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Properties of effect algebras based on sets of upper bounds

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Abstract We give an overview and clear up some relations between various properties of effect algebras based on properties of sets of upper bounds, e.g., completeness, orthocompleteness, weak orthocompleteness, maximality property, interpolation property.

Keywords effect algebra; upper bound; downward directed; complete; orthocomplete; weakly orthocomplete; interpolation property; maximality property

1 Introduction

Effect algebras become a basic notion for quantum structures (originated in quantum physics) as "unsharp" generalizations of orthomodular lattices, orthomodular posets and orthoalgebras [3, 4].

There are various properties of effect algebras based on properties of upper bounds for some sets of elements used in the literature. E.g., being a lattice, completeness, orthocompleteness, interpolation property (e.g., [5, 1]), weak orthocompleteness (e.g., [11, 14]), property (W+) (e.g., [2]), maximality property (e.g., [12, 13, 15]), property (CU) (e.g., [15]). Some of these properties were introduced in a different way (interpolation property, maximality property) but since an effect algebra is a de Morgan poset (with an antitone involution—the orthosupplement), we can use equivalent (in effect algebras) definitions using a common scheme.

We present a unifying attempt to the above mentioned properties introducing new ones. The properties might be classified according to the cardinality and the structure of considered systems (we should accept that elements might appear more than once in case of orthogonal systems) and according to properties of created sets of upper bounds:

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- We consider two-element systems, finite systems or systems of an arbitrary cardinality.
- We consider orthogonal systems, chains or sets. The set of majorants (upper bounds of finite sums) is used for orthogonal systems, the set of upper bounds is used otherwise.
- We consider the following properties of the set of upper bounds: has the least element; has a minimal element; is downward directed; either has the least element or no minimal element.

The paper is organized as follows: Basic notions and known properties are summarized in Section 2. In Section 3, the introduction of the general scheme of properties of effect algebras, connnections to notions already used in various papers and relations between the introduced properties that are easy to verify are given. Section 4 brings results concerning the relations between properties based on chains and on orthogonal systems. Examples showing differences between various properties (even for concrete orthomodular posets) are presented in Section 5. Most of the orthomodular posets used in these examples are known but some of their properties proved in this paper have not been verified yet.

2 Basic notions and properties

2.1 Definition. An *effect algebra* is an algebraic structure $(E, \oplus, \mathbf{0}, \mathbf{1})$ such that E is a set, $\mathbf{0}$ and $\mathbf{1}$ are different elements of E, and $\oplus$ is a partial binary operation on E such that for every $a, b, c \in E$ the following conditions hold:

(1) $a \oplus b = b \oplus a$, if one side exists;
(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, if one side exists;
(3) there is a unique *orthosupplement* a' such that $a \oplus a' = \mathbf{1}$;
(4) $a = \mathbf{0}$ whenever $a \oplus \mathbf{1}$ is defined.

For simplicity, we will use the notation E for an effect algebra. A partial ordering on an effect algebra E is defined by $a \leq b$ if there is a $c \in E$ such that $b = a \oplus c$. Such an element c is unique (if it exists) and is denoted by $b \ominus a$. In particular, $1 \ominus a = a'$. With respect to this partial ordering, $\mathbf{0}$ ($\mathbf{1}$, resp.) is the least (the greatest, resp.) element of E. The orthosupplementation is an antitone involution, i.e., for every $a, b \in E$, $a'' = a$ and $b' \leq a'$ whenever $a \leq b$. An *orthogonality* relation on E is defined by $a \perp b$ if $a \oplus b$ exists (that is if and only if $a \leq b'$). It can be shown that $a \oplus \mathbf{0} = a$ for every $a \in E$ and that the *cancellation law* is valid: if $a \oplus c \leq b \oplus c$ then $a \leq b$ (in particular, if $a \oplus c = b \oplus c$ then $a = b$). See, e.g., [3, 4].

2.2 Definition. Let P be a partially ordered set. An $a \in P$ is *the least* element of P if $a \leq P$ (i.e., $a \leq p$ for every $p \in P$). An $a \in P$ is *a minimal* element of P if there is no $p \in P \setminus \{a\}$ with $p \leq a$. An $a \in P$ is an *upper bound* of a set $S \subseteq P$ if $S \leq a$. P is *downward directed* if for every $a, b \in P$ there is a $c \in P$ such that $c \leq a, b$. A *chain* in P is a linearly (totally) ordered subset of P.

Obviously, the least element of a partially ordered set is its minimal element. On the other hand, there are partially ordered sets with more minimal elements (hence, without the least element and not downward directed).

2.3 Definition. An effect algebra is *complete* if every its subset has the least upper bound.

2.4 Definition. Let E be an effect algebra. A system $(a_i\colon i \in I)$ of elements of E is *orthogonal* if $\bigoplus(a_i\colon i \in F)$ is defined for every finite set $F \subseteq I$. A *majorant* of an orthogonal system is an upper bound of all its finite sums. The *sum* $\bigoplus(a_i\colon i \in I)$ of an orthogonal system $(a_i\colon i \in I)$ is its least majorant (if it exists). An effect algebra E is *orthocomplete* if the sum of every orthogonal system in E exists.

2.5 Definition. An effect algebra is *separable* if every orthogonal system of its distinct elements is countable. An effect algebra is *Archimedean* if there is no infinite orthogonal system consisting of the same nonzero element.

It is easy to see that an Archimedean effect algebra is separable if and only if every orthogonal system of its nonzero elements is countable. On the other hand, there is an uncountable orthogonal system of nonzero elements in every non-Archimedean effect algebra—for a nonzero non-Archimedean element a and an uncountable index set I we can take the orthogonal system $(a\colon i \in I)$.

2.6 Definition. Let P be a partially ordered set. An *ordered chain* in P is a system $(c_\beta\colon \beta \in \alpha)$ such that α is an ordinal, $c_\beta \in P$ for every $\beta \in \alpha$ and $c_\beta < c_\gamma$ for every $\beta, \gamma \in \alpha$ with $\beta < \gamma$.

Let us show that, for every chain, we can find an ordered subchain indexed by a cardinal with the same set of upper bounds (this seems to be known but we did not find a proper reference).

2.7 Lemma. *Let P be a partially ordered set and $C \subseteq P$ be a chain. Then there is a cardinal $\alpha \leq \operatorname{card} C$ such that there is an ordered subchain $(c_\beta\colon \beta \in \alpha)$ of C with the the same set of upper bounds as C.*

Proof. If $C = \emptyset$ then we put $\alpha = 0$. If C has a maximal element c then we put $\alpha = 1$ and $c_0 = c$. Let us suppose that C is nonempty and does not have a maximal element.

First, let us prove by induction that there is an ordinal α with the desired property. Let us choose a $c_0 \in C$. If β is an ordinal such that c_β has not been defined yet and c_γ has been defined for every $\gamma < \beta$ then for $C_\beta = \{c_\gamma\colon \gamma \in \beta\}$ we choose an upper bound $c_\beta \in C \setminus C_\beta$ of C_β (if it exists). The construction stops before reaching an ordinal with the cardinality greater then $\operatorname{card} C$.

Let α be the least ordinal with the desired property and let us prove that α is a cardinal. Let us denote $C' = \{c_\beta\colon \beta \in \alpha\}$, a bijection $f\colon \operatorname{card}\alpha \to \alpha$ and let us use the induction. Let us choose a $c'_0 \in C'$. If β is an ordinal such that c'_β has not been defined yet and c'_γ has been defined for every $\gamma < \beta$ then for $C_\beta = \bigcup_{\gamma \in \beta}\{c'_\gamma, c_{f(\gamma)}\}$ we choose an upper bound $c'_\beta \in C' \setminus C_\beta$ of C_β. The construction stops before reaching $\operatorname{card}\alpha + 1$. Since α is the least ordinal with the desired property, we obtain that $\alpha \leq \operatorname{card}\alpha$, i.e., α is a cardinal. $\square$

The following relations between chains and orthogonal systems will be used in the sequel. The second is a reformulation of the method used in the proof [8, Theorem 4.9].

2.8 Lemma. *Let $C = (c_\beta\colon \beta \in \alpha)$ be an ordered chain. Then the system $O = (c_{\beta+1} \ominus c_\beta\colon \beta + 1 \in \alpha)$ is orthogonal and every upper bound of C is a majorant of O.*

Proof. Let O' be a finite subsystem of O with $\operatorname{card} O' = n \in \mathbb{N}$. If $n = 0$ then the statement is true ($\bigoplus \emptyset = \mathbf{0}$). If $n > 0$ then there are ordinals $\beta_1 < \cdots < \beta_n$

such that $\beta_n + 1 \in \alpha$ and $O' = (c_{\beta_i+1} \ominus c_{\beta_i} : i \in \{1, \ldots, n\})$. Then $\bigoplus O' \leq c_{\beta_n+1} \ominus c_{\beta_1} \leq c_{\beta_n+1} \in C$. $\square$

2.9 Lemma. *Let $C = (c_\beta : \beta \in \alpha)$ be an ordered chain such that $\bigvee\{c_\gamma : \gamma \in \beta\}$ exists for every limit ordinal $\beta < \alpha$. Then there is an ordered chain $C' = (c'_\beta : \beta \in \alpha')$, $\alpha' \in \{\alpha, \alpha + 1\}$, with the same set of upper bounds as C such that $c'_\beta = \bigvee\{c'_\gamma : \gamma \in \beta\}$ for every limit ordinal $\beta < \alpha'$.*

Proof. For every $\beta \in \alpha$ we put $c'_\beta = \bigvee\{c_\gamma : \gamma \in \beta\}$ if β is a limit ordinal, $c'_\beta = c_\beta$ if $\beta < \operatorname{card} \mathbb{N}$, $c'_{\beta+1} = c_\beta$ otherwise. $\square$

3 Properties of effect algebras

We will introduce various properties of effect algebras in the form (XYZ) and will show basic relations between them.

3.1 Definition. We denote some properties of a partially ordered set P:
(L): P has the least element.
(M): P has a minimal element.
(D): P is downward directed.
(W): P either has the least element or no minimal element.

3.2 Definition. An effect algebra E has the property (XYZ) if every set of upper bounds specified by XY has the property (Z), $Z \in \{\mathrm{L}, \mathrm{M}, \mathrm{D}, \mathrm{W}\}$. The *specification* is as follows.

The symbol X gives the maximal cardinality of considered systems: $X = \mathrm{T}$ for two-valued, $X = \mathrm{F}$ for finite, empty X for no restriction.

The symbol Y gives the structure of considered systems and resulting set of upper bounds: $Y = \mathrm{S}$ for a set, $Y = \mathrm{C}$ for a chain, $Y = \mathrm{O}$ for an orthogonal system. We consider the set of majorants for $Y = \mathrm{O}$, the set of upper bounds otherwise.

3.3 Examples. Translations of notations introduced in Definition 3.2 to usual notions for some properties of effect algebras.
(TSL): E is a lattice.
(SL): E is a complete lattice.
(OL): orthocompleteness.
(TSD): *interpolation property.* Introduced by Goodearl [5] for abelian groups, by Bennet and Foulis [1] for effect algebras.
(OW): *weak orthocompleteness.* Introduced by Ovchinnikov [11] for orthomodular posets as *alternativity*, by Tkadlec [14] for effect algebras.
(OD): property (W+). Introduced by De Simone and Navara [2].
(TSM): *maximality property.* Introduced by Tkadlec [12].
(CD): property (CU). Introduced by Tkadlec [15].

There are obvious implications between properties of partially ordered sets: (L) $\Rightarrow$ (M), (L) $\Rightarrow$ (D) $\Rightarrow$ (W):

3.4 Proposition. *If a partially ordered set has the least element then it has a minimal element and is downward directed. Every downward directed partially ordered set either has the least element or no minimal element.*

Moreover, a partially ordered set has the property (L) if and only if it has both properties (M) and (W). As a corollary, we can add an arbitrary specification XY to obtain corresponding implications for properties (XYZ), $Z \in \{\mathrm{L}, \mathrm{M}, \mathrm{D}, \mathrm{W}\}$, of an effect algebra.

Obviously, a greater restriction to the cardinality of considered sets leads to a smaller restriction to the effect algebra, i.e., $(YZ) \Rightarrow (\mathrm{F}YZ) \Rightarrow (\mathrm{T}YZ)$ for every $Y \in \{\mathrm{S}, \mathrm{C}, \mathrm{O}\}$ and every $Z \in \{\mathrm{L}, \mathrm{M}, \mathrm{D}, \mathrm{W}\}$.

The least upper bound of a finite chain and the least majorant of a finite orthogonal system always exist. Hence, every effect algebra has all 'cardinality restricted' properties for chains and orthogonal systems discussed in this paper, i.e., (XYZ) for $X \in \{\mathrm{T}, \mathrm{F}\}$, $Y \in \{\mathrm{C}, \mathrm{O}\}$ and $Z \in \{\mathrm{L}, \mathrm{M}, \mathrm{D}, \mathrm{W}\}$.

It is well-known and easy to see that (TSL) $\Leftrightarrow$ (FSL) in partially ordered sets. The equivalence (TSD) $\Leftrightarrow$ (FSD) is proved in [5, Proposition 2.2] but we will present a bit modified proof to make the paper more self-contained.

3.5 Proposition. *Let P be a partially ordered set such that the set of upper bounds of every pair of its elements is downward directed (P has the interpolation property). Then for every pair of nonempty finite subsets A, B of P with $A \leq B$ (i.e., $a \leq b$ for every $a \in A$ and $b \in B$) there is an element $c \in P$ such that $A \leq c \leq B$.*

Proof. Let us use the induction for $\operatorname{card} A = n$. For $n = 1$ we can put c to be the element of A. Let $A = \{a_1, \ldots, a_n\}$ and there is a $c_{n-1} \in P$ such that $\{a_1, \ldots, a_{n-1}\} \leq c_{n-1} \leq B$. The set B is a subset of the set of upper bounds U of c_{n-1}, a_n. Since U is downward directed, we can use the induction to prove that there is an element $c_n \in U$ (and therefore $A \leq c_n$) with $c_n \leq B$. $\square$

We obtained that the only interesting (in the sense that they bring something new) 'cardinality restricted' versions of properties (XYZ) (i.e., with $X \in \{\mathrm{F}, \mathrm{T}\}$) are (FSL), (FSM), (FSD), (FSW), (TSM), (TSW).

In special cases we have some other relations between the considered properties or statements about them. Only cardinalities up to the cardinality of the effect algebra effectively applies, i.e., $(\mathrm{FS}Z) \Leftrightarrow (\mathrm{S}Z)$ for every $Z \in \{\mathrm{L}, \mathrm{M}, \mathrm{D}, \mathrm{W}\}$ in a finite effect algebra. Moreover, every chain-finite effect algebra has properties (SM) and (CL) (and their consequences).

Since chains are sets and considering orthogonal systems leads to considering sets of majorants, $(\mathrm{S}Z) \Rightarrow (\mathrm{C}Z), (\mathrm{O}Z)$ for every $Z \in \{\mathrm{L}, \mathrm{M}, \mathrm{D}, \mathrm{W}\}$.

4 Chains versus orthogonal systems

We will study relations between properties based on chains and on orthogonal systems using a natural correspondence between chains and orthogonal systems: using partial sums of an orthogonal system we obtain an ordered chain, using differences of subsequent elements of an ordered chain (according to Lemma 2.7, we can consider ordered chains only) we obtain an orthogonal system. The correspondence need not be one-to-one and the set of upper bounds of a chain need not be the set of majorants of an orthogonal system. Example 5.6 shows that we can obtain an upper bound of corresponding chain that is not a majorant of a given orthogonal system. On the other hand, according to Lemma 2.8, every upper bound of a chain is a majorant of corresponding orthogonal system. The reverse implication need not be true. The problem arises if an element ordered by a limit ordinal is not the least upper bound of smaller elements. This is solvable in orthocomplete effect algebras by adding such least upper bounds (see Lemma 2.9) and was done in [8, 9] considering the existence of least upper bounds of chains or sums of orthogonal systems up to a given cardinality.

4.1 Theorem. *The properties (CL) and (OL) are equivalent in every effect algebra.*

Proof. It is a consequence of [9, Theorem 3.2]. $\square$

The following proposition generalizes [8, Lemma 4.8] in the sense that the orthocompleteness assumption is omitted. (The cited result is stated in a more general context of positive cancellative partially abelian monoids, the following result might be reformulated to this context, too.) Let us remark that, in view of Lemma 2.9, instead of the equality $c_\beta = \bigvee(c_\gamma\colon\ \gamma \in \beta)$ (for limit ordinals $\beta < \alpha$) it would be sufficient to suppose the existence of the least upper bound (the description of the orthogonal system would be more complicated).

4.2 Proposition. *Let E be an effect algebra, $C = (c_\gamma\colon\ \gamma \in \alpha)$ be an ordered chain in E such that $\alpha > 0$, $c_0 = 0$ and $c_\beta = \bigvee(c_\gamma\colon\ \gamma \in \beta)$ for every limit ordinal $\beta < \alpha$. Then the set of upper bounds of the chain $C_\beta = (c_\gamma\colon\ \gamma \in \beta)$ is the set of majorants of the orthogonal system $O_\beta = (c_{\gamma+1} \ominus c_\gamma\colon\ \gamma + 1 \in \beta)$ for every $\beta \le \alpha$.*

In particular, $\bigvee C_\alpha = \bigoplus O_\alpha$ whenever $\bigvee C_\alpha$ exists (e.g., if α is not a limit ordinal) and $\bigvee C_\beta = \bigoplus O_\beta$ for every $\beta < \alpha$.

Proof. If $\beta = 0$ then $\bigvee C_0 = \mathbf{0} = \bigoplus O_0$.

Let us suppose $0 < \beta \le \alpha$. According to the assumptions, $\bigvee C_\beta = c_\beta$ for a limit ordinal $\beta < \alpha$, $\bigvee C_\beta = c_{\beta-1}$ for a nonlimit ordinal $\beta \le \alpha$. According to Lemma 2.8, the system O_β is orthogonal and every upper bound of C_β is a majorant of O_β. Let us use the induction to prove that every every majorant of O_β is an upper bound of C_β. It is true for $\beta = 1$ ($\bigvee C_1 = c_0 = \mathbf{0} = \bigoplus O_1$). Let us suppose that $\beta > 1$, $\bigvee C_\gamma = \bigoplus O_\gamma$ for every $\gamma < \beta$ and a is a majorant of O_β. We distinguish three cases.

1) If β is a limit ordinal then, for every $\gamma \in \beta$, we have $\gamma + 1 \in \beta$ and therefore $c_\gamma = \bigvee C_{\gamma+1} = \bigoplus O_{\gamma+1} \le a$.

2) If β is not a limit ordinal and $\beta - 1$ is a limit ordinal then $\bigvee C_\beta = c_{\beta-1} = \bigvee C_{\beta-1} = \bigoplus O_{\beta-1} \le a$.

3) If neither β nor $\beta - 1$ are limit ordinals then $\bigvee C_\beta = c_{\beta-1} = c_{\beta-2} \oplus (c_{\beta-1} \ominus c_{\beta-2}) = \bigvee C_{\beta-1} \oplus (c_{\beta-1} \ominus c_{\beta-2}) = \bigoplus O_{\beta-1} \oplus (c_{\beta-1} \ominus c_{\beta-2}) \le a$. $\square$

There is no problem in case we can skip limit ordinals, i.e., if we can use only countable chains or orthogonal system. We will show that the conditions (CZ) and (OZ) are equivalent in separable Archimedean effect algebras for every $Z \in \{\mathrm{M}, \mathrm{D}, \mathrm{W}\}$ (see [16, 17] for properties (CD) and (OD)).

4.3 Proposition. *Let E be a separable effect algebra.*

1) *For every orthogonal system O in E there is a chain C in E such that the set of majorants of O is the set of upper bounds of C.*

2) *If E is Archimedean then for every chain C in E there is an orthogonal system O in E such that the set of upper bounds of C is the set of majorants of O.*

Proof. 1) Since it is not important to distinguish infinite multiplicities of elements in orthogonal systems, we may suppose that they are countable. Since the effect algebra is separable, the number of elements in O is countable and we can put $O = (a_i\colon\ i \in I)$, $I = \mathbb{N}$ or $I = \{1, \dots, n\}$ for some $n \in \mathbb{N}$. The set of upper bounds of the chain $\{\bigoplus_{i=1}^{k} a_i\colon\ k \in I\}$ is the set of majorants of O.

2) According to Lemma 2.7, there is a cardinal α and an ordered subchain $\{c_\beta\colon\ \beta \in \alpha\}$ of C with the same set of upper bounds as for C. The system $O = \big(c_{\beta+1} \ominus c_\beta\colon\ \beta{+}1 \in \alpha\big)$ is orthogonal. Since E is separable and Archimedean, O is countable and therefore $\alpha = \operatorname{card}\mathbb{N}$ and the set of majorants of O is the set of upper bounds of C. □

4.4 Corollary. *Let E be a separable effect algebra, $Z \in \{M, D, W\}$.*

1) *If E has the property* (CZ) *then it has the property* (OZ).

2) *If E is Archimedean and has the property* (OZ) *then it has the property* (CZ).

5 Examples

Let us present examples showing that some implications between given properties are not true. There is a plenty of such implications (see, e.g., [15, 16]), let us take care to clear up the differences between properties (L), (M), (D) and (W) for the same specification FS, S, C or O and the differences between specifications FS and S, C and O for these properties.

The examples were already used (some of them are well-known), however, usually for other properties than those studied in this paper. E.g., Example 5.1 was used in [6, Example 1], Example 5.3 was used in [7], Example 5.4 (as well as Example 5.3) was used in [10, proof of Theorem 3.4], Example 5.6 is a modification of [11, Example 3.9] where one set was considered countable.

More related appearances are the following. Example 5.4 was used in [2, Example 3.3] to show that (OW) $\not\Rightarrow$ (OL), (TSL), (OD) and in [15, Example 3.8] as an example of an effect algebra without the property (TSM). Example 5.5 (a bit different version) was used in [2, Example 3.5] as an example of an orthomodular poset without the property (OW). Example 5.6 was used in [16, Example 2.6] to show that (CD) $\not\Rightarrow$ (OW).

All examples in this section are the so-called concrete (set-representable) orthomodular posets. In particular, the $\oplus$ operation is the union of disjoint sets, the least element is the empty set, the partial ordering is the inclusion and the orthosupplement is the set-theoretic complement.

5.1 Example. Let E be the family of even-element subsets of a 6-element set X, $A \oplus B = A \cup B$ for disjoint $A, B \in E$. Then $(E, \oplus, \emptyset, X)$ is an effect algebra.

Since E is finite, it is orthocomplete and every set of elements has a minimal upper bound. There are pairs of elements in E (e.g., different nondisjoint two-element sets) with more minimal upper bounds.

I.e., (XSM) $\not\Rightarrow$ (XSW) for X empty or $X \in \{\mathrm{F}, \mathrm{T}\}$, (CL) $\not\Rightarrow$ (FSL), (OL) $\not\Rightarrow$ (FSL).

5.2 Example. Let E be a family of finite and cofinite subsets of an infinite set X, $A \oplus B = A \cup B$ for disjoint elements of E. Then $(E, \oplus, \emptyset, X)$ is an effect algebra.

The set of upper bounds of every set of elements is downward directed. There is an infinite orthogonal set (e.g., of singletons) with a coinfinite union and therefore without a minimal majorant. There is an infinite chain (e.g., of finite sets) with a coinfinite union and therefore without a minimal upper bound. E is a non-complete lattice.

I.e., $(Y\mathrm{D}) \not\Rightarrow (Y\mathrm{M})$ for $Y \in \{\mathrm{S}, \mathrm{C}, \mathrm{O}\}$, (FSL) $\not\Rightarrow$ (SL).

5.3 Example. Let E be the family of (Lebesgue) measurable subsets of $[0,1]$ with a rational measure, $A \oplus B = A \cup B$ for disjoint elements of E. Then $(E, \oplus, \emptyset, [0,1])$ is an effect algebra.

E has the property (SW): If $S \subseteq E$ then either $\bigcup S \in E$ ($\bigcup S$ is the least upper bound of S) or for every upper bound A of S there is an $x \in A \setminus \bigcup S$ and therefore there is a smaller upper bound $A \setminus \{x\}$ of S.

E does not have the property (FSD): Let $x \in (0, 0.3)$ be an irrational number, $A = (0.3, 0.6)$, $B = (0.3 + x, 0.6 + x)$, $C = (x, 0.6 + x)$, $D = (0.3, 0.9)$. Then C, D are upper bounds of $\{A, B\}$ with no smaller upper bound of $\{A, B\}$.

I.e., (FSW) $\not\Rightarrow$ (FSD), (SW) $\not\Rightarrow$ (SD).

5.4 Example. Let X_1, X_2, X_3, X_4 be mutually disjoint infinite sets, $X = \bigcup_{i=1}^{4} X_i$,

$$
\begin{aligned}
E_0 &= \{\emptyset, X_1 \cup X_2, X_2 \cup X_3, X_3 \cup X_4, X_4 \cup X_1, X\}, \\
E &= \{(A \setminus F) \cup (F \setminus A) \colon\ A \in E_0,\ F \subseteq X \text{ is finite}\},
\end{aligned}
$$

$A \oplus B = A \cup B$ for disjoint $A, B \in E$. Then $(E, \oplus, \emptyset, X)$ is an effect algebra.

E has the property (SW): Let $S \subseteq E$. If $\bigcup S$ is infinite and coinfinite in some X_n, $n \in \{1,2,3,4\}$, and A is an upper bound of S then A is cofinite in X_n, there is an $x \in X_n \setminus \bigcup S$, $A \setminus \{x\}$ is an upper bound of S, hence there is no minimal upper bound of S. Let us suppose that $\bigcup S$ is finite or cofinite in every X_i, $i \in \{1,2,3,4\}$. If the set of indices $i \in \{1,2,3,4\}$ such that $\bigcup S$ is cofinite in X_i is empty, 4-element or belongs to $\big\{\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}\big\}$ then $\bigcup S$ is the least upper bound of S. For remaining possibilities and for an upper bound A of S, there is an $n \in \{1,2,3,4\}$ such that A is cofinite in X_n and $X_n \cap \bigcup S$ is finite, there is an $x \in X_n \cap (A \setminus \bigcup S)$, $A \setminus \{x\}$ is an upper bound of S, hence there is no minimal upper bound of S.

The set of upper bounds of every finite set of elements is downward directed. There are pairs of elements in E (e.g., $X_1 \cup X_2$, $X_1 \cup X_4$) without a minimal upper bound. $X_1 \cup X_2, X_1 \cup X_4$ are majorants of $\big\{\{x\} \colon\ x \in X_1\big\}$ with no smaller majorant of this orthogonal set and upper bounds of infinite chains $C \subseteq X_1$ with no smaller upper bound of C.

I.e., (FSD) $\not\Rightarrow$ (TSM), $(Y\mathrm{W}) \not\Rightarrow (Y\mathrm{D})$ for $Y \in \{\mathrm{S}, \mathrm{C}, \mathrm{O}\}$), (FSD) $\not\Rightarrow$ (SD).

5.5 Example. Let X_1, X_2, X_3, X_4 be mutually disjoint infinite sets $X = \bigcup_{i=1}^{4} X_i$,

$$
\begin{aligned}
E_0 &= \{\emptyset, X_1 \cup X_2, X_2 \cup X_3, X_3 \cup X_4, X_4 \cup X_1, X\}, \\
E &= \{(A \setminus F) \cup (F \setminus A) \colon\ A \in E_0,\ F \subseteq X_1 \text{ is finite}\},
\end{aligned}
$$

$A \oplus B = A \cup B$ for disjoint $A, B \in E$. Then $(E, \oplus, \emptyset, X)$ is an effect algebra.

The set of upper bounds of every finite set of elements is downward directed. $X_1 \cup X_2, X_1 \cup X_4$ are minimal upper bounds of $\big\{\{x_1\} \colon\ x_1 \in X_1\big\}$.

I.e., (FSW) $\not\Rightarrow$ (SW).

5.6 Example. Let X, Y be disjoint uncountable sets of the same cardinality,

$$
\begin{aligned}
E_0 &= \{A \subseteq (X \cup Y) \colon\ \mathrm{card}(A \cap X) = \mathrm{card}(A \cap Y) \text{ is finite}\}, \\
E &= E_0 \cup \{(X \cup Y) \setminus A \colon\ A \in E_0\},
\end{aligned}
$$

$A \oplus B = A \cup B$ for disjoint $A, B \in E$. Then $(E, \oplus, \emptyset, X \cup Y)$ is an effect algebra.

E has the property (FSM): Is $S \subseteq E$ is finite then $\bigcup S$ is either finite or cofinite. In both cases we can add a finite subset of one of the sets X, Y to obtain a minimal upper bound of S.

E does not have the property (SM): There is an $S \subseteq E$ consisting of an infinite number of disjoint two-element sets such that their union is coinfinite. Every upper bound of S is cofinite, hence there is a smaller upper bound of S.

E has the property (CD): Let $\mathcal{C}$ be a chain in E, $A, B \in E$ be upper bounds of $\mathcal{C}$. If $\mathcal{C}$ contains a cofinite subset of $X \cup Y$ then $\mathcal{C}$ has a maximal element and therefore the least upper bound. If $\mathcal{C}$ does not contain an infinite set then the set $\bigcup \mathcal{C}$ is countable and there is an element $C \in E$ such that $\bigcup \mathcal{C} \subseteq C \subseteq A \cap B$.

E does not have the property (OW): Let $x_0 \in X$, $y_0 \in Y$. There is a bijection $f\colon X \to Y \setminus \{y_0\}$. The orthogonal set $\big\{\{x, f(x)\}\colon x \in X \setminus \{x_0\}\big\}$ has different minimal upper bounds $(X \cup Y) \setminus \{x_0, f(x_0)\}$ and $(X \cup Y) \setminus \{x_0, y_0\}$.

I.e., (FSM) $\not\Rightarrow$ (SM), (CD) $\not\Rightarrow$ (OD), (CW) $\not\Rightarrow$ (OW).

The following questions seems to be open.

5.7 Questions. Is there an effect algebra with some of the following properties:

1) Every pair of its elements has a minimal upper bound and there is a finite set of its elements without a minimal upper bound, i.e., (TSM) $\not\Rightarrow$ (FSM)?

2) Every pair of its elements either has the least upper bound or no minimal majorant and there is a finite set of its elements with a minimal upper bound that is not its least upper bound, i.e., (TSW) $\not\Rightarrow$ (FSW)?

3) Every its chain (orthogonal system, resp.) has a minimal upper bound (majorant, resp.) and there is a chain (orthogonal system, resp.) without the least upper bound (majorant, resp.), i.e., (CM) $\not\Rightarrow$ (CW), (OM) $\not\Rightarrow$ (OW)?

4) Every its chain has a minimal upper bound and there is an orthogonal system without a minimal majorant, i.e., (CM) $\not\Rightarrow$ (OM)?

5) For every its orthogonal system the set of majorants has a minimal element (is downward directed, has the least element or no minimal element, resp.) and there is a chain such that the set of its upper bounds does not have a minimal element (is not downward directed, has a minimal element that is not the least element, resp.), i.e., (OZ) $\not\Rightarrow$ (CZ) for $Z \in \{\mathrm{M}, \mathrm{D}, \mathrm{W}\}$?

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