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Interpolations in posets and effect algebras

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Abstract

We study various types of the interpolation property in posets and effect algebras. We present connections to other properties of posets and effect algebras (completeness, orthocompleteness, maximality property) and a theorem about preserving compatibility to suprema and infima using an interpolation property.

Keywords effect algebra; poset; interpolation; orthocomplete; weakly orthocomplete; maximality property; pseudolattice; compatibility

1 Introduction

Effect algebras (and their equivalents D-posets) become a basic notion for quantum structures (originated in quantum physics) as "unsharp" generalizations of orthomodular lattices, orthomodular posets and orthoalgebras [2, 3].

There are various properties of effect algebras based on properties of upper bounds for some sets of elements used in the literature. E.g., being a lattice, completeness, orthocompleteness, interpolation property (e.g., [1, 4]), weak orthocompleteness (e.g., [6, 8]) maximality property (e.g., [7, 9]).

We study various types the interpolation property in the sense that we consider also infinite sets. We present connections to other properties of posets and effect algebras and present several generalizations of previous results.

The paper is organized as follows: Basic notions and known properties are summarized in Section 2. In Section 3 we introduce various types of interpolation properties and show connections between them. Section 4 brings results concerning connections to other notions used in theories of partially ordered sets and quantum structures, such as completeness, orthocompleteness, the maximality property. The result about preserving compatibility by the operations of suprema and infima is presented in Section 5.

2 Basic notions and properties

2.1 Definition. An *effect algebra* is an algebraic structure $(E, \oplus, \mathbf{0}, \mathbf{1})$ such that E is a set, $\mathbf{0}$ and $\mathbf{1}$ are different elements of E, and $\oplus$ is a partial binary operation on E such that for every $a, b, c \in E$ the following conditions hold:

(1) $a \oplus b = b \oplus a$, if one side exists;

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(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, if one side exists;
(3) there is a unique *orthosupplement* a' such that $a \oplus a' = \mathbf{1}$;
(4) $a = \mathbf{0}$ whenever $a \oplus \mathbf{1}$ is defined.

For simplicity, we will use the notation E for an effect algebra. A partial ordering on an effect algebra E is defined by $a \leq b$ if there is a $c \in E$ such that $b = a \oplus c$. Such an element c is unique (if it exists) and is denoted by $b \ominus a$. In particular, $1 \ominus a = a'$. With respect to this partial ordering, $\mathbf{0}$ ($\mathbf{1}$, resp.) is the least (the greatest, resp.) element of E. The orthosupplementation is an antitone involution, i.e., for every $a, b \in E$, $a'' = a$ and $b' \leq a'$ whenever $a \leq b$. An *orthogonality* relation on E is defined by $a \perp b$ if $a \oplus b$ exists (that is if and only if $a \leq b'$). It can be shown that $a \oplus \mathbf{0} = a$ for every $a \in E$ and that the *cancellation law* is valid: if $a \oplus c \leq b \oplus c$ then $a \leq b$ (in particular, if $a \oplus c = b \oplus c$ then $a = b$). See, e.g., [2, 3].

Some notions and results will be formulated for posets (partially ordered sets). To simplify some notations we will use sets of elements instead of elements as arguments of relations and operations in a usual way (considering all possible choices of elements). E.g., if A, B are subsets of a poset then by $A \leq B$ we mean that $a \leq b$ for every $a \in A$ and every $b \in B$.

2.2 Definition. Let $(P, \leq)$ be a poset. An $a \in P$ is *the least* element of P if $a \leq P$. An $a \in P$ is *a minimal* element of P if there is no $p \in P \setminus \{a\}$ with $p \leq a$. An $a \in P$ is an *upper bound* (*lower bound*) of a set $A \subseteq P$ if $A \leq a$ ($a \leq A$). P is *downward directed* if for every $a, b \in P$ there is a $c \in P$ such that $c \leq a, b$.

Obviously, the least element of a poset is its minimal element. On the other hand, there are posets with more minimal elements (hence, without the least element and not downward directed).

2.3 Definition. A *bounded poset* is a structure $(P, \leq, \mathbf{0}, \mathbf{1})$ such that $(P, \leq)$ is a poset, $\mathbf{0}$ is the least and $\mathbf{1}$ is the greatest element of P.

A *de Morgan poset* is a structure $(P, \leq, ')$ where $(P, \leq)$ is a poset and $'$ is an antitone involution on P, i.e., it is a mapping $P \to P$ such that (1) $b' \leq a'$ whenever $a, b \in P$ with $a \leq b$, and (2) $(a')' = a$ for every $a \in P$.

The following definition generalizes the maximality property introduced by Tkadlec [7]. We need a slight generalization for posets that are not de Morgan.

2.4 Definition. A poset is a *pseudolattice* if every pair of its elements has a minimal upper bound and a maximal lower bound.

For simplicity, we will use the notation P for a (bounded, de Morgan, resp.) poset. Every effect algebra forms a bounded de Morgan poset. For every set A in a de Morgan poset we have $\bigvee A = (\bigwedge A')'$ ($\bigwedge A = (\bigvee A')'$, resp.) whenever one side of the equality exists.

We will deal with cardinals to make sharp upper bounds for cardinalities of some sets. We accept that the proper class of all set cardinals is a cardinal (the greatest one). Using this cardinal in fact adds no restriction, because every set has smaller cardinality—usually the reference to this cardinal is omitted ("property" instead of "α-property"). The least infinite cardinal (countable) is the cardinality of the set of natural numbers and is denoted by ω_0.

3 Interpolation properties

The interpolation property was introduced by Goodearl [4] for partially ordered abelian groups and by Bennet and Foulis [1] for effect algebras.

3.1 Definition. A poset P has the *interpolation property* if for every $a, b, c, d \in P$ with $\{a, b\} \leq \{c, d\}$ there is an $e \in P$ such that $\{a, b\} \leq e \leq \{c, d\}$.

In other words, the set of upper bounds of a two-element set is downward directed. We generalize this notion using a general cardinality of involved sets.

3.2 Definition. Let $\alpha, \beta > 1$ be cardinals. A poset P is (α, β)*-interpolated* if for every nonempty $A, B \subset P$ with $A \leq B$, $\operatorname{card} A < \alpha$ and $\operatorname{card} B < \beta$ there is an element $c \in P$ such that $A \leq c \leq B$ (c is an *interpolation* of A, B). A poset is α*-interpolated* if it is (α, α)-interpolated.

The notion of the 3-interpolated poset coincides with the original interpolation property. This is the least nontrivial situation (if some of the sets is one-element then its element is an interpolation). Obviously, if an interpolation exists for some cardinals then it exists for smaller cardinals (greater then 1), too. Let us remark that accepting empty sets in the definition above we obtain a different notion (with "extrapolations", i.e., lower and upper bounds). E.g., in the real interval $(0, 1)$ every pair of nonempty sets A, B with $A \leq B$ has an interpolation (e.g., $\bigvee A$) but, e.g., the pair $\emptyset, (0, 1)$ has no interpolation. No difference appears in bounded posets (e.g., in effect algebras).

Let us observe some properties. The first is a slight generalization of [4, Proposition 2.2].

3.3 Proposition. *Let $\alpha > 1$ be a cardinal. Every $(\alpha, 3)$-interpolated poset is (α, ω_0)-interpolated. Every $(3, \alpha)$-interpolated poset is (ω_0, α)-interpolated.*

Proof. Let P be an $(\alpha, 3)$-interpolated poset, $A, B \subset P$ be nonempty, B be finite. We will use the induction according to the number n of elements of the set B. According to the assumption, the interpolation of A, B exists for $n \leq 2$. Let us suppose that it exists for a given natural number $n \geq 2$ and let us consider $B = \{b_1, \ldots, b_{n+1}\}$. According to the assumption, there are interpolations d_i of $A, \{b_i, b_{n+1}\}$ for every $i \in \{1, \ldots, n\}$. For $D = \big\{d_i \colon i \in \{1, \ldots, n\}\big\}$ we obtain $A \leq D$ and $\operatorname{card} D \leq n$, hence, according to the induction assumption, there is a $c \in E$ such that $A \leq c \leq D$ and therefore $A \leq c \leq B$.

The second part can be proved analogously. □

3.4 Corollary. *Every 3-interpolated poset is ω_0-interpolated.*

According to previous statements, we may (and will) consider only infinite cardinals for (α, β)-interpolated posets.

3.5 Proposition. *Let α, β be infinite cardinals. A de Morgan poset is (α, β)-interpolated if and only if it is (β, α)-interpolated.*

Proof. Let P be a (α, β)-interpolated de Morgan poset, $A, B \subset P$ be nonempty sets such that $B \leq A$, $\operatorname{card} B \leq \beta$ and $\operatorname{card} A \leq \alpha$. Then $A' \leq B'$, $\operatorname{card} A' \leq \alpha$, $\operatorname{card} B' \leq \beta$ and there is a $c \in P$ such that $A' \leq c \leq B'$, i.e., $B \leq c' \leq A$. Hence P is (β, α)-interpolated. The reverse implication can be proved analogously. □

3.6 Proposition. *Let α, β be infinite cardinals, β be a limit cardinal. Then a poset is (α, β)-interpolated ((β, α)-interpolated, resp.) if and only if it is (α, γ)-interpolated ((γ, α)-interpolated resp.) for every infinite cardinal $\gamma < \beta$.*

Proof. Obvious. □

According to the previous proposition, for a limit cardinal we have the interpolation if and only if we have the interpolation for all smaller cardinals. The following example shows that this is not true for any nonlimit cardinal.

3.7 Example. Let X_1, X_2, X_3, X_4 be disjoint sets of an infinite set cardinality α, $X = \bigcup_{i=1}^{4} X_i$. Let us put

$$E_0 = \{\emptyset, X_1 \cup X_2, X_2 \cup X_3, X_3 \cup X_4, X_4 \cup X_1, X\},$$
$$E = \left\{A \subset X\colon \operatorname{card}\big((A \setminus A_0) \cup (A_0 \setminus A)\big) < \alpha \text{ for some } A_0 \in E_0\right\}.$$

Then $(E, \subset, {}^{\mathrm{c}}, \emptyset, X)$ with $A^{\mathrm{c}} = X \setminus A$ for every $A \in E$ is an effect algebra.

Let us show that E is α-interpolated. Let $F, G \subset E$ be nonempty, $F \leq G$, $\operatorname{card} F, \operatorname{card} G < \alpha$. If one of the sets is one-element then this element is an interpolation of F, G. Let us suppose that both F, G are at least two-element. Every element of E is derived from some element of E_0 using a symmetric difference with a subset of X of cardinality less than α. If a nonempty set of elements of E derived from $A \in E_0$ has the cardinality less than α then both its union and intersection is an element of E derived from A. If there are at least two elements of $E_0 \setminus \{\emptyset, X\}$ used for elements of F or some element of F is derived from X then every element of G is derived from X and therefore $\bigwedge G = \bigcap G$ is an interpolation of F, G. If there is at most one element of $E_0 \setminus \{\emptyset, X\}$ used for elements of F then $\bigvee F = \bigcup F$ is an interpolation of F, G.

Let us consider $F = \big\{\{x\}\colon\ x \in X_1\big\}$, $G = \{X_1 \cup X_2, X_1 \cup X_4\}$. Then $F \leq G$, $\operatorname{card} F = \alpha$, $\operatorname{card} G = 2$, every upper bound of F has the cardinality α, every lower bound of G has the cardinality less than α. Hence, E is not $(\alpha + 1, \omega_0)$-interpolated.

4 Interpolation and completeness

Let us show some relations of α-interpolated posets to posets with some sort of completeness. The notion of weak orthocompleteness was introduced by Ovchinnikov [6] in case of orthomodular lattices and generalized by Tkadlec [8] in case of effect algebras. We present here a more general definition involving cardinalities and also a weak form of completeness.

4.1 Definition. Let α be an infinite cardinal. A poset $(P, \leq)$ is *α-complete* if for every nonempty set $A \subset P$ both $\bigvee A$ and $\bigwedge A$ exist. It is *weakly α-complete* if for every nonempty set $A \subset P$ (1) either $\bigvee A$ exists or A has no minimal upper bound; (2) either $\bigwedge A$ exists or A has no maximal lower bound.

There are standard special cases: ω_0-complete poset is called a *lattice*; a lattice that is α-complete for the greatest cardinal α, i.e., that is β-complete for every set cardinal β, is called *complete*. It suffices to check only one condition (e.g., for $\bigvee A$) in de Morgan posets.

4.2 Definition. Let E be an effect algebra. A system $(a_i\colon\ i \in I)$ of elements of E is *orthogonal* if $\bigoplus_{i\in F} a_i$ is defined for every finite set $F \subset I$.

A *majorant* of an orthogonal system is an upper bound of all its finite sums.

The *sum* $\bigoplus(a_i\colon\ i \in I)$ of an orthogonal system is its least majorant (if it exists).

Le α be an infinite cardinal. An effect algebra E is *α-orthocomplete* if every its orthogonal system $(a_i\colon\ i \in I)$ with $\operatorname{card} I < \alpha$ has the sum. An effect algebra E is *weakly α-orthocomplete* if every its orthogonal system $(a_i\colon\ i \in I)$ with $\operatorname{card} I < \alpha$ either has the sum or has no minimal majorant.

Even a nonzero element of an effect algebra might be orthogonal to itself, hence we need to consider systems (containing possibly some elements more than once) instead of sets. Every pair of elements of an orthogonal system is orthogonal. On the other hand, there are mutually orthogonal elements that do not form an orthogonal system if the effect algebra is not an orthomodular poset. Every effect algebra is ω_0-orthocomplete.

The second part of the following statement was proved by Bennett and Foulis [1, Lemma 2.6 (i)].

4.3 Proposition. *Let α be an infinite cardinal. Every α-complete poset is α-interpolated. In particular, every lattice is ω_0-interpolated.*

Proof. Let P be an α-complete poset $A, B \subset P$ be nonempty, $A \le B$, $\operatorname{card} A, \operatorname{card} B < \alpha$. Then $\bigvee A$ ($\bigwedge B$) is an interpolation of A, B. □

4.4 Proposition. *Let α be an infinite cardinal. Every α-interpolated poset is weakly α-complete.*

Proof. Let P be an α-interpolated poset, $A \subset P$ be nonempty with $\operatorname{card} A < \alpha$. Then the set of upper bounds of A is downward directed, hence every its minimal element is its least element. Analogously for the set of lower bounds of A. □

4.5 Proposition. *Let α be an infinite cardinal. Every (α, ω_0)-interpolated effect algebra is weakly α-orthocomplete.*

Proof. Let E be an (α, ω_0)-interpolated effect algebra and $O = (a_i\colon\ a_i \in I)$ be an orthogonal system in E with $\operatorname{card} I < \alpha$. Let us denote by A the set of finite sums of O. Then $\operatorname{card} A < \alpha$ and, according to the assumption, the set of majorants of O (i.e., upper bounds of A) is downward directed. Hence it either has the least element (the sum of O) or no minimal element. □

The following result generalizes the result of Bennett and Foulis [1, Lemma 2.6 (ii)] stated for bounded finite posets with the interpolation property.

4.6 Proposition. *Every ω_0-interpolated pseudolattice is a lattice.*

Proof. Let P be na ω_0-interpolated pseudolattice, $a, b \in P$. Since P is a pseudolattice, there is a maximal lower bound c of $\{a, b\}$. For every lower bound d of $\{a, b\}$ we have $\{c, d\} \le \{a, b\}$ and therefore there is an interpolation e of these two sets; since c is maximal, $e = c$ and therefore $d \le c$. It means that $c = a \wedge b$. The existence of $a \vee b$ can be proved analogously. □

5 Interpolation and compatibility

First, let us recall the definition of compatibility and some known properties of effect algebras.

5.1 Definition. Elements a, b of an effect algebra E are *compatible* (denoted by $a \leftrightarrow b$) if there is an orthogonal system (*Mackey decomposition* of a, b) (a_1, b_1, c) in E such that $a = a_1 \oplus c$ and $b = b_1 \oplus c$.

5.2 Lemma. *Elements a, b of an effect algebra E are compatible if and only if there is an element $c \in E$ such that $c \leq a, b$ and $a \perp (b \ominus c)$.*

Proof. $\Rightarrow$: Let $a, b \in E$ are compatible. Then there are is a Mackey decomposition (a_1, b_1, c) of a, b and therefore $a = (a_1 \oplus c) \perp b_1 = (b \ominus c)$.

$\Leftarrow$: Let $a, b, c \in E$ such that $c \leq a, b$ and $a \perp (b \ominus c)$. Then $a \oplus (b \ominus c) = (a \ominus c) \oplus c \oplus (b \ominus c)$ is defined and therefore $(a \ominus c, b \ominus c, c)$ is a Mackey decomposition of a, b. $\square$

5.3 Lemma. *Let E be an effect algebra, $a, b, c, d \in E$, $B \subset E$.*

(1) *If $b \perp (a \ominus c)$ and $c \leq d \leq a$ then $b \perp (a \ominus d)$.*

(2) *If $a \perp B$ and $\bigvee B$ exists then $a \perp \bigvee B$.*

(3) *If $a \leftrightarrow b$ then $a \leftrightarrow b'$.*

Proof. (1) If $c \leq d \leq a$ then $c \oplus (a \ominus c) = a = d \oplus (a \ominus d) = c \oplus (d \ominus c) \oplus (a \ominus d)$. Using the cancellation law we obtain $a \ominus c = (d \ominus c) \oplus (a \ominus d)$, therefore $a \ominus d \leq a \ominus c$. If $b \perp (a \ominus c)$ then $(a \ominus c) \leq b'$, therefore $(a \ominus d) \leq b'$, i.e., $b \perp (a \ominus d)$.

(2) For every $b \in B$, if $a \perp b$ then $b \leq a'$. Hence $\bigvee B \leq a'$, i.e., $a \perp \bigvee B$.

(3) If $a \leftrightarrow b$ then, according to Lemma 5.2 there is a $c \in E$ such that $c \leq a, b$ and $(a \ominus c) \perp b$. Then $(a \ominus c) \leq a, b'$ and, since $c \leq b$, $a \ominus (a \ominus c) = c \perp b'$. According to Lemma 5.2, $a \leftrightarrow b'$. $\square$

The following theorem is an analogue of the result of Jenča and Riečanová [5, Theorem 2.1], which was stated for lattice (instead of interpolated) effect algebras.

5.4 Theorem. *Let E be an effect algebra, $a \in E$, $B \subset E$, $a \leftrightarrow B$ and E is $(\operatorname{card} B + 1, \omega_0)$-interpolated. If $\bigvee B$ ($\bigwedge B$, resp.) exists, then $a \leftrightarrow \bigvee B$ ($a \leftrightarrow \bigwedge B$, resp.).*

Proof. We obtain for every $b \in B$: since $a \leftrightarrow b$ then, according to Lemma 5.2, there is a $c_b \in E$ such that $c_b \leq \{a, b\}$ and $b \perp (a \ominus c_b)$. Hence $C = \{c_b : b \in B\} \leq \{a, \bigvee B\}$ and, according to the assumptions, there is a $c \in E$ such that $C \leq c \leq \{a, \bigvee B\}$. According to Lemma 5.3 (1), $b \perp (a \ominus c)$, according to Lemma 5.3 (2), $\bigvee B \perp (a \ominus c)$, according to Lemma 5.2, $a \leftrightarrow \bigvee B$.

If $\bigwedge B$ exists then $\bigvee B' = (\bigwedge B)'$ exists and, according to Lemma 5.3 (3), $a \leftrightarrow B'$. Using the first part of the theorem we obtain that $a \leftrightarrow \bigvee B'$ and, according to Lemma 5.3 (3), $a \leftrightarrow (\bigvee B')' = \bigwedge B$. $\square$

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