathscr{L}_U(D_{B_0} \cup B_0) = \mathscr{L}_U D_{B_0} + \mathscr{L}_U B_0$, and the statement follows by rearrangement. $\square$

Now we shall estimate the so-called packing density of the family $\mathfrak{B}_U$ in U. Since U is a finite dimensional subspace of H, we identify it with $\mathbb{R}^n$, $n = \dim U$. We put

$$Q_k = [-k,k]^n,$$

the n-dimensional cube in U of the side $2k$. With this notation we can state the following

Lemma 4. *There is $\delta_0 > 0$ such that for every finite dimensional subspace $U \cong \mathbb{R}^n$ and every $r > 0$*

$$\limsup_{k\to\infty}\ \sup\Big\{\frac{\mathscr{L}_n(Q_k \cap \bigcup \mathfrak{B}_U)}{\mathscr{L}_n Q_k}\ \Big|\ \mathfrak{B} \in \mathfrak{B}(r)\Big\} \leq \frac{1}{\frac{1}{2}(1+\delta)^n - 1}$$

for any $0 < \delta \leq \delta_0$ and $n \in \mathbb{N}$ with $\frac{1}{2}(1+\delta)^n - 1 > 0$.

Proof. Let $\mathfrak{B} \in \mathfrak{B}(r)$ be arbitrary and let $\delta_0 > 0$ be as in the Lemma 3. We denote by $\mathcal{R}$ the family of all balls in $\mathfrak{B}_U$ such that $(1+\delta)$ enlargement of B is still contained in the cube Q_k,

$$\mathcal{R} = \{B \in \mathfrak{B}_U \mid (1+\delta)B \subset Q_k\}.$$

Then

$$\begin{aligned} \mathscr{L}_n(Q_k \cap \bigcup \mathfrak{B}_U) &= \sum_{B \in \mathfrak{B}_U} \mathscr{L}_n(Q_k \cap B) \\ &\leq \sum_{B \in \mathcal{R}} \mathscr{L}_n B + \mathscr{L}_n\big(Q_k \setminus Q_{k-2r(1+\delta)}\big) \qquad (11) \\ &= \sum_{B \in \mathcal{R}} \mathscr{L}_n B + \mathscr{L}_n Q_k \Big[1 - \Big(1 - \frac{2r(1+\delta)}{k}\Big)^n\Big] \end{aligned}$$

provided $k > 2r(1+\delta)$. By Corollary 1,

$$\mathscr{L}_n B \leq \frac{1}{\frac{1}{2}(1+\delta)^n - 1} \mathscr{L}_n D_B \qquad (12)$$

for n with $\frac{1}{2}(1+\delta)^n - 1 > 0$. Also, $D_B \subset Q_k$ for any $B \in \mathcal{R}$. Since the sets D_B are disjoint for different B's we may sum up the estimates in (12) to get

$$\sum_{B \in \mathcal{R}} \mathscr{L}_n B \leq \frac{1}{\frac{1}{2}(1+\delta)^n - 1} \sum_{B \in \mathcal{R}} \mathscr{L}_n D_B \leq \frac{1}{\frac{1}{2}(1+\delta)^n - 1} \mathscr{L}_n Q_k. \qquad (13)$$

Looking back to (11) one has

$$\mathscr{L}_n(Q_k \cap \bigcup \mathfrak{B}_U) \leq \frac{1}{\frac{1}{2}(1+\delta)^n - 1} \mathscr{L}_n Q_k + \mathscr{L}_n Q_k \Big[1 - \Big(1 - \frac{2r(1+\delta)}{k}\Big)^n\Big].$$

Since the expression on the right hand side does not depend on $\mathfrak{B}$ the same estimate holds true also for the supremum over all $\mathfrak{B} \in \mathfrak{B}(r)$. Hence

$$\begin{aligned} &\limsup_{k \to \infty} \sup \Big\{ \frac{\mathscr{L}_n(Q_k \cap \bigcup \mathfrak{B}_U)}{\mathscr{L}_n Q_k} \;\Big|\; \mathfrak{B} \in \mathfrak{B}(r) \Big\} \\ &\qquad \leq \frac{1}{\frac{1}{2}(1+\delta)^n - 1} + \limsup_{k \to \infty} \Big[1 - \Big(1 - \frac{2r(1+\delta)}{k}\Big)^n\Big] \\ &\qquad = \frac{1}{\frac{1}{2}(1+\delta)^n - 1} \end{aligned}$$

and the lemma is proved. □

The straightforward reformulation of the statement of Lemma 4 is the following:

For any cube $Q \subset U \cong \mathbb{R}^n$

$$\limsup_{r\to 0} \sup\Big\{ \frac{\mathscr{L}_n(Q \cap \bigcup \mathfrak{B}_U)}{\mathscr{L}_n Q} \;\Big|\; \mathfrak{B} \in \mathfrak{B}(r) \Big\} \le \frac{1}{\frac{1}{2}(1+\delta)^n - 1} \tag{14}$$

for any $0 < \delta \le \delta_0$ and all $n \in \mathbb{N}$ sufficiently big.

Till now we have used only Lebesgue measure. The next (easy) lemma allows to get the estimates for any other measure absolutely continuous with respect to the Lebesgue measure.

Lemma 5. *Let $f \in L^1(\mathbb{R}^n)$ and let (K_r), $r > 0$ be a system of measurable sets $K_r \subset \mathbb{R}^n$ satisfying the following condition:*

There is $\sigma > 0$ such that for every cube $Q \subset \mathbb{R}^n$

$$\limsup_{r\to 0} \frac{\mathscr{L}_n(Q \cap K_r)}{\mathscr{L}_n Q} \le \sigma.$$

Then

$$\limsup_{r\to 0} \int_{K_r} f \, d\mathscr{L}_n \le \sigma \, \|f\|_{L^1}.$$

Proof. Let $\varepsilon > 0$. There is a continuous function $g\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ with the compact support such that $\|f - g\|_{L^1} \le \varepsilon$. Further, by the uniform continuity of g there is $\delta > 0$ such that

$$|g(x) - g(y)| \le \varepsilon$$

for any $x, y \in \mathbb{R}^n$ satisfying $\|x - y\| \le \delta$.

Let $Q \subset \mathbb{R}^n$ be a cube containing the support of g. We partition the cube Q into finite family $\mathcal{P}$ of subcubes of diameter at most δ and then we choose in each $P \in \mathcal{P}$ a point $x_P \in P$, for example the centre. Now

$$
\begin{aligned}
\int_{K_r} f \, d\mathscr{L}_n &\leq \|f - g\|_{L^1} + \int_{K_r} g \, d\mathscr{L}_n \leq \varepsilon + \sum_{P \in \mathcal{P}} \int_{P \cap K_r} g \, d\mathscr{L}_n \\
&\leq \varepsilon + \sum_{P \in \mathcal{P}} \int_{P \cap K_r} (g - g(x_P)) \, d\mathscr{L}_n + \sum_{P \in \mathcal{P}} g(x_P) \, \mathscr{L}_n(P \cap K_r) \\
&\leq \varepsilon + \varepsilon \sum_{P \in \mathcal{P}} \mathscr{L}_n(P \cap K_r) + \sum_{P \in \mathcal{P}} g(x_P) \, \mathscr{L}_n(P \cap K_r) \\
&\leq \varepsilon + \varepsilon \mathscr{L}_n Q + \sum_{P \in \mathcal{P}} g(x_P) \, \mathscr{L}_n(P \cap K_r). \qquad (15)
\end{aligned}
$$

By the assumption we can choose $r > 0$ small enough to guarantee that

$$
\mathscr{L}_n(P \cap K_r) \leq (\sigma + \varepsilon) \, \mathscr{L}_n P
$$

for all $P \in \mathcal{P}$. Then the last sum in (15) can be estimated by

$$
\begin{aligned}
\sum_{P \in \mathcal{P}} g(x_P) \, \mathscr{L}_n(P \cap K_r) &\leq (\sigma + \varepsilon) \sum_{P \in \mathcal{P}} g(x_P) \, \mathscr{L}_n P \\
&\leq (\sigma + \varepsilon) \Big(\int_Q g \, d\mathscr{L}_n + \sum_{P \in \mathcal{P}} \int_P (g(x_P) - g) \, d\mathscr{L}_n \Big) \\
&\leq (\sigma + \varepsilon) \Big(\int_Q g \, d\mathscr{L}_n + \varepsilon \mathscr{L}_n Q \Big) \\
&\leq (\sigma + \varepsilon) \Big(\|f - g\|_{L^1} + \|f\|_{L^1} + \varepsilon \mathscr{L}_n Q \Big) \\
&\leq (\sigma + \varepsilon) \Big(\varepsilon + \|f\|_{L^1} + \varepsilon \mathscr{L}_n Q \Big)
\end{aligned}
$$

Combining this estimate with the (15) we obtain that

$$
\limsup_{r \to 0} \int_{K_r} f \, d\mathscr{L}_n \leq \varepsilon + \varepsilon \mathscr{L}_n Q + (\sigma + \varepsilon) \Big(\varepsilon + \|f\|_{L^1} + \varepsilon \mathscr{L}_n Q \Big).
$$

Since $\varepsilon > 0$ is arbitrarily small we conclude that

$$
\limsup_{r \to 0} \int_{K_r} f \, d\mathscr{L}_n \leq \sigma \|f\|_{L^1},
$$

which completes the proof. $\square$

Proof of Proposition 1. Let $\varepsilon > 0$ be given. We choose $\delta \in (0, \delta_0]$ such that the conclusion of Lemma 4 holds true. Also, let $n \in \mathbb{N}$ be large enough to satisfy

$$\frac{1}{\frac{1}{2}(1+\delta)^n - 1} \leq \frac{\varepsilon}{3}. \tag{16}$$

By assumption, there is an finite dimensional space U, $\dim U \geq n$ such that μ_U is absolutely continuous with respect to the Lebesgue measure $\mathscr{L}_U$. We denote

$$f = \frac{d\mu_U}{d\mathscr{L}_U}.$$

For every $r > 0$ there is a $\mathfrak{B}^{(r)} \in \mathfrak{B}(r)$ such that

$$\mu_U \bigcup \mathfrak{B}_U^{(r)} \geq \frac{1}{2} \sup\Big\{\mu_U \bigcup \mathfrak{B}_U \mid \mathfrak{B} \in \mathfrak{B}(r)\Big\}. \tag{17}$$

We put $K_r = \bigcup \mathfrak{B}_U^{(r)}$, $r > 0$. For this choice of K_r the assumption of Lemma 5 is satisfied: Let $Q \subset U$ be a cube. Then by Lemma 4 in the form (14) and by (16) we have

$$\limsup_{r\to 0} \frac{\mathscr{L}_U(Q \cap \bigcup \mathfrak{B}_U^{(r)})}{\mathscr{L}_U Q} \leq \limsup_{r\to 0} \sup_{\mathfrak{B}\in\mathfrak{B}(r)} \frac{\mathscr{L}_U(Q \cap \bigcup \mathfrak{B}_U)}{\mathscr{L}_U Q} \leq \frac{\varepsilon}{3}.$$

So Lemma 5 implies

$$\limsup_{r\to 0} \mu_U \bigcup \mathfrak{B}_U^{(r)} = \limsup_{r\to 0} \int_{\bigcup \mathfrak{B}_U^{(r)}} f \, d\mathscr{L}_U \leq \frac{\varepsilon}{3}.$$

In combination with (17) we get

$$\limsup_{r\to 0} \sup\Big\{\mu_U \bigcup \mathfrak{B}_U \mid \mathfrak{B} \in \mathfrak{B}(r)\Big\} \leq \frac{2\varepsilon}{3}.$$

It follows that there is $r_0 > 0$ such that for all $0 < r \leq r_0$ we have

$$\sup\Big\{\mu_U \bigcup \mathfrak{B}_U \mid \mathfrak{B} \in \mathfrak{B}(r)\Big\} \leq \varepsilon. \tag{18}$$

Now we are ready to estimate the measure $\mu \bigcup \mathfrak{B}$ for any $\mathfrak{B} \in \mathfrak{B}(r)$, and $0 < r \leq r_0$. By the condition (iii) in Proposition 1 and (18)

$$\begin{aligned}
\mu \bigcup \mathfrak{B} \leq (\mu_U \times \mu_{U^\perp}) \bigcup \mathfrak{B} &= \int_{U^\perp} \mu_U\Big(U \cap \big(x + \bigcup \mathfrak{B}\big)\Big) \, d\mu_{U^\perp}(x) \\
&\leq \int_{U^\perp} \sup\Big\{\mu_U \bigcup \mathfrak{B}_U \mid \mathfrak{B} \in \mathfrak{B}(r)\Big\} \, d\mu_{U^\perp} \leq \int_{U^\perp} \varepsilon \, d\mu_{U^\perp} = \varepsilon.
\end{aligned}$$

Since this is true for all $\mathfrak{B} \in \mathfrak{B}(r)$ we may conclude

$$\sup\Big\{\mu\bigcup\mathfrak{B} \mid \mathfrak{B} \in \mathfrak{B}(r)\Big\} \leq \varepsilon,$$

provided $0 < r \leq r_0$ and the Proposition 1 is proved. □

It may be of some interest to make the following final remark. Although the classical version of Vitali Covering theorem fails for e.g. all infinite dimensional Gaussian measures there is still a weaker statement of the covering type which holds true. The validity of Differentiation theorem is in fact equivalent to such weak covering theorem. For details see e.g. Hayes and Pauc [2], or the deep paper of M. Talagrand [7], where this connection is treated in considerable generality.

Based on the already mentioned positive differentiation result [8] for some class $\mathcal{G}$ of Gaussian measures we have the following:

Given $\gamma \in \mathcal{G}$, $1 < p < \infty$, $\varepsilon > 0$, and Vitali system $\mathcal{V}$ on a set A in a separable Hilbert space, there is a countable subsystem $\mathcal{S} \subset \mathcal{V}$ such that

(i) $\gamma(A \setminus \bigcup \mathcal{S}) = 0$,

(ii) $\Big\| \sum_{B\in\mathcal{S}} \chi_B - \chi_{\bigcup\mathcal{S}} \Big\|_{L^p(\gamma)} < \varepsilon$.

The condition (ii) means that instead of disjointness we are only able to make the overlap of sets in $\mathcal{S}$ arbitrarily small in the given L^p norm.

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