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Note on generalizations of orthocomplete and lattice effect algebras

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Abstract It is proved that two different common generalizations of orthocomplete and lattice effect algebras coincide within the class of separable Archimedean effect algebras.

Keywords Effect algebra, orthocomplete, lattice, separable, Archimedean

Ovchinnikov [4] introduced weakly orthocomplete orthomodular posets (he called them alternative) as a common generalization of orthocomplete orthomodular posets and orthomodular lattices and showed that they are disjunctive. Weak orthocompleteness is useful in the study of orthoatomisticity and disjunctivity might be used to characterize atomisticity [4, 8]. Weak orthocompleteness was generalized by De Simone and Navara [1] to the so-called property (W+), which was generalized for effect algebras by Tkadlec [10].

Tkadlec [5] introduced the class of orthomodular posets with the maximality property as another common generalization of orthocomplete orthomodular posets and orthomodular lattices. He showed various consequences of this property and generalized it to the so-called property (CU) [5, 6, 7, 9].

Both properties (W+) and (CU) are generalizations of orthocomplete effect algebras and lattice effect algebras [9, 10]. We show that in the class of separable Archimedean effect algebras these properties are equivalent.

1 Basic notions and properties

Definition 1.1 An *effect algebra* is an algebraic structure $(E, \oplus, \mathbf{0}, \mathbf{1})$ such that E is a set, $\mathbf{0}$ and $\mathbf{1}$ are different elements of E and $\oplus$ is a partial binary operation on E such that for every $a, b, c \in E$ the following conditions hold:

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(1) $a \oplus b = b \oplus a$ if $a \oplus b$ exists,
(2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if $(a \oplus b) \oplus c$ exists,
(3) there is a unique $a' \in E$ such that $a \oplus a' = \mathbf{1}$ (*orthosupplement*),
(4) $a = \mathbf{0}$ if $a \oplus \mathbf{1}$ is defined.

For simplicity, we use the notation E for an effect algebra. A partial ordering on an effect algebra E is defined by $a \leq b$ if there is a $c \in E$ such that $b = a \oplus c$. Such an element c is unique (if it exists) and is denoted by $b \ominus a$. $\mathbf{0}$ ($\mathbf{1}$, resp.) is the least (the greatest, resp.) element of E with respect to this partial ordering. For every $a, b \in E$, $a'' = a$ and $b' \leq a'$ whenever $a \leq b$. An *orthogonality* relation on E is defined by $a \perp b$ if $a \oplus b$ exists. See, e.g., Dvurečenskij and Pulmannová [2], Foulis and Bennett [3].

Definition 1.2 Let E be an effect algebra. The *isotropic index* $i(a)$ of an element $a \in E$ is $\sup\{n \in \mathbb{N} \colon na \text{ is defined}\}$, where $na = \bigoplus_{i=1}^{n} a$ is the sum of n copies of a.

An effect algebra E is *Archimedean* if every nonzero element has a finite isotropic index.

The isotropic index of $\mathbf{0}$ is ∞ in every effect algebra.

Definition 1.3 Let E be an effect algebra.

A nonempty system $(a_i)_{i \in I}$ of (not necessarily distinct) elements of E is called *orthogonal*, if sums of all finite subsystems are defined.

An element $a \in E$ is a *majorant* of an orthogonal system O if it is an upper bound of all sums of finite subsystems of O.

Definition 1.4 An effect algebra is *separable* if every orthogonal system of its distinct elements is countable.

It is easy to see that an Archimedean effect algebra is separable if and only if every orthogonal system of its nonzero elements is countable. On the other hand, there is an uncountable orthogonal system of nonzero elements in every non-Archimedean effect algebra—for a nonzero non-Archimedean element a and an uncountable index set I we can take the orthogonal system $(a)_{i \in I}$.

2 Results

First, let us define our main notions.

Definition 2.1 Let E be a partially ordered set.

A *chain* in E is a nonempty linearly (totally) ordered subset of E.

An element $a \in E$ is an *upper bound* of a set $S \subseteq E$ if $s \leq a$ for every $s \in S$.

A set $S \subseteq E$ is *downward directed* if for every $a, b \in S$ there is a $c \in S$ such that $c \leq a, b$.

Definition 2.2 An effect algebra E fulfills the condition (CU) if the set of upper bounds of every chain in E is downward directed.

An effect algebra E fulfills the condition (W+) if the set of majorants of every orthogonal system in E is downward directed.

Let us remark that these properties are not equivalent [10, Example 2.7]. The next theorem was proved in [10]:

Theorem 2.3 *Every separable effect algebra fulfilling the condition (CU) fulfills the condition (W+).*

Let us present auxiliary results concerning general properties of chains. These results seem to be known but we did not find a proper reference to them.

Let us say that a chain $\{c_\beta\colon\ \beta \in \alpha\}$ for some ordinal α is *ordered* if $c_\beta < c_\gamma$ for every $\beta, \gamma \in \alpha$ with $\beta < \gamma$.

Lemma 2.4 *Let E be a partially ordered set and $C \subseteq E$ be a chain. Then there is an ordinal α and an ordered subchain $\{c_\beta\colon\ \beta \in \alpha\}$ of C such that for every $c \in C$ there is a $\beta \in \alpha$ with $c \leq c_\beta$.*

Proof We will construct the desired ordered subchain by the transfinite induction. Let us take an arbitrary $c_0 \in C$. Let α be an ordinal such that c_α has not been defined yet and c_β has been already defined for every $\beta \in \alpha$. If the set $C_\alpha = \{c_\beta\colon\ \beta \in \alpha\}$ has an upper bound $c \in C \setminus C_\alpha$ then we put $c_\alpha = c$. If there is no such c, the construction is complete. This construction stops before we reach an ordinal with cardinality greater then the cardinality of C. □

The main meaning of the previous lemma is that we are able to find an ordered subchain indexed by an ordinal with the same set of upper bounds. Now let us show that in the countable case (i.e. finite or with the cardinality of the set of natural numbers $\mathbb{N} = \{0, 1, \dots\}$) the indexing ordinal might be the least infinite ordinal ω or a finite ordinal (natural number). In the latter case it can be even the number 1.

Lemma 2.5 *Let E be a partially ordered set and $C \subseteq E$ be a countable chain. Then there is an ordinal $\alpha \leq \omega$ and an ordered subchain $\{c_\beta\colon\ \beta \in \alpha\}$ of C such that for every $c \in C$ there is a $\beta \in \alpha$ with $c \leq c_\beta$.*

Proof If C has a maximal element c then we can take $\alpha = 1$ and $c_0 = c$. Let us suppose that C does not have a maximal element.

C is infinite and therefore there is a bijection $f\colon\ \mathbb{N} \to C$. For every $k \in \mathbb{N}$, we will put $c_k = f(n_k)$ for an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers defined by the following induction. Let us put $n_0 = 0$. Let k be a natural number such that c_{k+1} has not been defined yet and c_k has been already defined. Since the set C does not have a maximal element, the set $\{c \in C\colon\ c > f(n_k)\}$ is infinite and therefore the set $N_k = \{n \in \mathbb{N}\colon\ n > n_k,\ f(n) > f(n_k)\}$ is nonempty. We put $n_{k+1} = \min N_k$.

The chain $\{c_k\colon\ k \in \mathbb{N}\}$ is ordered. Let us check the desired property. For every $c \in C$ there is an $n \in \mathbb{N}$ such that $c = f(n)$ and, since $(n_k)_{k\in\mathbb{N}}$ is increasing, a $k \in \mathbb{N}$ such that $n_k \leq n < n_{k+1}$. Due to our construction, $c = f(n) \leq f(n_k) = c_k$. □

Theorem 2.6 *Every separable Archimedean effect algebra fulfilling the condition (W+) fulfills the condition (CU).*

Proof Let E be a separable Archimedean effect algebra and let C be a chain in E. According to Lemma 2.4 there is an ordinal α and an ordered subchain $\{c_\beta\colon\ \beta \in \alpha\}$ with the same set of upper bounds. The system $O = \big(c_{\beta+1} \ominus c_\beta \colon \beta, (\beta+1) \in \alpha\big)$ is orthogonal. Since E is separable and Archimedean, O is countable and therefore α is countable. Hence, according to Lemma 2.5, there is an ordinal $\alpha' \le \omega$ and an ordered subchain $\{c'_\beta \colon \beta \in \alpha'\}$ with the same set of upper bounds. Let a, b be upper bounds of C. Then a, b are majorants of the orthogonal system $O' = (c'_0) \cup \big(c'_{\beta+1} \ominus c'_\beta \colon \beta, (\beta+1) \in \alpha'\big)$. According to the condition (W+), there is a majorant $d \le a, b$ of O'. Hence d is an upper bound of $\{c'_\beta \colon \beta \in \alpha'\}$ and therefore of C. □

Corollary 2.7 *The conditions (CU) and (W+) are equivalent in separable Archimedean effect algebras.*

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References

1. De Simone, A., Navara, M.: Yosida-Hewitt and Lebesgue decompositions of states on orthomodular posets. Tech. Report no. 40, Università "Federico II" Napoli, Italy (1997)
2. Dvurečenskij, A., Pulmannová, S.: New Trends in Quantum Structures. Kluwer Academic, Bratislava (2000)
3. Foulis, D. J., Bennett, M. K.: Effect algebras and unsharp quantum logics. Found. Phys. **24**, 1331–1352 (1994). doi: 10.1007/BF02283036
4. Ovchinnikov, P. G.: On alternative orthomodular posets. Demonstr. Math. **27**, 89–93 (1994)
5. Tkadlec, J.: Conditions that force an orthomodular poset to be a Boolean algebra. Tatra Mt. Math. Publ. **10**, 55–62 (1997)
6. Tkadlec, J.: Central elements of effect algebras. Int. J. Theor. Phys. **43**, 1363–1369 (2004). doi: 10.1023/B:IJTP.0000048621.17418.bb
7. Tkadlec, J.: Atomic sequential effect algebras. Int. J. Theor. Phys. **47**, 185–192 (2008). doi: 10.1007/s10773-007-9492-1
8. Tkadlec, J.: Atomistic and orthoatomistic effect algebras. J. Math. Phys. **49**, 053505 (2008). doi: 10.1063/1.2912228
9. Tkadlec, J.: Effect algebras with the maximality property. Algebra Universalis **61**, 187–194 (2009). doi: 10.1007/s00012-009-0013-3
10. Tkadlec, J.: Common generalizations of orthocomplete and lattice effect algebras. Int. J. Theor. Phys. **49**, 3279–3285 (2010). doi: 10.1007/s10773-009-0108-9