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PARTIALLY ADDITIVE STATES ON ORTHOMODULAR POSETS

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We fix a Boolean subalgebra B of an orthomodular poset P and study the mappings $s : P \to [0,1]$ which respect the ordering and the orthocomplementation in P and which are additive on B. We call such functions B-states on P. We first show that every P possesses "enough" two-valued B-states. This improves the main result in [13], where B is the centre of P. Moreover, it allows us to construct a closure-space representation of orthomodular lattices. We do this in the third section. This result may also be viewed as a generalization of [6]. Then we prove an extension theorem for B-states giving, as a by-product, a topological proof of a classical Boolean result.

1. Basic definitions and preliminaries.

1.1. Definition. An *orthomodular poset* (abbr. an OMP) is a triple $(P, \leq, ')$ such that

(1) $(P, \leq)$ is a partially ordered set with a least element 0 and a greatest element 1,
(2) the operation $' : P \to P$ is an *orthocomplementation*, i.e. for every $a, b \in P$ we have $a'' = a$ and $b' \leq a'$ whenever $a \leq b$,
(3) the least upper bound exists for every pair of orthogonal elements in P ($a, b \in P$ are *orthogonal*, $a \perp b$, if $a \leq b'$),
(4) the *orthomodular law* is valid in P: $b = a \vee (b \wedge a')$ whenever $a \leq b$ ($a, b \in P$).

A typical example of an OMP is the lattice of all projections in a Hilbert space or, of course, a Boolean algebra. (We do not assume that P is a lattice. If it is, we call it an *orthomodular lattice*.)

Throughout the paper, P will be an arbitrary OMP and B an arbitrary Boolean subalgebra of P. (By a *Boolean subalgebra* of P we mean a subset of P which forms a Boolean algebra with respect to $\leq$ and $'$ inherited from P, see also [4], [7].) Let us state our basic definition.

1.2. Definition. Let B be a Boolean subalgebra of P. A *partially additive state with respect to* B (abbr. a B-state) is a mapping $s : P \to [0,1]$ such that

(1) if $a \leq b$ then $s(a) \leq s(b)$ $(a, b \in P)$,
(2) $s(a') = 1 - s(a)$ $(a \in P)$,
(3) $s(a \vee b) = s(a) + s(b)$ provided $a \perp b$ and $a, b \in B$.

Let us denote the set of all B-states on P by $S_B(P)$. Thus $S_B(P) \subset [0,1]^P$. In what follows we will make use of the following observations: The set $S_B(P)$ viewed as a subset of $[0,1]^P$ is a convex compact. Indeed, the convexity is obvious and the compactness is a standard consequence of the Tikhonov theorem ($[0,1]^P$ is considered with the pointwise topology). It should be noted that the "ordinary" state on P is exactly an element of the intersection $\bigcap S_B(P)$, where B runs over all Boolean subalgebras of P.

2. Two-valued B-states. B-ideals. Let us denote by $S_B^2(P)$ the set of all two-valued B-states on P. We will show in this section that $S_B^2(P)$ is rich enough to determine the ordering in P. This extends [13] which contains the same result in the much easier situation of B being the centre of P.

Let us first introduce an auxiliary notion.

2.1. DEFINITION. Let B be a Boolean subalgebra of P. A *partial ideal I on P with respect to B* (abbr. a B-ideal) is a nonempty subset of P such that

(A) if $a \in I$ and $b \leq a$ then $b \in I$ $(a, b \in P)$,
(B) $a \vee b \in I$ provided $a, b \in I \cap B$.

Further, we call a B-ideal I *proper* if

(C1) $a \in I$ implies $a' \notin I$.

Finally, we call a proper B-ideal I a *B-prime ideal* if

(C2) $a \in P \setminus I$ implies $a' \in I$.

In what follows we will sometime replace without noticing the condition (B) by the apparently weaker condition (B') equivalent to (B):

(B') $a \vee b \in I$ provided $a, b \in I \cap B$ and $a \perp b$.

The link between two-valued B-states and B-ideals is presented in the following simple proposition.

2.2. PROPOSITION. *There is a one-to-one correspondence between two-valued B-states and B-prime ideals given by the mapping $s \mapsto s^{-1}(0)$.*

Proof. Obvious.

In the course of the following propositions we will show that any pair of noncomparable elements in P is separated by a B-prime ideal.

2.3. Proposition. *Let* $\{I_\alpha;\ \alpha \in A\}$ *be a collection of* B*-ideals in* P*. Then the least* B*-ideal containing all* I_α $(\alpha \in A)$ *is* $J = \bigcup\{I_\alpha;\ \alpha \in A\} \cup \{a \in P;\ a \le b_1 \vee \cdots \vee b_n,$ *where* $b_k \in I_{\alpha_k} \cap B$ *for any* $k \in \{1, \ldots, n\}\}$*.*

Proof. The proof requires only a verification of the properties from the definition of a B-ideal.

Let us agree to call the B-ideal J from Proposition 2.3 the *B-ideal generated by* $\{I_\alpha;\ \alpha \in A\}$.

Prior to the next propositions, observe that the elements b_k $(k \in \{1, \ldots, n\})$ in Proposition 2.3 can be chosen pairwise orthogonal.

2.4. Proposition. *Let* $I \subset P$ *be a proper* B*-ideal. Suppose that* $\{a, a'\} \cap I = \emptyset$ *for an* $a \in P$*. Then the* B*-ideal generated by* $\{I, [0, a]\}$ *is proper.*

Proof. Suppose that the B-ideal J generated by $\{I, [0, a]\}$ is not proper and seek a contradiction. If J is not proper, then there is an $e \in P$ such that $\{e, e'\} \subset J$. Observe that $\{e, e'\} \not\subset I \cup [0, a]$. Indeed, both e, e' cannot be in I and if $e \in [0, a]$ then $a' \le e'$; hence $e' \notin I$.

According to Proposition 2.3 we may assume that $e \le b_1 \vee b_2$, $b_1 \in I \cap B$, $b_2 \in [0, a] \cap B$ and $b_1 \perp b_2$. We may also assume without any loss of generality that $e = b_1 \vee b_2$. Hence $e' \in J \cap B$ and therefore there are $b_3 \in I \cap B$, $b_4 \in [0, a] \cap B$ such that $b_3 \perp b_4$ and $e' = b_3 \vee b_4$. Then b_1, b_2, b_3, b_4 are pairwise orthogonal and, moreover, $1 = e \vee e' = b_1 \vee b_2 \vee b_3 \vee b_4$. Thus, $a' \le (b_2 \vee b_4)' = b_1 \vee b_3 \in I$, a contradiction.

2.5. Proposition. *Each proper* B*-ideal is contained in a* B*-prime ideal.*

Proof. By Zorn's lemma, each proper B-ideal is contained in a maximal proper B-ideal. By Proposition 2.4, each maximal proper B-ideal is a B-prime ideal.

2.6. Proposition. *Suppose that* $a \not\le b$ $(a, b \in P)$*. Then there exists a* B*-prime ideal* I *such that* $a \notin I$ *and* $b \in I$*.*

Proof. By Proposition 2.4, the B-ideal generated by $\{[0, b], [0, a']\}$ is proper. The rest follows from Proposition 2.5.

2.7. Theorem. *Let* B *be a Boolean subalgebra of* P*. Suppose that* $a \not\le b$ $(a, b \in P)$*. Then there exists a two-valued* B*-state* $s \in S_B^2(P)$ *such that* $s(a) = 1$ *and* $s(b) = 0$*.*

Proof. This follows immediately from Propositions 2.6 and 2.2.

In the next section we will need the following result.

2.8. THEOREM. *Let B, B_1 be Boolean subalgebras of P. Let s_1 be a two-valued state on B_1. Then there exists a two-valued B-state s on P such that $s|B_1 = s_1$.*

Proof. Put $I_1 = s_1^{-1}(0)$. Put further $J = \{b \in P;$ there exists $a \in I_1$ with $b \le a\}$. Then J is a proper B-ideal and, according to Proposition 2.5, J is contained in a B-prime ideal I. I_1 is a prime ideal on B_1, hence $I \cap B = I_1$. The rest follows from Proposition 2.2.

As the following example (due to Mirko Navara) shows, Theorem 2.7 cannot be improved in such a way that $s \in S_{B_1}(P) \cap S_{B_2}(P)$ for given Boolean subalgebras B_1, B_2 of P.

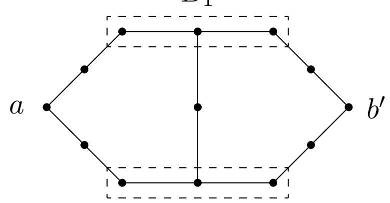

Fig. 1

2.9. EXAMPLE. Figure 1 shows the Greechie diagram (see [3]) of an orthomodular lattice P. The elements $a, b' \in P$ are not orthogonal, hence $a \not\le b$, but there is no $s \in S_{B_1}(P) \cap S_{B_2}(P)$ such that $s(a) = 1$ and $s(b) = 0$.

3. A representation theorem for orthomodular lattices. The main result in this section is a representation of P by means of clopen sets in a compact Hausdorff closure space (a generalized Stone representation). We will show as an improvement of [6] (where B is the centre of P) that if P is a lattice and if we are given a Boolean subalgebra B in P, we can ensure that the restriction of the representation to B becomes the Stone representation.

First we reformulate results of the previous section in a way convenient for our representation theorem.

3.1. PROPOSITION. *Let $\mathcal{P}$ be the set of all B-prime ideals in P. Let the mapping $i : P \to \exp \mathcal{P}$ be defined by $i(a) = \{I \in \mathcal{P};\ a \notin I\}$. Finally, write $\overline{\mathcal{A}} = \bigcap\{i(b);\ b \in P$ and $\mathcal{A} \subset i(b)\}$ for any $\mathcal{A} \subset \mathcal{P}$. Then*

1) *$i(0) = \emptyset$, $i(1) = \mathcal{P}$ and $i : (P, \le, ') \to (i(P), \subset, ')$ is an isomorphism,*

2) *if $\mathcal{A}_\alpha \in i(P)$ $(\alpha \in A)$ and $\bigvee_{\alpha \in A} \mathcal{A}_\alpha$ exists in $(i(P), \subset, ')$, then $\bigvee_{\alpha\in A} \mathcal{A}_\alpha = \overline{\bigcup_{\alpha \in A} \mathcal{A}_\alpha}$,*

3) *if $\mathcal{A}, \mathcal{B} \in i(B)$ then $\mathcal{A} \vee \mathcal{B} = \mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \wedge \mathcal{B} = \mathcal{A} \cap \mathcal{B}$.*

P r o o f. The first property follows from the definition of i and from Theorem 2.7. As for the second property, we know that $\bigvee_{\alpha\in A}\mathcal{A}_\alpha \in i(P)$ contains all $\mathcal{A}_\alpha$ $(\alpha \in A)$ and therefore $\bigcup_{\alpha\in a}\mathcal{A}_\alpha \subset \bigvee_{\alpha\in A}\mathcal{A}_\alpha$. Then the equality $\overline{\bigcup_{\alpha\in a}\mathcal{A}_\alpha} = \bigvee_{\alpha\in A}\mathcal{A}_\alpha$ follows from the definition of the "bar" operation. Finally, suppose that $I \in i(a\vee b)$, $a, b \in B$. Then $a \vee b \notin I$ and therefore either $a \notin I$ or $b \notin I$. Hence $I \in i(a) \cup i(b)$. Thus, $i(a \vee b) = i(a) \cup i(b)$ and we have $\mathcal{A} \vee \mathcal{B} = \mathcal{A} \cup \mathcal{B}$. Dually, $\mathcal{A} \wedge \mathcal{B} = (\mathcal{A}' \vee \mathcal{B}')' = (\mathcal{A}' \cup \mathcal{B}')' = \mathcal{A} \cap \mathcal{B}$.

Prior to stating our main result in this section let us shortly review basic facts on closure spaces (see [2], [6]). By a *closure space* we mean a pair $(X, \overline{\ })$, where X is a nonempty set and $\overline{\ } : \exp X \to \exp X$ is an operation which has the following four properties:

(1) $\overline{\emptyset} = \emptyset$,
(2) $A \subset \overline{A}$ for any $A \subset X$,
(3) $A \subset B$ implies $\overline{A} \subset \overline{B}$ $(A, B \subset X)$,
(4) $\overline{\overline{A}} = \overline{A}$ for any $A \subset X$.

A set $A \subset X$ is called *closed* in $(X, \overline{\ })$ if $\overline{A} = A$ and $B \subset X$ is called *open* if $X \setminus B$ is closed. A closure space $(X, \overline{\ })$ is called *Hausdorff* if any pair of points in X can be separated by disjoint open sets, and $(X, \overline{\ })$ is called *compact* if any open covering of X has a finite subcovering. It should be noted that the intersection of any collection of closed sets is again a closed set. However, the union of two closed sets need not be closed.

Let us agree to write $CO(X)$ for the collection of all subsets of X which are simultaneously closed and open.

3.2. THEOREM. *Let $\mathcal{P}$, i and $\overline{\ }$ have the same meaning as in Proposition 3.1. Then $\mathcal{P}$ is a compact Hausdorff closure space and $i(P) \subset CO(\mathcal{P})$. If P is a lattice, then $i(P) = CO(\mathcal{P})$.*

P r o o f. One verifies easily that $\mathcal{P}$ is a closure space. Suppose that $a \in P$. Then $i(a) = \overline{i(a)}$ and therefore $i(a)$ is closed. Also, $i(a) = i(a'') = i(a')'$ and therefore $i(a)$ is open. Thus $i(P) \subset CO(\mathcal{P})$. This allows us to prove that $(\mathcal{P}, \overline{\ })$ is Hausdorff and compact. Indeed, if $I_1, I_2 \in \mathcal{P}$ and $I_1 \neq I_2$, then there is an element $a \in P$ such that $a \in I_1 \setminus I_2$ $(a' \in I_2 \setminus I_1)$. We therefore have two disjoint open sets $i(a), i(a')$ which separate I_1, I_2.

To show that $\mathcal{P}$ is compact, consider an open covering $\{\mathcal{A}_\alpha;\ \alpha \in A\}$ of P. Since every closed set in $\mathcal{P}$ is an intersection of elements of $i(P)$, every open set is a union of elements of $i(P)$. We therefore may (and will) suppose that $\mathcal{A}_\alpha = i(a_\alpha)$ $(a_\alpha \in P, \alpha \in A)$. Hence there is no B-prime ideal I such that $I \supset \{a_\alpha;\ \alpha \in A\}$. This means that the B-ideal J generated by $\{[0, a_\alpha];\ \alpha \in A\}$ is not proper, It follows that for some $d \in P$ we have one of the following possibilities (see Proposition 2.3): Either $d \in [0, a_{\alpha_1}]$, $d' \in [0, a_{\alpha_2}]$ $(\alpha_1, \alpha_2 \in A)$ or $d \le b_1 \vee \cdots \vee b_n$ for $b_k \in B \cap [0, a_{\alpha_k}]$ $(\alpha_k \in A$,

$k \in \{1, \ldots, n\}$), $d' \in J$. In the former case $a'_{\alpha_1} \leq d' \leq a_{\alpha_2}$ and therefore $\mathcal{P} = i(a_{\alpha_1}) \cup i(a'_{\alpha_1}) \subset i(a_{\alpha_1}) \cup i(a_{\alpha_2})$. In the latter case we may (and will) assume the equality instead of the inequality. thus, we have $d \in B$. Hence $d' \in J \cap B$ and therefore we can write $d' = \tilde{b}_1 \vee \cdots \vee \tilde{b}_m$ ($b_k \in [0, a_{\alpha_k}] \cap B$, $\alpha_k \in A$, $k \in \{1, \ldots, m\}$). Then we have

$$\begin{aligned} \mathcal{P} &= i(d \vee d') = i(b_1 \vee \cdots \vee b_n \vee \tilde{b}_1 \vee \cdots \vee \tilde{b}_m) \\ &= i(b_1) \cup \cdots \cup i(b_n) \cup i(\tilde{b}_1) \cup \cdots \cup i(\tilde{b}_m) \\ &\subset i(a_{\alpha_1}) \cup \cdots \cup i(a_{\alpha_m}). \end{aligned}$$

Thus, in both cases we have found a finite subcovering of $\{\mathcal{A}_\alpha;\ \alpha \in A\}$.

Suppose now that P is a lattice and $\mathcal{A} \in CO(\mathcal{P})$. According to the definition of the closure operation we may write $\mathcal{A} = \bigcup_{\alpha \in A} i(a_\alpha)$ for some $a_\alpha \in P$. Making use of the compactness of $\mathcal{P}$ we have $\mathcal{A} = \bigcup_{k=1}^n i(a_{\alpha_k})$ ($\alpha_k \in A$, $k \in \{1, \ldots, n\}$). Thus, $\mathcal{A} = \bigvee_{k=1}^n i(a_{\alpha_k}) = i(\bigvee_{k=1}^n a_{\alpha_k}) \in i(P)$.

Before we state our last result in this section, recall that a mapping $f : L_1 \to L_2$ between two orthomodular lattices is called *orthoisomorphism* if f is one-to-one and respects ordering and orthocomplementation.

3.3. THEOREM. *Let B be a Boolean subalgebra of an orthomodular lattice P. Then there exists a compact Hausdorff closure space $\mathcal{P}$ such that P is orthoisomorphic to $CO(\mathcal{P})$. Moreover, the orthoisomorphism $f : P \to CO(\mathcal{P})$ can be taken such that $f(B)$ is the Stone representation of B.*

Proof. This follows from Theorems 3.2 and 2.8.

4. Extensions of B-states. It is obvious that a trace of a B-state on B is a state. It is natural to ask whether any state on B is a trace of a B-state, i.e. whether the restriction $r : S_B(P) \to S(B)$ is onto. In Theorem 2.8 we have showed that this is true for two-valued states. Here we generalize this result to arbitrary states on B.

4.1. THEOREM. *Let B, B_1 be Boolean subalgebras of P. If s_1 is a state on B_1, then there exists a B-state s on P such that $s|B_1 = s_1$.*

Proof. We use the compactness of $S = S_B(P) \cap S_{B_1}(P)$. In some places we partially utilize the technique of [11] and [10].

Let s_1 be a state on B_1 and let $D = \{d_1, \ldots, d_n\}$ be a partition of B_1. Thus, $\bigvee_{k=1}^n d_k = 1$ and $d_i \perp d_j$ for $i \neq j$ ($i, j \in \{1, \ldots, n\}$). Put $F_D = \{s \in S;\ s|D = s_1|D\}$. Let $\mathcal{D}$ denote the set of all partitions of B_1. We will show that $\mathcal{F} = \{F_D;\ D \in \mathcal{D}\}$ is a filter base consisting of nonempty closed sets in S. First, every set F_D is closed by the definition of the topology in S ("pointwise convergence"). Let now D_1, D_2 be two partitions of B_1. Then $F_{D_1} \cap F_{D_2} \supset F_{D_1 \wedge D_2}$, where $D_1 \wedge D_2 = \{d_1 \wedge d_2;\ d_1 \in D_1 \text{ and } d_2 \in D_2\}$ is a partition of B_1. Finally, let D be a partition of B_1. For every $d \in D \setminus \{0\}$ take a state $s_d \in S^2_{B_1}(B_1)$ such that $s_d(d) = 1$ (Theorem 2.7). According to Theorem 2.8, for every $d \in D \setminus \{0\}$ there exists a B-state $\tilde{s}_d \in S^2_B(P)$ such that $\tilde{s}_d | B_1 = s_d$. hence $\tilde{s}_d \in S$ and $s = \sum_{d \in D \setminus \{0\}} s_1(d) \tilde{s}_d \in F_D$. Thus, $\mathcal{F}$ is a centred system. Since S is compact, we have a B-state s such that $s \in \bigcap \mathcal{F}$. It follows immediately from the definition of $\mathcal{F}$ that s extends s_1. The proof is complete.

It may be of independent interest to note the following corollary of the previous result which might be viewed as a topological proof of a classical Boolean result (see [5], [11], compare also [8]).

4.2. Corollary. *Let B_1 be a Boolean subalgebra of a Boolean algebra B. Then every state on B_1 extends over B.*

5. Open questions. Another concept of partial additivity of states (also stronger than in [13]) is studied in [12] and [1], where a theorem analogous to Theorem 2.7 is proved. The definition of the so-called *central state* (abbr. *c-state*) differs from the definition of B-state in the third condition:

(3^c) $s(a \vee b) = s(a) + s(b)$ provided $a \perp b$ and $a \in C(P)$, $b \in P$,

where $C(P)$ is the centre of P.

It is an open problem whether results analogous to those in this paper are valid for B-states that are simultaneously c-states.

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