Časopis pro pěstování matematiky, **113** (1988), No. 4, 435–436

A NOTE ON DETERMINACY OF MEASURES

Pavel Pták, Josef Tkadlec, Praha

(Received October 23, 1986)

Summary. In the article it is shown that the Cramér–Wold theorem implies a stronger form of the Christensen theorem.

Keywords: determining set, probability measure.

AMS Classification: 28A05.

Let $\mathscr{B}(R^n)$ denote the collection of all Borel subsets of R^n and let $\mathscr{C}$ be a subset of $\mathscr{B}(R^n)$. Let $\mathscr{C}$ be called *determining* when the following statement holds: If μ_1, μ_2 are two probability measures on $\mathscr{B}(R^n)$ which agree on $\mathscr{C}$ then they are necessarily identical. The theorem of Christensen ([3]) says that the collection of all open balls is determining and the theorem of Cramér and Wold ([2]) says that the collection of all open half-spaces is determining. In this note we observe that the Cramér–Wold theorem implies a stronger form of the Christensen theorem. (As a by-product we obtain another proof of the Christensen theorem. For further discussion on the determinacy of measures, the reader is referred to [1], [4], [5], [6] and [7].)

Theorem. *Let p be a point in R^n $(n \in N)$ and let $\mathscr{C}$ denote the collection of all open balls having p on the boundary. Then $\mathscr{C}$ is determining.*

P r o o f. Let μ_1, μ_2 agree on $\mathscr{C}$. Applying a suitable transformation and multiple if necessary, we may assume that $p = 0 \in R^n$ and $\mu_1\{0\} = \mu_2\{0\} = 0$. Let $\mathscr{C}_1$ denote the collection of all open half-spaces which have 0 on the boundary. Put $\mathscr{D} = \mathscr{C} \cup \mathscr{C}_1$. Then μ_1, μ_2 agree on $\mathscr{D}$. Indeed, each open half-space in $\mathscr{C}_1$ can be obtained as a union of an increasing sequence of balls in $\mathscr{C}$. Hence μ_1, μ_2 have to agree on $\mathscr{C}_1$ in view of their monotone continuity.

Let now $\varphi\colon R^n \to R^n$ be a mapping such that $\varphi(0) = 0$ and $\varphi(x) = x/||x||^2$ otherwise. Then φ is obviously a Borel isomorphism. One can easily show that $\varphi(\mathscr{D})$ is exactly the collection of all open half-subspaces in R^n. By our assumption, the measures $\mu_1\varphi^{-1}, \mu_2\varphi^{-1}$ agree on $\varphi(\mathscr{D})$ and therefore $\mu_1\varphi^{-1} = \mu_2\varphi^{-1}$ (the Cramér–Wold theorem). This means that $\mu_1 = \mu_2$ and the proof is complete.

References

[1] *P. Billingsley:* Convergence of Probability Measures. Willey, New York 1968.

[2] *H. Cramér, H. Wold:* Some theorems on distribution functions. J. London Math. Soc. *11* (1936), 290–295.

[3] *J. P. R. Christensen:* On some measures analogous to Haar measure. Math. Scand. *26* (1970), 103–106.

[4] *R. O. Davies:* Can two different measures agree on balls? (Good news and bad news). Mimeographed notes, 1972.

[5] *R. O. Davies:* Measures not approximable or not specifiable by means of balls. Mathematika *18* (1971), 157–160.

[6] *T. Neubrunn:* A note on quantum probability spaces. Proc. Amer. Math. Soc. *25* (1970), 672–675.

[7] *F. H. Ruymgaart:* A note on the concept of joint distributions of pairs of observables. The Indian Journal of Statistics, Ser. A, Vol *45* (1983), 38–43.

Souhrn

POZNMKA O URENOSTI MR

Pavel Pták, Josef Tkadlec

V článku je ukázáno, že Cramérova–Woldova vta implikuje silnější verzi Christensenovy věty.

Резюме

ЗАМЕЧАНИЕ ОБ ОПРЕДЕЛЕННОСТИ МЕР

Pavel Pták, Josef Tkadlec

В работе показано, что теорема Крамэра–Волда влечет за собой более сильный вариант теоремы Христенсена.

Authors' address: Katedra matematiky FEL ČVUT, Suchbátarova 2, 166 27 Praha 6.