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Constructions of Some Non-σ-porous Sets on the Real Line

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The class of σ-porous sets introduced by E. P. Dolzenko [1] often appears as a description of exceptional sets in case these sets are of measure zero and of the first category. The fact that the class of σ-porous sets is strictly contained in the class $\mathcal{A}$ of sets which are of measure zero and of the first category was demonstrated by L. Zajíček [4]. Since the class $\mathcal{A}$ has the properties

(i) if a Borel set A does not belong to $\mathcal{A}$, then $A+A$ contains an interval, and

(ii) each disjoint family of Borel sets not belonging to $\mathcal{A}$ is countable,

it is natural to ask if these properties also hold with $\mathcal{A}$ replaced by the class of σ-porous sets. These problems were posed by P. D. Humke [3] and W. Wilczynski at the symposium "Real Analysis" held in August 1982 in Esztergom, Hungary. J. Foran and P. D. Humke [2] showed some "enveloping" properties of σ-porous sets and posed a problem whether there exists a porous set contained in no σ-porous G_δ set.

Here we give positive answer to the last question and prove that the class of σ-porous sets has neither of the properties (i) or (ii) even for perfect sets. To construct the corresponding examples we give a general method of the construction of perfect non-σ-porous sets, a special case of which has been used by L. Zajíček [4] in his construction of perfect non-σ-porous set of measure zero.

For a subset S of the real line we define the set

$$P(S) = \left\{ x \in S;\ \limsup_{\delta \to 0_+} \frac{l(S,x,\delta)}{\delta} > 0 \right\},$$

where $l(S,x,\delta)$ is the length of the longest subinterval of $(x-\delta, x+\delta)$ disjoint from S. The set S is said to be porous if $P(S) = S$ and is said to be σ-porous if it can be written as a countable union of porous sets. The Lebesgue measure of the set S will be denoted by $|S|$. By an open (closed) interval we mean any nonempty open (closed) connected subset of the real line. If x is a positive real number and I an open (closed) interval, then $x * I$ is the open (closed) interval with the same centre as I and with length $|x * I| = x \cdot |I|$. Through the paper we will distinguish the set of positive integers and the set of natural numbers (containing the number zero).

A general method of construction of perfect non-σ-porous sets. Assume that

(a) $(k_n)_{n=1}^{\infty}$ is an arbitrary nondecreasing sequence of natural numbers such that $\lim_{n\to\infty} k_n = \infty$, and
(b) for every closed interval R and every positive integer n, a finite system $\mathcal{D}_n(R)$ of closed subintervals of R is given.

For every closed interval R and every positive integer n we define the system $\mathcal{R}_n(R)$ of closed, non-overlapping subintervals of R as follows: The set E of all those endpoints of the intervals $2^k * D$, $k = 0, \dots, k_n$, $D \in \mathcal{D}_n(R)$, which belong to $\operatorname{Int} R$ decompose R into $1 + \operatorname{card} E$ closed, non-overlapping (necessarily non-degenerated) subintervals of R. The system $\mathcal{R}_n(R)$ is the system of all such subintervals of R which are not subsets of any element of $\mathcal{D}_n(R)$.

Let $a < b$ be real numbers. By induction we define systems $\mathcal{R}_n$ of closed, non-overlapping intervals such that $\mathcal{R}_0 = \{[a, b]\}$ and $\mathcal{R}_n = \bigcup\{\mathcal{R}_n(R);\ R \in \mathcal{R}_{n-1}\}$ for every positive integer n.

Proposition. *Suppose that for every positive integer n and every closed interval R the following conditions hold:*

(C1) *Whenever $D \in \mathcal{D}_n(R)$, then $2^{k_n+1} * D \subset R$.*
(C2) *Whenever $k \in \{0, \dots, k_n\}$ and $D_1, D_2 \in \mathcal{D}_n(R)$ are such that $(2^k * D_1) \cap (2^k * D_2) \neq \emptyset$, then there is a $D \in \mathcal{D}_n(R)$ such that $(2^k * D_1) \cup (2^k * D_2) \subset (2^k * D)$.*

Then the set $S = \bigcap_{n=0}^{\infty} \bigcup\{R;\ R \in \mathcal{R}_n\}$ is perfect and non-σ-porous.

Moreover, if the set S is nowhere dense and $G \supset P(S)$ is a G_δ set, then G is non-σ-porous.

Proof. It is easy to see that S is nonempty and perfect. Hence we need only to prove the second part of the proposition. Denote

$$\begin{aligned} \mathcal{D} &= \bigcup_{n=0}^{\infty} \{\operatorname{Int} D;\ D \in \mathcal{D}_{n+1}(R) \text{ and } R \in \mathcal{R}_n\} \setminus \{\emptyset\}, \\ G &= \bigcap_{m=1}^{\infty} G_m, \end{aligned}$$

where G_m $(m = 1, 2, \dots)$ are open sets.

Assume that G is σ-porous. Then there exists a sequence $(P_m)_{m=1}^{\infty}$ of porous sets such that

$$G = \bigcup_{m=1}^{\infty} P_m \tag{1}$$

and such that for every positive integer m, for every $x \in P_m$ and every $\delta > 0$, there exists an open interval $I \subset (x - \delta, x + \delta) \setminus P_m$ with $x \in 2 * I$ (this immediately follows from [4], Theorem 4.5). We will construct a sequence $(F_m)_{m=0}^{\infty}$ of nonempty, perfect sets such that for every positive integer m, $F_m \cap P_m = \emptyset$ and $F_m \subset F_{m-1} \cap G_m$, which obviously contradicts (1). The sets F_m will be given by

$$F_m = R_m \setminus \bigcup \{2^m * D;\ D \subset R_m \text{ and } D \in \mathcal{D}\}, \tag{2}$$

where $R_m \subset G_m$ belongs to some $\mathcal{R}_{r_m}$ with

$$k_{r_m} \geq m + 1. \tag{3}$$

From (2), (3) and the conditions (C1) and (C2) it is clear that the sets F_m will be nonempty and perfect.

Let r_0 be a positive integer such that $k_{r_0} \geq 1$ and let $R_0 \in \mathcal{R}_{r_0}$. We put

$$F_0 = R_0 \setminus \bigcup \{D;\ D \subset R_0 \text{ and } D \in \mathcal{D}\} = R_0 \cap S.$$

Suppose now that m is a positive integer and that F_{m-1} has been already defined. The set S is nowhere dense, hence $P(S) \cap \operatorname{Int} R_{m-1} \cap F_{m-1} \neq \emptyset$ and we can find a positive integer $r'_m \geq r_{m-1}$ and an $R'_m \in \mathcal{R}_{r'_m}$ such that $R' \subset R_{m-1} \cap G_m$ and such that the set

$$F'_m = R'_m \setminus \bigcup \{2^{m-1} * D;\ D \subset R'_m \text{ and } D \in \mathcal{D}\} \subset F_{m-1}$$

is nonempty and perfect. We distinguish two cases:

1) $\overline{P}_m \not\supset F'_m$. Then there exists a positive integer r_m and an $R_m \in \mathcal{R}_{r_m}$ such that (3) holds, $R_m \subset R'_m \setminus \overline{P}_m$ and $R_m \cap F'_m$ is infinite. We define F_m by (2).

2) $\overline{P}_m \supset F'_m$. Because

$$P_m \cap \operatorname{Int} R'_m \subset \bigcup \{2 * I;\ I \subset R'_m \setminus P_m \text{ is an open interval}\}$$

and the components of $R'_m \setminus F'_m$ are $2^{m-1} * D$, where $D \subset R'_m$ and $d \in \mathcal{D}$, it follows that

$$P_m \cap \operatorname{Int} R'_m \subset \bigcup \{2^m * D;\ D \subset R'_m \text{ and } D \in \mathcal{D}\}.$$

The set

$$F''_m = R'_m \setminus \bigcup \{2_m * D;\ D \subset R'_m \text{ and } D \in \mathcal{D}\}$$

is disjoint from $P_m \cap \operatorname{Int} R'_m$, nonempty and perfect. There exist a positive integer r_m and an $R_m \in \mathcal{R}_{r_m}$ such that (3) holds, $R_m \subset \operatorname{Int} R'_m$ and $R_m \cap F''_m$ is infinite. We define F_m by (2).

Corollary 1. *There exists a porous set contained in no σ-porous G_δ set.*

Theorem 1. *There exists a perfect non-σ-porous set S such that for every finite sequence $(c_1, \dots, c_i)$ the set $\sum_{j=1}^{i} c_j S$ is of measure zero; hence for every countable set C the set*

$$\left\{ \sum_{j=1}^{i} c_j s_j;\ i \text{ is a natural number, } c_j \in C \text{ and } s_j \in S \right\}$$

does not contain any interval.

Proof. First we associate with every positive integer n, every closed interval $R = [c, d]$ and every positive integer N a system $\mathcal{D}_n(R, N)$ of closed subintervals of R and polynomial (not depending on R, N) P_n of one variable such that $|R \setminus \bigcup \mathcal{D}_n(R, N)| \le n \cdot 3^{-N}|R|$ as follows.

Define the points d_m, $m = 0, \pm 1, \dots, \pm N, N + 1$, by

$$\begin{aligned} [d_0, d_1] &= \frac{1}{2} * [c, d] \\ d_{-m} - c &= \frac{1}{3}(d_{-m+1} - c), \quad m = 1, \dots, N \\ d - d_m &= \frac{1}{3}(d - d_{m-1}), \quad m = 2, \dots, N + 1 \end{aligned}$$

It follows that

$$|(c, d_{-N})| = |(d_{N+1}, d)| = \frac{1}{4} \cdot 3^{-N}|R|.$$

Hence it is possible to find in each interval (d_m, d_{m+1}), $m = 0, \pm 1, \dots, \pm N$, a closed subinterval such that the system $\mathcal{D}_1(R, N)$ of all these intervals fulfils

$$|R \setminus \bigcup \mathcal{D}_1(R, N)| \le 3^{-N}|R|.$$

The number of intervals in $\mathcal{D}_1(R, N)$ is $P_1(N) = 2N + 1$.

If $n > 1$ is a positive integer and for every closed interval R' and every positive integer N the system $\mathcal{D}_{n-1}(R', N)$ has been defined, then we define $\mathcal{D}_n(R, N)$ as follows: For every D from $\mathcal{D}_1(R, N)$ the endpoints of the intervals $2^{-k} * D$, $k = 0, \dots, n - 1$, decompose D into $2n - 1$ non-overlapping closed subintervals. For each such subinterval I we constructed the system $\mathcal{D}_{n-1}(I, N)$. The system $\mathcal{D}_n(R, N)$ is the union of all such systems $\mathcal{D}_{n-1}(I, N)$ and of the set $\{2^{-n-1} * D;\ D \in \mathcal{D}_1(R, N)\}$. Then

$$\operatorname{card} \mathcal{D}_n(R, N) = (2N + 1) \cdot \big(1 + (2n - 1) \cdot P_{n-1}(N)\big) = P_n(N)$$

$$|R \setminus \bigcup \mathcal{D}_n(R, N)| \le (3^{-N} + (n - 1) \cdot 3^{-N}) \cdot |R| \le n \cdot 3^{-N} \cdot |R|$$

We select real numbers $a < b$, put $k_n = n - 1$ for every positive integer n and construct the set S by our construction, where we put $\mathcal{D}_n(R) = \mathcal{D}_n(R, N_n)$ for suitable N_n such that

$$(\operatorname{card} \mathcal{R}_n)^n \cdot |\bigcup \mathcal{R}_n| \le \frac{1}{n} \tag{4}$$

for every positive integer n. This is possible, because

$$(\operatorname{card}\mathcal{R}_n)^n \cdot |\bigcup \mathcal{R}_n| \le \big((2nP_n(N_n)+1)\cdot \operatorname{card}\mathcal{R}_{n-1}\big)^n \cdot n \cdot 3^{-N_n} \cdot |\bigcup \mathcal{R}_{n-1}|.$$

According to Proposition the set S is non-σ-porous. From (4) it follows that for every positive integer n

$$\begin{aligned}
|\sum_{j1}^{i} c_j S| &\le |\sum_{j=1}^{i} c_j\Big(\bigcup \mathcal{R}_n\Big)| = |\bigcup_{R_1\in\mathcal{R}_n} \cdots \bigcup_{R_i\in\mathcal{R}_n} \sum_{j=1}^{i} c_j R_j| \\
&\le \sum_{R_1\in\mathcal{R}_n} \cdots \sum_{R_i\in\mathcal{R}_n} \sum_{j=1}^{i} |c_j|\cdot|R_j| \\
&\le i\cdot \max\{|c_j|;\ j=1,\dots,i\}\cdot(\operatorname{card}\mathcal{R}_n)^i \cdot |\bigcup \mathcal{R}_n| \\
&\le \frac{i}{n}\cdot \max\{|c_j|;\ j=1,\dots,i\}\cdot(\operatorname{card}\mathcal{R}_n)^{i-n},
\end{aligned}$$

hence $|\sum_{j=1}^{i} c_j S| = 0$.

Theorem 2. *Let K be of the first category. Then there exists a perfect, non-σ-porous set S of measure zero disjoint from K.*

Proof. We need only to prove that for every F_σ set K of the first category and of full measure there exists a perfect, non-σ-porous set S disjoint from K. Denote $K = \bigcup_{m=1}^{\infty} F_m$, where F_m ($m = 1, 2, \dots$) are closed and nowhere dense. First we associate with every positive integer n, every closed interval R such that $K \cap \operatorname{bdry} R = \emptyset$ and with every positive integer m a finite system $\mathcal{D}_n(R, m)$ of closed subintervals of R such that

$$(F_m \cap R) \subset \bigcup\{\operatorname{Int} D;\ D \in \mathcal{D}_n(R, m)\} \tag{5}$$

as follows.

Because $F_m \cap \operatorname{bdry} R = \emptyset$ and because F_m is closed and nowhere dense, there exists a finite disjoint system $\mathcal{D}_1(R, m)$ of closed non-degenerated subintervals of R such that (5) holds and that $2 * D \subset R$ whenever $D \in \mathcal{D}_1(R, m)$. Because the set K is of the first category it is possible to choose the system $\mathcal{D}_1(R, m)$ such that

$$K \cap \bigcup\{\operatorname{bdry} 2^k * D;\ k \text{ is an integer and } D \in \mathcal{D}_1(R, m)\} = \emptyset.$$

If $n > 1$ is a positive integer and for every closed interval R' with $K \cap \operatorname{bdry} R' = \emptyset$ and for every positive integer m the system $\mathcal{D}_{n-1}(R, m)$ has been defined, then we define $\mathcal{D}_n(R, m)$ as follows: For every D from $\mathcal{D}_1(R, m)$ the endpoints of the intervals $2^{-k} * D$, $k = 0, \dots, n-1$, decompose D

into $2n-1$ non-overlapping closed subintervals $I_j(D)$, $j=0,\ldots,2n-2$, $I_0(D)=2^{-n+1}*D$. We define

$$\begin{aligned}\mathcal{D}_n(R,m) &= \bigcup\{\mathcal{D}_{n-1}(I_j(D),m);\ j=1,\ldots,2n-2 \text{ and } D\in\mathcal{D}_1(R,m)\}\\ &\quad \cup\{2^{-n+1}*D;\ D\in\mathcal{D}_1(R,m)\}.\end{aligned}$$

We select real numbers $a<b$ not belonging to K and put $k_n=n-1$ for every positive integer n, $\mathcal{D}_n(R)=\mathcal{D}_n(R,n)$ for every closed interval R with $K\cap \operatorname{bdry} R=\emptyset$ and for every positive integer n. We construct the set S by our construction. It is easy to see that the conditions (C1) and (C2) hold and that for every positive integer n, $(\bigcup\mathcal{R}_{n+1})\cap F_n=\emptyset$. Therefore $S\cap K=\emptyset$ and. according to Proposition, the set S is non-σ-porous.

Corollary 2. *There exists an uncountable family of disjoint non-σ-porous perfect subsets of the real line.*

Remark. By combining the constructions of $\mathcal{D}_n(R)$ from proofs of Theorems 1 and 2 it is possible to construct the set S from Theorem 1 disjoint from a given set K of the first category.

References

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