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Distributivity and associativity in effect algebras

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Abstract

We prove "large associativity" of the partial sum in effect algebras and present an overview of distributivity-like properties of partial operations $\oplus$ and $\ominus$ in effect algebras with respect to (possibly infinite) suprema and infima and vice versa generalizing several previous results.

Keywords: Effect algebra; distributivity; associativity.

1. Introduction

Effect algebras [4] and equivalent D-posets [6] were introduced in the nineties of the twentieth century as "unsharp" generalizations of "sharp" quantum logics (orthomodular lattices, orthomodular posets, orthoalgebras) incorporating some fuzzy logics (MV-algebras). E.g., consider the effect algebra $([0,1],\oplus,0,1)$ with the real unit interval $[0,1]$ and the partial operation $\oplus$ defined as the sum of real numbers whenever this sum belongs to $[0,1]$. This effect algebra corresponds to MV-algebra with the Łukasiewicz t-conorm $\oplus$ if we extend the definition of $\oplus$ by $a \oplus b = 1$ whenever $a + b > 1$.

Effect algebras are partially ordered by a natural way. The distributivity-like properties of suprema and infima (possibly infinite) with respect to partial operations $\oplus$ and $\ominus$ and vice versa were studied by Bennett and Foulis [1] in the context of effect algebras (sometime assuming that they form a lattice) and by Chovanec and Kôpka [2] in the context of D-posets (for two-element sets assuming that the D-posets form a lattice). We present a unified overview of generalizations of these results.

A "large associativity" (also for infinite number of elements) of the partial operation $\oplus$ was studied by Riečanová [7] in the context of abelian RI-posets for complete lattices and by Ji [5] for orthocomplete effect algebras. We generalize these results for effect algebras.

We present examples showing that these results cannot be improved to obtain distributivity (associativity, resp.) in all cases.

Our results can be useful in the study of effect algebras (quantum and fuzzy structures)—see, e.g., [1, 3, 5]. They seem to be ultimate because we were able to omit all assumptions for the underlying structure and for the cardinalities of considered sets.

2. Basic notions and properties

Let us start with a review of basic notions and properties.

2.1 Definition. An *effect algebra* is an algebraic structure $(E, \oplus, \mathbf{0}, \mathbf{1})$ such that E is a set, $\mathbf{0}$ and $\mathbf{1}$ are different elements of E, and $\oplus$ is a partial binary operation on E such that for every $a, b, c \in E$ the following conditions hold:

(1) $a \oplus b = b \oplus a$, if one side exists;
(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, if one side exists;
(3) there is a unique *orthosupplement* a' such that $a \oplus a' = \mathbf{1}$;
(4) $a = \mathbf{0}$ whenever $a \oplus \mathbf{1}$ is defined.

For simplicity, we will use the notation E for an effect algebra. A partial ordering on an effect algebra E is defined by $a \le b$ if there is a $c \in E$ such that $b = a \oplus c$. Such an element c is unique (if it exists), is equal to $(a \oplus b')'$ and is denoted by $b \ominus a$. In particular, $a' = 1 \ominus a$. With respect to this partial ordering, $\mathbf{0}$ ($\mathbf{1}$, resp.) is the least (the greatest, resp.) element of E. The orthosupplementation is an antitone involution, i.e., for every $a, b \in E$, $a'' = a$ and $b' \le a'$ whenever $a \le b$. An *orthogonality* relation on E is defined by $a \perp b$ if $a \oplus b$ exists (that is if and only if $a \le b'$). It can be shown that $a \oplus \mathbf{0} = a$ for every $a \in E$ and that the *cancellation law* is valid: if $a \oplus c \le b \oplus c$ then $a \le b$ (in particular, if $a \oplus c = b \oplus c$ then $a = b$). See, e.g., [3, 4].

An equivalent notion (in the sense of a natural correspondence) of a *D-poset* defined by the properties of the partial operation $\ominus$ is used sometimes. See, e.g., [3, 6].

2.2 Definition. Let E be an effect algebra. A system $(a_i)_{i \in I}$ of elements of E is *orthogonal* if $\bigoplus_{i \in F} a_i$ is defined for every finite set $F \subseteq I$. A *majorant* of an orthogonal system is an upper bound of all its finite sums. The *sum* $\bigoplus_{i \in I} a_i$ of an orthogonal system $(a_i)_{i \in I}$ is its least majorant (if it exists).

A finite system is orthogonal if and only if the sum of all its elements is defined. Every subsystem of an orthogonal system is orthogonal. The empty system is orthogonal and its sum is the least element $\mathbf{0}$. Every pair of elements in an orthogonal system is orthogonal. On the other hand there are nonorthogonal systems of pairwise orthogonal elements if (and only if) the effect algebra does not form an orthomodular poset.

A simple example of an effect algebra is the structure $([0, 1], \oplus, 0, 1)$ where $[0, 1]$ is the interval of real numbers and $\oplus$ is defined by $a \oplus b = a + b$ for $a + b \le 1$. Then $a \ominus b = a - b$ (whenever it is defined).

Let us summarize some properties of the operations $\oplus$ and $\ominus$ showing that these partial operations behave very much like the real operations $+$ and $-$. The basic difference is that we have to take care whether they are defined.

2.3 Lemma. *Let E be an effect algebra, $a, b, c, a_i \in E$, $i \in I$, I is finite:*

(1) *If $b = \bigoplus_{i \in I} a_i$ then $b \ge \bigoplus_{i \in J} a_i$ and $b \ominus \bigoplus_{i \in J} a_i = \bigoplus_{i \in I \setminus J} a_i$ for every $J \subseteq I$. In particular, $(a \oplus b) \ominus b = a$ whenever $a \perp b$.*
(2) *If $a \le b$ then $a \oplus (b \ominus a) = b$, $b \ominus (b \ominus a) = a$ and $b \ominus a = a' \ominus b'$.*
(3) *If $a \le b \perp c$ then $a \oplus c \le b \oplus c$ and $b \oplus c = (a \oplus c) \oplus (b \ominus a)$, i.e., $(b \oplus c) \ominus a = (b \ominus a) \oplus c$.*
(4) *If $a \le b \le c$ then $c \ominus a = (b \ominus a) \oplus (c \ominus b)$, i.e., $b \ominus a \le c \ominus a$ and $c \ominus b \le c \ominus a$.*
(5) *$c \ominus (a \oplus b) = (c \ominus a) \ominus b = (c \ominus b) \ominus a$ whenever one of the compared expressions exists.*
(6) *If $a \perp b$ then $(a \oplus b)' = a' \ominus b = b' \ominus a$.*

Proof. (1) It is a consequence of the commutativity and associativity of $\oplus$ and of the definition of $\ominus$.

(2) The first equality is the definition of $b \ominus a$, the second follows using part (1), the third follows from the cancellation law and from the equality (we use $b' \le a'$) $a \oplus (b \ominus a) \oplus b' = b \oplus b' = \mathbf{1} = a \oplus a' = a \oplus b' \oplus (a' \ominus b')$.

(3) $b \oplus c = a \oplus (b \ominus a) \oplus c = (a \oplus c) \oplus (b \ominus a)$.

(4) Since $a \oplus (c \ominus a) = c = b \oplus (c \ominus b) = a \oplus (b \ominus a) \oplus (c \ominus b)$, using the cancellation law we obtain $c \ominus a = (b \ominus a) \oplus (c \ominus b)$.

(5) $a \oplus b \leq c$ if and only if there is a $d \in E$ such that $c = a \oplus b \oplus d$; this is equivalent to $a \leq c \ominus b$. Then $c = a \oplus (c \ominus a) = a \oplus b \oplus \big((c \ominus a) \ominus b\big)$. According to part (1), $c \ominus (a \oplus b) = (c \ominus a) \ominus b$. The rest follows from the symmetry of a, b.

(6) $a' \ominus b = (a'' \oplus b)' = (a \oplus b)'$. The rest is analogous. $\square$

To simplify some notations we will use sets of elements instead of elements as arguments of relations and operations in a usual way. E.g., if a is an element and B is a set of elements of an effect algebra then by $a \leq B$ we mean that $a \leq b$ for every $b \in B$ and by $a \oplus B$ we mean the set $\{a \oplus b \colon b \in B\}$.

3. Distributivity

Let us present distributivity properties of $\oplus$ and $\ominus$ with respect to (possibly infinite) suprema and infima and vice versa. Some of them are known and are presented here (with independent proofs) to obtain a complete pattern of three equivalent theorems—Theorems 3.1, 3.2 and 3.3 (each part of every of these theorems is equivalent to the same part of remaining ones).

Theorem 3.1 is a generalization of [2, Propositions 2.3 and 2.4] proved for lattices and 2-element sets (in the context of D-posets). Theorem 3.2 is a generalization of [2, Propositions 2.6 and 2.9] proved for lattices and 2-element sets (in the context of D-posets). Theorem 3.2 (1) was proved in [1, Corollary 2.3]. Theorem 3.3 (2) is a generalization of [1, Theorem 3.2] stated for lattices. Theorem 3.3 (3) was proved in [1, Theorem 2.2].

Since the orthosupplement is an antitone involution, every effect algebra forms a de Morgan poset, i.e., for every its subset B, if $\bigvee B$ ($\bigwedge B$, resp.) exists then $(\bigvee B)' = \bigwedge B'$ ($(\bigwedge B)' = \bigvee B'$, resp.). These *de Morgan laws* might be formulated as follows: $\mathbf{1} \ominus \bigvee B = \bigwedge(\mathbf{1} \ominus B)$, $\mathbf{1} \ominus \bigwedge B = \bigvee(\mathbf{1} \ominus B)$ for every subset B of an effect algebra whenever one side of the respective equality exists. Hence, the following theorem is a generalization of de Morgan laws.

3.1 Theorem. *Let E be an effect algebra, $a \in E$, $B \subseteq E$, $B \leq a$.*
(1) *If $\bigvee B$ exists then $a \ominus \bigvee B = \bigwedge(a \ominus B)$.*
(2) *If $\bigwedge(a \ominus B)$ exists then $a \ominus \bigwedge(a \ominus B)$ is a minimal upper bound of B.*
(3) *If $\bigvee(a \ominus B)$ exists then $a \ominus \bigwedge B = \bigvee(a \ominus B)$.*
(4) *If $\bigwedge B$ exists then $a \ominus \bigwedge B$ is a minimal upper bound of $a \ominus B$.*

Proof. (1) We have $B \leq \bigvee B \leq a$ and therefore $a \ominus \bigvee B \leq a \ominus B$, i.e., $a \ominus \bigvee B$ is a lower bound of $a \ominus B$. Let $c \in E$ be a lower bound of $a \ominus B$. We have $c \leq a \ominus B$ and therefore $B \leq a \ominus c$, hence $\bigvee B \leq a \ominus c$ and therefore $c \leq a \ominus \bigvee B$, i.e., $a \ominus \bigvee B$ is the greatest lower bound of $a \ominus B$.

(2) We have $\bigwedge(a \ominus B) \leq a \ominus B$ and therefore $B \leq a \ominus \bigwedge(a \ominus B)$, i.e., $a \ominus \bigwedge(a \ominus B)$ is an upper bound of B. Let $c \in E$ be an upper bound of B such that $c \leq a \ominus \bigwedge(a \ominus B)$. We have $B \leq c \leq a$ and therefore $a \ominus c \leq a \ominus B$, hence $a \ominus c \leq \bigwedge(a \ominus B)$ and therefore $c \geq a \ominus \bigwedge(a \ominus B)$, i.e., $a \ominus \bigwedge(a \ominus B)$ is a minimal upper bound of B.

(3), (4) follows from parts (1) and (2) if we replace B by $a \ominus B$. $\square$

The following theorem is a reformulation of Theorem 3.1 for a' instead of a and B' instead of B.

3.2 Theorem. *Let E be an effect algebra, $a \in E$, $B \subseteq E$, $a \leq B$.*
(1) *If $\bigwedge B$ exists then $\bigwedge B \ominus a = \bigwedge(B \ominus a)$.*
(2) *If $\bigwedge(B \ominus a)$ exists then $a \oplus \bigwedge(B \ominus a)$ is a maximal lower bound of B.*
(3) *If $\bigvee(B \ominus a)$ exists then $\bigvee B \ominus a = \bigvee(B \ominus a)$.*
(4) *If $\bigvee B$ exists then $\bigvee B \ominus a$ is a minimal upper bound of $B \ominus a$.*

The following theorem is a reformulation of Theorem 3.2 for $a \oplus B$ instead of B.

3.3 Theorem. *Let E be an effect algebra, $a \in E$, $B \subseteq E$, $a \perp B$.*

(1) *If $\bigwedge(a \oplus B)$ exists then $a \oplus \bigwedge B = \bigwedge(a \oplus B)$.*

(2) *If $\bigwedge B$ exists then $a \oplus \bigwedge B$ is a maximal lower bound of $a \oplus B$.*

(3) *If $\bigvee B$ exists then $a \oplus \bigvee B = \bigvee(a \oplus B)$.*

(4) *If $\bigvee(a \oplus B)$ exists then $\bigvee(a \oplus B) \ominus a$ is a minimal upper bound of B.*

The following example shows that that minimal upper bounds (maximal lower bounds, resp.) in Theorems 3.1, 3.2 and 3.3 could not be replaced by suprema (infima, resp.) in general. (The example concerns Theorem 3.1 (2) but other examples might be derived easily.)

3.4 Example. Let $X = \{1, 2, 3, 4, 5, 6\}$, E be the family of even-element subsets of X, $\oplus$ be the union of disjoint sets. $(E, \oplus, \emptyset, X)$ is an effect algebra (forms an orthomodular poset), the partial ordering is the inclusion. Then $a = \{1, 2, 3, 4\}$, $B = \big\{\{1, 2\}, \{2, 3\}\big\}$ fulfills the assumptions of Theorem 3.1 (2) but $\bigvee B$ does not exist.

Theorems 3.1, 3.2 and 3.3 might be simplified in effect algebras where a minimal upper bound has to be a supremum (e.g., in lattice effect algebras). It suffices to have this property only for sets with the same cardinality as the set B.

3.5 Corollary. *Let E be an effect algebra such that every minimal upper bound of a set is its supremum, $a \in E$, $B \subseteq E$.*

(1) *If $B \leq a$ then $a \ominus \bigvee B = \bigwedge(a \ominus B)$ and $a \ominus \bigwedge B = \bigvee(a \ominus B)$ whenever one side of the respective equality exists.*

(2) *If $a \leq B$ then $\bigvee B \ominus a = \bigvee(B \ominus a)$ and $\bigwedge B \ominus a = \bigwedge(B \ominus a)$ whenever one side of the respective equality exists.*

(3) *If $a \perp B$ then $a \oplus \bigvee B = \bigvee(a \oplus B)$ and $a \oplus \bigwedge B = \bigwedge(a \oplus B)$ whenever one side of the respective equality exists.*

3.6 Corollary. *Let E be an effect algebra, $a, b \in E$ such that $a \perp b$ and $a \vee b$ exists. Then $a \wedge b$ exists and $a \oplus b = (a \vee b) \oplus (a \wedge b)$. In particular, $a \oplus b \geq a \vee b$ and the equality is valid if and only if $a \wedge b = \mathbf{0}$.*

Proof. It follows from Theorem 3.1, part (1), for $\{a, b\} \leq a \oplus b$. □

Let us remark that the inequality $a \oplus b \geq a \vee b$ in the above statement is obvious.

If we put $a = \bigvee B$ in Theorem 3.1 (1), we obtain $\bigwedge(a \ominus B) = \mathbf{0}$. If we put $a = \bigwedge B$ in Theorem 3.2 (1), we obtain $\bigwedge(B \ominus a) = \mathbf{0}$—this was proved in [1, Corollary 2.4]. Let us present stronger results.

3.7 Theorem. *Let E be an effect algebra, $a \in E$, $B \subseteq E$.*

(1) *$\bigwedge(B \ominus a) = \mathbf{0}$ if and only if a is a maximal lower bound of B.*

(2) *$\bigwedge(a \ominus B) = \mathbf{0}$ if and only if a is a minimal upper bound of B.*

Proof. (1) For every $b \in E$, $b \leq B \ominus a$ if and only if $a \oplus b \leq B$. Hence, $\bigwedge(B \ominus a) = \mathbf{0}$ if and only if there is no greater lower bound of B than a.

(2) For every $b \in E$, $b \leq a \ominus B$ if and only if $B \leq a \ominus b$. Hence, $\bigwedge(a \ominus B) = \mathbf{0}$ if and only if there is no smaller upper bound of B than a. □

4. Associativity

The partial operation $\oplus$ is associative considering finite sums. We will consider "large associativity" including also infinite sums, i.e. (using the commutativity of $\oplus$), $\bigoplus_{i \in I} a_i = \bigoplus_{j \in J} \bigoplus_{i \in I_j} a_i$ for an orthogonal system $(a_i)_{i \in I}$ where I is a disjoint union of I_j, $j \in J$. This was proved for two-element set J (it might be easily generalized for finite J) by Riečanová [7, Theorem 1.6 (iv)] in the context of abelian RI-posets for complete lattices with the assumption that the right side exists and by Ji [5, Lemma 3.2] for orthocomplete effect algebras (the existence of all considered sums is ensured). We have to assume the existence of at least one side of the large associativity equation in general effect algebras.

Let us start with properties of "disjoint" subsums of an orthogonal system.

4.1 Proposition. *Let E be an effect algebra, $(a_i)_{i\in I}$ be an orthogonal system in E, I be a disjoint union of I_j, $j \in J$. If $\bigoplus_{i\in I_j} a_i$ exists for every $j \in J$ then the system $(\bigoplus_{i\in I_j} a_i)_{j\in J}$ is orthogonal and the set of its majorants is the set of majorants of the system $(a_i)_{i\in I}$.*

Proof. Let us denote by $\mathcal{F}$ the family of finite subsets of I, $b_{j,F} = \bigoplus_{i\in I_j\cap F} a_i$, $b_j = \bigoplus_{i\in I_j} a_i$, $J_F = \{j \in J\colon I_j \cap F \neq \emptyset\}$ for every $F \in \mathcal{F}$ and every $j \in J$.

Let c be a majorant of $(a_i)_{i\in I}$, $G \subseteq J$ be finite, $F_j \subseteq I_j$ be finite for every $j \in G$. Then $c \geq \bigoplus_{i\in\bigcup\{F_j\colon j\in G\}} a_i = \bigoplus_{j\in G} b_{j,F_j}$. For every $k \in G$, we consecutively obtain $c \ominus \bigoplus_{j\in G\setminus\{k\}} b_{j,F_j} \geq b_{k,F_k}$, $c \ominus \bigoplus_{j\in G\setminus\{k\}} b_{j,F_j} \geq b_k$, $c \geq b_k \oplus \bigoplus_{j\in G\setminus\{k\}} b_{j,F_j}$. Repeating this procedure, we obtain $c \geq \bigoplus_{j\in G} b_j$, i.e., the system $(b_j)_{j\in J}$ is orthogonal (we can consider $c = \mathbf{1}$) and c is its majorant.

Let c be a majorant of $(b_j)_{j\in J}$. Then, for every $F \in \mathcal{F}$, $\bigoplus_{i\in F} a_i = \bigoplus_{j\in J_F} b_{j,F} \leq \bigoplus_{j\in J_F} b_j \leq c$, i.e., c is a majorant of $(a_i)_{i\in I}$. □

4.2 Theorem. *Let E be an effect algebra, $(a_i)_{i\in I}$ be an orthogonal system in E, I be a disjoint union of I_j, $j \in J$, K be a subset of J.*

(1) If $\bigoplus_{j\in J}\bigoplus_{i\in I_j} a_i$ exists then it is equal to $\bigoplus_{i\in I} a_i$.

(2) If $\bigoplus_{i\in I} a_i$ and $\bigoplus_{j\in K}\bigoplus_{i\in I_j} a_i$ exist then $\bigoplus_{i\in I} a_i \ominus \bigoplus_{j\in K}\bigoplus_{i\in I_j} a_i$ is a minimal majorant of $(a_i)_{i\in\bigcup\{I_j\colon j\in J\setminus K\}}$, i.e., a minimal majorant of the system $\big(\bigoplus_{i\in I_j} a_i\big)_{j\in J\setminus K}$ if its sums exist.

Proof. (1) It is a consequence of Proposition 4.1.

(2) According to part (1), $\bigoplus_{j\in K}\bigoplus_{i\in I_j} a_i = \bigoplus_{i\in\bigcup\{I_j\colon j\in K\}} a_i$, hence, without a loss of generality we may (and will) assume that $J = \{1, 2\}$ and $K = \{1\}$. Let us denote by $\mathcal{F}$ the family of finite subsets of I, $a_F = \bigoplus_{i\in F} a_i$, $a = \bigoplus_{i\in I} a_i$, $b_{j,F} = \bigoplus_{i\in I_j\cap F} a_i$, for every $F \in \mathcal{F}$ and every $j \in J$, $b_1 = \bigoplus_{i\in I_1} a_i$.

Let $F_1 \subseteq I_1$ and $F_2 \subseteq I_2$ be finite. We consecutively obtain $b_{1,F_1} \oplus b_{2,F_2} = a_{F_1\cup F_2} \leq a$, $b_{1,F_1} \leq a \ominus b_{2,F_2}$, $b_1 \leq a \ominus b_{2,F_2}$, $b_{2,F_2} \leq a \ominus b_1$. Hence $a \ominus b_1$ is a majorant of $(a_i)_{i\in I_2}$. Let c be a majorant of $(a_i)_{i\in I_2}$ with $c \leq a \ominus b_1$. Then $a \ominus \big((a \ominus b_1) \ominus c\big) = a \ominus \big(a \ominus (b_1 \oplus c)\big) = b_1 \oplus c \geq b_{1,F} \oplus b_{2,F} = a_F$ for every $F \in \mathcal{F}$ and therefore $a \ominus \big((a \ominus b_1) \ominus c\big) \geq a$, i.e., $c = a \ominus b_1$. Hence, $a \ominus b_1$ is a minimal majorant of $(a_i)_{i\in I_2}$. □

Let us remark that it suffices to assume the existence of $\bigoplus_{i\in I_j} a_i$ for every $j \in J$ in Theorem 4.2 (1) if the set J is finite—according to Proposition 4.1, the system $(\bigoplus_{i\in I_j} a_i)_{j\in J}$ is orthogonal, hence summable.

The following examples show that the existence of $\bigoplus_{i\in I} a_i$ is not sufficient to obtain the large associativity in Theorem 4.2 (2)—a summable system might be divided to two nonsummable subsystems, the minimal majorant need not be the sum (the least majorant).

4.3 Example. Let X be an infinite set, E be the set of finite and cofinite subsets of X, $\oplus$ be the union of disjoint sets, $Y \subseteq X$ be such that Y and $X \setminus Y$ are infinite. Then $(E, \oplus, \emptyset, X)$ is an effect algebra (forms a Boolean algebra) with a summable orthogonal system $(\{x\})_{x\in X}$ (its sum is X) which can be divided to two nonsummable subsystems $(\{x\})_{x\in Y}$ and $(\{x\})_{x\in X\setminus Y}$.

4.4 Example. Let $X = \{x_i\colon i \in \mathbb{N}\}$, $Y = \{y_i\colon i \in \mathbb{N}\}$ be disjoint sets, $Z = X \cup Y$, $E = \{A \subseteq Z\colon \operatorname{card}(X \cap A) = \operatorname{card}(Y \cap A)$ is finite or $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus A)$ is finite$\}$, $\oplus$ be the union of disjoint sets, $A_i = \{x_i, y_{i+1}\}$ for $i \in \mathbb{N}$. Then $(E, \oplus, \emptyset, Z)$ is an effect algebra with an orthogonal system $(A_i)_{i\in\mathbb{N}}$, $\bigoplus_{i\in\mathbb{N}} A_i = Z$, $\bigoplus_{i\in\{0\}} A_i = A_0$, the orthogonal system $(A_i)_{i\in\mathbb{N}\setminus\{0\}}$ has two minimal majorants $Z \setminus A_0$ and $Z \setminus \{x_0, y_0\}$.

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