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Common generalizations of orthocomplete and lattice effect algebras

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Abstract Connections between the weak orthocompleteness and the maximality property in effect algebras are presented. It is proved that an orthomodular poset with the maximality property is disjunctive. A characterization of Archimedean weakly orthocomplete effect algebras is given.

Keywords Effect algebra · weakly orthocomplete · maximality property · disjunctive · Archimedean · separable

Ovchinnikov [7] introduced weakly orthocomplete orthomodular posets (he called them alternative) as a common generalization of orthocomplete orthomodular posets and orthomodular lattices and showed that they are disjunctive. Weak orthocompleteness is useful in the study of orthoatomisticity and disjunctivity might be used to characterize atomisticity [7, 11]. Weak orthocompleteness was generalized by De Simone and Navara [1].

Tkadlec [8] introduced the class of orthomodular posets with the maximality property as another common generalization of orthocomplete orthomodular posets and orthomodular lattices. He showed various consequences of this property and generalize it [8, 9, 10, 12].

We show that these two notions are incomparable, that maximality property also implies disjunctivity in orthomodular posets and present a characterization of Archimedean weakly orthocomplete effect algebras. We show also some other relations within the class of effect algebras.

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1 Basic notions and properties

Definition 1.1 An *effect algebra* is an algebraic structure $(E, \oplus, \mathbf{0}, \mathbf{1})$ such that E is a set, $\mathbf{0}$ and $\mathbf{1}$ are different elements of E and $\oplus$ is a partial binary operation on E such that for every $a, b, c \in E$ the following conditions hold:

(1) $a \oplus b = b \oplus a$ if $a \oplus b$ exists,
(2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if $(a \oplus b) \oplus c$ exists,
(3) there is a unique $a' \in E$ such that $a \oplus a' = \mathbf{1}$ (*orthosupplement*),
(4) $a = \mathbf{0}$ if $a \oplus \mathbf{1}$ is defined.

For simplicity, we use the notation E for an effect algebra. A partial ordering on an effect algebra E is defined by $a \leq b$ if there is a $c \in E$ such that $b = a \oplus c$. Such an element c is unique (if it exists) and is denoted by $b \ominus a$. $\mathbf{0}$ ($\mathbf{1}$, resp.) is the least (the greatest, resp.) element of E with respect to this partial ordering. For every $a, b \in E$, $a'' = a$ and $b' \leq a'$ whenever $a \leq b$. It can be shown that $a \oplus \mathbf{0} = a$ for every $a \in E$ and that a *cancellation law* is valid: for every $a, b, c \in E$ with $a \oplus b \leq a \oplus c$ we have $b \leq c$. An *orthogonality* relation on E is defined by $a \perp b$ if $a \oplus b$ exists. See, e.g., Dvurečenskij and Pulmannová [2], Foulis and Bennett [3].

Let E be a partially ordered set. For every $a, b \in E$ with $a \leq b$ we denote $[a, b] = \{c \in E : a \leq c \leq b\}$. A *chain* in E is a nonempty linearly (totally) ordered subset of E.

Obviously, if $a \perp b$ and $a \vee b$ exist in an effect algebra, then $a \vee b \leq a \oplus b$. The reverse inequality need not be true (it holds in orthomodular posets).

Definition 1.2 Let E be an effect algebra. An element $a \in E$ is *principal* if $b \oplus c \leq a$ for every $b, c \in E$ such that $b, c \leq a$ and $b \perp c$.

Definition 1.3 An *orthoalgebra* is an effect algebra E in which, for every $a \in E$, $a = \mathbf{0}$ if $a \oplus a$ is defined.

An *orthomodular poset* is an effect algebra in which every element is principal.

Every orthomodular poset is an orthoalgebra. Indeed, if $a \oplus a$ is defined then $a \oplus a \leq a = a \oplus \mathbf{0}$ and, according to the cancellation law, $a \leq \mathbf{0}$ and therefore $a = \mathbf{0}$.

Orthomodular posets are characterized as effect algebras such that $a \oplus b = a \vee b$ for every orthogonal pair a, b. Let us remark that an orthomodular poset is usually defined as a bounded partially ordered set with an orthocomplementation in which the orthomodular law is valid. (See [3, 4])

Definition 1.4 Let E be an effect algebra. The *isotropic index* $i(a)$ of an element $a \in E$ is $\sup\{n \in \mathbb{N} : na \text{ is defined}\}$, where $na = \bigoplus_{i=1}^{n} a$ is the sum of n copies of a.

An effect algebra E is *Archimedean* if every nonzero element has a finite isotropic index.

The isotropic index of $\mathbf{0}$ is ∞. In an orthoalgebra, $a \oplus a$ is defined only for $a = \mathbf{0}$, hence the isotropic index of every nonzero element is 1. Therefore every orthoalgebra is Archimedean.

Definition 1.5 Let E be an effect algebra.

A nonempty system $(a_i)_{i \in I}$ of (not necessarily distinct) elements of E is called *orthogonal*, if sums of all finite subsystems are defined.

An element $a \in E$ is a *majorant* of an orthogonal system O if it is an upper bound of all sums of finite subsystems of O.

The *sum* $\bigoplus O$ of an orthogonal system O is the least majorant of O (if it exists).

E is *orthocomplete* if every orthogonal system has the sum.

E is *weakly orthocomplete* if every orthogonal system either has the sum or has no minimal majorant.

Every pair of elements of an orthogonal system is orthogonal. On the other hand, there are mutually orthogonal elements that do not form an orthogonal system if the effect algebra is not an orthomodular poset. Since only $\mathbf{0}$ is orthogonal to itself in an orthoalgebra and since the multiplicity of $\mathbf{0}$ and the order of elements in an orthogonal system do not play an important role, we may consider sets instead of systems in orthoalgebras. Every majorant of an orthogonal system is its upper bound, these notions coincide just in orthomodular posets. It is easy to see that an effect algebra is weakly orthocomplete if and only if for every orthogonal system every its minimal majorant is its least majorant (the sum).

Let us denote by $\mathcal{O}(E)$ the family of all orthogonal systems in an effect algebra E and let us define an equivalence $\sim$ on $\mathcal{O}(E)$ as follows: for every $O_1, O_2 \in \mathcal{O}(E)$, $O_1 \sim O_2$ if for every $a \in E \setminus \{\mathbf{0}\}$ the multiplicities of a in O_1, O_2 are the same or both infinite. (We ignore the difference in the order of elements, in the multiplicity of $\mathbf{0}$, in infinite multiplicities.) Let us denote by $[O]$ the class of orthogonal systems equivalent to the orthogonal system O and by $\mathcal{O}(E)|_\sim = \{[O]\colon O \in \mathcal{O}(E)\}$ the set of equivalence classes. We define a partial ordering $\preceq$ in $\mathcal{O}(E)|_\sim$ as follows: for every $O_1, O_2 \in \mathcal{O}(E)$, $[O_1] \preceq [O_2]$ if for every $a \in E \setminus \{\mathbf{0}\}$ the multiplicity of a in O_1 is less then or equal to the multiplicity of a in O_2 or both these multiplicities are infinite.

Definition 1.6 Let E be an effect algebra and $S \subseteq E$. An orthogonal system O in E is a *maximal* orthogonal system majorated by S, if O is majorated by S (every element of S is a majorant of O) and there is no orthogonal system O' majorated by S such that $[O] \prec [O']$.

Let us remark that the "maximality" refers to the partial ordering of the equivalence classes. If a maximal orthogonal system O majorated by a set S contains some nonzero element with an infinite multiplicity (this element has an infinite isotropic index, the effect algebra is not Archimedean) then we can add this element to O and the resulting orthogonal system will be majorated by S, too. This is impossible in Archimedean effect algebras.

Lemma 1.7 *Let E be an effect algebra and $S \subseteq E$. For every orthogonal system O majorated by S there is a maximal orthogonal system M majorated by S such that $[O] \preceq [M]$.*

Proof If an orthogonal system O' is majorated by S then every orthogonal system from the class $[O']$ is majorated by S. Let us consider the family of classes $[O']$ of orthogonal systems majorated by S such that $[O] \preceq [O']$. This is a nonempty family such that every chain in it has an upper bound. According to Zorn's lemma, there is a maximal class in this family and we can take an arbitrary its element to get the desired M. □

Definition 1.8 An effect algebra E has the *maximality property* if $\{a, b\}$ has a maximal lower bound for every $a, b \in E$.

It is easy to see (going to orthosupplements) that an effect algebra E has the maximality property if and only if $\{a, b\}$ has a minimal upper bound for every $a, b \in E$.

2 General relations

We will study connections between various properties. First, let us present a useful notion.

Definition 2.1 A set S in an effect algebra is *downward directed* if for every $a, b \in S$ there is a $c \in S$ such that $c \leq a, b$.

It is easy to see that every minimal element in a downward directed set is its least element and that every set with the least element is downward directed.

Theorem 2.2 *Let E be an effect algebra. Consider the following properties:*

(L) *E is a lattice.*
(OC) *E is orthocomplete.*
(W+) *For every orthogonal system in E, the set of its majorants is downward directed.*
(WOC) *E is weakly orthocomplete.*
(CU) *For every chain in E, the set of its upper bounds is downward directed.*
(M) *E has the maximality property.*

Then the following implications hold: (L),(OC)$\Rightarrow$(W+)$\Rightarrow$(WOC); (L),(OC)$\Rightarrow$(CU)$\Rightarrow$(M).

Proof (L)$\Rightarrow$(W+): If a, b are majorants of an orthogonal system O then $a \wedge b \leq a, b$ is a majorant of O.

(OC)$\Rightarrow$(W+): Obvious.

(W+)$\Rightarrow$(WOC): Every minimal majorant of an orthogonal system is its least majorant.

(L)$\Rightarrow$(CU) [12]: If $a, b \in E$ are upper bounds of a chain $C \subseteq E$, then $a \wedge b \leq a, b$ is an upper bound of C.

(OC)$\Rightarrow$(CU) [12]: According to [6, Theorem 3.2], every chain in an orthocomplete effect algebra has the least upper bound.

(CU)$\Rightarrow$(M) [12]: Let $a, b \in E$. Since $[\mathbf{0}, a] \cap [\mathbf{0}, b] \supseteq \{\mathbf{0}\}$, the family of chains in $[\mathbf{0}, a] \cap [\mathbf{0}, b]$ is nonempty. According to Zorn's lemma, there is a maximal chain C in $[\mathbf{0}, a] \cap [\mathbf{0}, b]$. According to our assumption, there is an upper bound $c \leq a, b$ of C. Since the chain C is maximal, $c \in C$ is a maximal element of $[\mathbf{0}, a] \cap [\mathbf{0}, b]$. □

Let us remark that the condition (W+) was introduced by De Simone and Navara [1], the condition (WOC) was introduced by Ovchinnikov [7], the conditions (M) and (CU) were introduced by Tkadlec [8, 12].

It seems to be an open problem whether the condition (W+) implies the maximality property (it is true in Archimedean effect algebras—see Theorem 3.1) or the condition (CU). Let us present examples showing that no other implication between properties from Theorem 2.2 holds except those just mentioned, stated in Theorem 2.2 and direct consequences of the transitivity.

Example 2.3 Let X be a countable infinite set. Let E be the family of finite and cofinite subsets of X with the $\oplus$ operation defined as the union of disjoint sets. Then $(E, \oplus, \emptyset, X)$ is an orthomodular lattice (it forms a Boolean algebra) that is not orthocomplete.

Example 2.4 Let X be a 6-element set. Let E be the family of even-element subsets of X with the $\oplus$ operation defined as the union of disjoint sets from E. Then $(E, \oplus, \emptyset, X)$ is a finite (hence orthocomplete) orthomodular poset that is not a lattice.

Example 2.5 Let X_1, X_2, X_3, X_4 be mutually disjoint infinite sets, $X = \bigcup_{i=1}^4 X_i$,

$$E_0 = \{\emptyset, X_1 \cup X_2, X_2 \cup X_3, X_3 \cup X_4, X_4 \cup X_1, X\},$$
$$E = \{(A \setminus F) \cup (F \setminus A) \colon\ F \subseteq X \text{ is finite},\ A \in E_0\},$$

$A \oplus B = A \cup B$ for disjoint $A, B \in E$. Then $(E, \oplus, \emptyset, X)$ is a weakly orthocomplete orthomodular poset. Indeed, since singletons are elements of E then every element of E is the union and therefore the sum of every maximal orthogonal system it majorates (see Theorem 3.8). $(E, \oplus, \emptyset, X)$ does not fulfill the condition (W+) from Theorem 2.2: the orthogonal set $\{\{x\} \colon\ x \in X_1\}$ has majorants $X_1 \cup X_2$ and $X_1 \cup X_4$ but no majorant less then or equal to both of them. $(E, \oplus, \emptyset, X)$ does not fulfill the maximality property: lower bounds of $\{X_1 \cup X_2, X_4 \cup X_1\}$ are finite subsets of X_1, hence there is no maximal lower bound.

Example 2.6 ([12]) Let X, Y be disjoint infinite countable sets,

$$E_0 = \{A \subseteq (X \cup Y) \colon\ \operatorname{card}(A \cap X) = \operatorname{card}(A \cap Y) \text{ is finite}\},$$
$$E = E_0 \cup \{(X \cup Y) \setminus A \colon\ A \in E_0\},$$

$A \oplus B = A \cup B$ for disjoint $A, B \in E$. Then $(E, \oplus, \emptyset, X \cup Y)$ is an orthomodular poset with the maximality property. Let $X = \{x_n \colon\ n \in \mathbb{N}\}$, $y_0 \in Y$, $f \colon\ X \to Y \setminus \{y_0\}$ be a bijection. Then the chain $\{\{x_2, \ldots, x_n, f(x_2), \ldots, f(x_n)\} \colon\ n \in \mathbb{N} \setminus \{1\}\}$ has two minimal upper bounds $(X \cup Y) \setminus \{x_1, f(x_1)\}$ and $(X \cup Y) \setminus \{x_1, y_0\}$, hence the condition (CU) from Theorem 2.2 is not fulfilled.

Example 2.7 Let X, Y be disjoint uncountable sets of the same cardinality,

$$E_0 = \{A \subseteq (X \cup Y) \colon\ \operatorname{card}(A \cap X) = \operatorname{card}(A \cap Y) \text{ is finite}\},$$
$$E = E_0 \cup \{(X \cup Y) \setminus A \colon\ A \in E_0\},$$

$A \oplus B = A \cup B$ for disjoint $A, B \in E$. Then $(E, \oplus, \emptyset, X \cup Y)$ is an orthomodular poset fulfilling the condition (CU) from Theorem 2.2. Indeed, let $\mathcal{C}$ be a chain in E and $A, B \in E$ its upper bounds. If $\mathcal{C}$ contains a cofinite subset of $X \cup Y$ then $\mathcal{C}$ has a maximal element. If $\mathcal{C}$ does not contain an infinite set then the set $\bigcup \mathcal{C}$ is countable and there is an element $C \in E$ such that $\bigcup \mathcal{C} \subseteq C \subseteq A \cap B$.

The orthomodular poset is not weakly orthocomplete, because for $x_0 \in X$, $y_0 \in Y$ there is a bijection $f \colon\ X \to Y \setminus \{y_0\}$ and the orthogonal set $\{\{x, f(x)\} \colon\ x \in X \setminus \{x_0\}\}$ has different minimal upper bounds $(X \cup Y) \setminus \{x_0, f(x_0)\}$ and $(X \cup Y) \setminus \{x_0, y_0\}$.

3 Relations in special cases

We will show connection concerning conditions from Theorem 2.2 in some cases.

Theorem 3.1 *Every Archimedean effect algebra fulfilling the condition (W+) from Theorem 2.2 has the maximality property.*

Proof Let E be an Archimedean effect algebra fulfilling the condition (W+) from Theorem 2.2 and let $a, b \in E$. The orthogonal system $(\mathbf{0})$ is majorated by $\{a, b\}$. According to Lemma 1.7, there is a maximal orthogonal system M majorated by $\{a, b\}$. According to the condition (W+), there is a majorant $c \leq a, b$ of M. Since E is Archimedean, c is a maximal lower bound of $\{a, b\}$ (otherwise there is a $d \in E$ such that $c < d \leq a, b$ and we can add nonzero $d \ominus c$ to M). □

Definition 3.2 An effect algebra is *separable* if every orthogonal system of its distinct elements is countable.

It is easy to see that an Archimedean effect algebra is separable if and only if every orthogonal system of its nonzero elements is countable. On the other hand, there is an uncountable orthogonal system of nonzero elements in every non-Archimedean effect algebras.

Theorem 3.3 *Every separable effect algebra fulfilling the condition (CU) from Theorem 2.2 fulfills the condition (W+) from Theorem 2.2.*

Proof Let O be an orthogonal system in a separable effect algebra E fulfilling the condition (CU) from Theorem 2.2. Since it is not important to distinguish infinite multiplicities of elements in orthogonal systems, we may suppose that they are countable. Since the effect algebra is separable, the number of elements in O is countable and we can put $O = (a_i)_{i\in I}$ ($I = \mathbb{N}$ or $I = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$). Let a, b be majorants of O. Then the chain $\{\bigoplus_{i=1}^{k} a_i : k \in I\} \subseteq [0, a] \cap [0, b]$ has an upper bound $c \leq a, b$ which is a majorant of O. □

Let us remark that the above theorem cannot be strengthened by replacing the condition (CU) by the maximality property: the orthomodular poset with the maximality property in Example 2.6 is separable but not even weakly orthocomplete—the orthogonal set $\big\{\{x_n, f(x_n)\} : n \in \mathbb{N} \setminus \{1\}\big\}$ has two minimal upper bounds $(X \cup Y) \setminus \{x_1, f(x_1)\}$ and $(X \cup Y) \setminus \{x_1, y_0\}$.

Both weak orthocompleteness and maximality property implies disjunctivity (see, e.g., [5]) in orthomodular posets.

Definition 3.4 An effect algebra E is *disjunctive* if for every $a, b \in E$ with $a \not\leq b$ there is a nonzero $c \in E$ such that $c \leq a$ and $c \wedge b = \mathbf{0}$.

Theorem 3.5 *Every weakly orthocomplete orthomodular poset and every orthomodular poset with the maximality property is disjunctive.*

Proof For the case of weakly orthocomplete orthomodular posets see [7].

Let E be an orthomodular poset with the maximality property and let $a, b \in E$ such that $a \not\leq b$. Since E has the maximality property, there is a maximal element c in $[\mathbf{0}, a] \cap [\mathbf{0}, b]$. For the element $d = a \ominus c$ we obtain $d \leq a$ and, since $a \not\leq b$, $d \neq \mathbf{0}$. It suffice to prove that $d \wedge b = \mathbf{0}$. Indeed, if it is not true then there is a nonzero element $e \in E$ such that $e \leq b, d$. Hence $e \perp c$ and $c < c \oplus e \leq a, b$ (we use the principality of both a and b)—this contradicts to the maximality of c. □

Let us remark that Examples 2.5 and 2.7 show that a disjunctive orthomodular poset need not be weakly orthocomplete nor have the maximality property. Let us show examples that the previous theorem cannot be strengthened to orthoalgebras or lattice effect algebras.

Example 3.6 The so called Wright triangle [4, Example 2.13] is an orthoalgebra that is finite (hence orthocomplete and therefore weakly orthocomplete and with the maximality property) and not disjunctive.

Example 3.7 $C_3 = \{\mathbf{0}, a, \mathbf{1}\}$ with $a \oplus a = \mathbf{1}$ and $x \oplus \mathbf{0} = x$ for every $x \in C_3$ is a lattice effect algebra (hence weakly orthocomplete and with the maximality property) that is not disjunctive: $\mathbf{1} \not\leq a$ but there is no nonzero $b \in C_3$ such that $b \wedge a = \mathbf{0}$.

Let us present a characterization of Archimedean weakly orthocomplete effect algebras.

Theorem 3.8 *Let E be an effect algebra.*

(1) If every element of E is the sum of every maximal orthogonal system it majorates, then E is weakly orthocomplete.

(2) If E is weakly orthocomplete and Archimedean, then every element of E is the sum of every maximal orthogonal system it majorates.

Proof (1) Let O be an orthogonal system in E and let a be a minimal majorant of O. Then O is a maximal orthogonal system majorated by a (otherwise we can add a nonzero element $b \leq a$ and therefore the element $a \ominus b < a$ majorates O) and therefore $\bigoplus O = a$ exists.

(2) Let $a \in E$ and let O be a maximal orthogonal system majorated by a. Let us suppose that a is not the sum of O and seek a contradiction. Either $\bigoplus M$ does not exist or $\bigoplus M < a$. In both cases a is not a minimal majorant of O, hence there is a majorant $b < a$ of O. Then $a \ominus b$ might be added to O and, since E is Archimedean, this contradicts to the maximality of O. □

The following example shows that the assumption of Archimedeanity in the part (2) of the previous theorem cannot be omitted.

Example 3.9 Let $E = \{0, 1, 2, \dots, n, \dots, n', \dots, 2', 1', 0'\}$ with $m \oplus n = m + n$ for every $m, n \in \mathbb{N} \cup \{0\}$ and $m \oplus n' = (n - m)'$ for every $m, n \in \mathbb{N} \cup \{0\}$ with $m \leq n$. Then $(E, \oplus, 0, 0')$ is a lattice effect algebra (it forms a chain) and therefore weakly orthocomplete. Neither element n', $n \in \mathbb{N} \cup \{0\}$, is the sum of the maximal orthogonal system $\prod_{n \in \mathbb{N}} \{n\}^{\mathbb{N}}$ it majorates.

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