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Properties of Boolean Orthoposets

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A Boolean orthoposet is the orthoposet P fulfilling the following condition: If $a, b \in P$ and $a \wedge b = 0$ then $a \perp b$. This condition seems to be a sound generalization of distributivity in orthoposets. Also, the class of (orthomodular) Boolean orthoposets may play an interesting role in quantum logic theory. This class is wide enough and, on the other hand, enjoys some properties of Boolean algebras. In this paper we summarize results on Boolean orthoposets involving distributivity, set representation, properties of the state space, existence of Jauch–Piron states, and results concerning orthocompleteness and completion.

1. BASIC NOTIONS

Definition 1.1. An *orthoposet* is a triple $(P, \leq, ')$ such that:

1. $(P, \leq)$ is a partially ordered set with a least element 0 and a greatest element 1.
2. $' : P \to P$ is an *orthocomplementation*, i.e., (i) $a'' = a$, (ii) $a \leq b \Rightarrow b' \leq a'$, (iii) $a \wedge a' = 0$ for every $a, b \in P$.

Elements a, b of P are called *orthogonal* (denoted by $a \perp b$) if $a \leq b'$. An orthoposet $(P, \leq, ')$ is called *Boolean* if $a \perp b$ whenever $a \wedge b = 0$.

Definition 1.2. Let α be a cardinal number. An orthoposet P is called *α-orthocomplete* if every set of cardinality less than α consisting of mutually orthogonal elements of P has a supremum. An orthoposet is called *orthocomplete* if it is α-orthocomplete for every cardinal number α.

Definition 1.3. An ω_0-orthocomplete (ω_0 denotes the first infinite cardinal) orthoposet is called *orthomodular* if $b = a \vee (b \wedge a')$ for every $a, b \in P$ with $a \leq b$.

2. DISTRIBUTIVITY

The following theorem [10] shows some distributivity property of Boolean orthoposets. Let us note that analogous results (but only in one-way) can be found in [2, Lemma 3.7] (for orthomodular posets) and in [8, Proposition

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1.3.10] (for orthomodular lattices) provided some compatibility conditions are fulfilled.

Theorem 2.1. Suppose that P is a Boolean orthoposet and that $S_1 \cup \ldots \cup S_n \subset P$ such that $S_1 \wedge \ldots \wedge S_n$, $\bigvee S_1, \ldots, \bigvee S_n$ exist in P. Then

$$\bigvee(S_1 \wedge \ldots \wedge S_n) = (\bigvee S_1) \wedge \ldots \wedge (\bigvee S_n)$$

if at least one side of this equality exists.

We use the following abbreviation $S_1 \wedge \ldots \wedge S_n = \{s_1 \wedge \ldots \wedge s_n;\ s_1 \in S_1, \ldots, s_n \in S_n\}$.

Corollary 2.2. Every ω_0-orthocomplete (ω_0 denotes the first infinite cardinal) Boolean orthoposet is orthomodular.

3. ORTHOCOMPLETENESS

In Corollary 2.2 we have shown that every ω_0-orthocomplete Boolean orthoposet is orthomodular. The following result [13] is a generalization of various results from [4, 10, 6, 5].

Theorem 3.1. Every orthocomplete Boolean orthoposet is a Boolean algebra.

Let us note that an orthocomplete Boolean algebra is complete. As the following proposition shows, the condition of orthocompleteness in Theorem 3.1 cannot be weakened to α-orthocompleteness for any cardinal number α.

Proposition 3.2. For every cardinal number α there is an α-complete orthomodular Boolean orthoposet that is not a Boolean algebra.

4. COMPLETION

An important problem in quantum theories is determining which orthomodular posets can be completed to an orthomodular lattice (see, e.g., [1]). Here we give a partial solution to this problem [12].

Definition 4.1. An ortholattice L is called *complete* if $\bigvee A$ exists in L for every $A \subset L$.

Let P be an orthoposet. A complete ortholattice $L \supset P$ is called the *MacNeille completion of* P if the following hold:

1. P is a suborthoposet of L, i.e., basic relations and operations on P (0, 1, $\leq$, $'$) are restrictions of those on L.

2. $\bigvee_P A = \bigvee_L A$ for every $A \subset P$ such that $\bigvee_P A$ exists.
3. For every $a \in L$ there are $A_1, A_2 \subset P$ such that $a = \bigvee A_1 = \bigwedge A_2$.

Theorem 4.2. The MacNeille completion of a Boolean orthoposet is a Boolean algebra.

5. STATE SPACE

In this section we will show that the state space of a Boolean orthoposet is rich enough [9, 13] and give a characterization of Boolean orthoposets by means of two-valued states [11].

Definition 5.1. A *state* on an orthoposet P is a mapping $s: P \to [0,1]$ such that:

1. $s(1) = 1$.
2. $s(a) \leq s(b)$ whenever $a \leq b$.
3. $s(\bigvee F) = \sum_{a \in F} s(a)$ for every finite set $F \in P$ of mutually orthogonal elements such that $\bigvee F$ exists in P.

Definition 5.2. A set S of (not necessarily all) states on an orthoposet P is called: *unital* if for every $a \in P \setminus \{0\}$ there is a state $s \in S$ such that $s(a) = 1$; *full* if for every pair $a, b \in P$ with $a \not\leq b$ there is a state $s \in S$ such that $s(a) \not\leq s(b)$.

Theorem 5.3. The set of two-valued states on a Boolean orthoposet is full.

Theorem 5.4. Let P be an orthoposet. The following two properties are equivalent:

1. P is a Boolean orthoposet.
2. The orthoposet P has a unital set of two-valued states and every unital set of two-valued states on P is full.

6. JAUCH–PIRON STATES

In quantum logic theory an important role is played by so-called Jauch–Piron states [3, 7, 8]. In this section we will clear up the connection between Boolean orthoposets and orthoposets with enough two-valued Jauch–Piron states [11].

Definition 6.1. A state s on P is called *Jauch–Piron* if for every pair $a, b \in P$ with $s(a) = s(b) = 1$ there is a $c \in P$ with $s(c) = 1$ such that $c \leq a, b$.

Theorem 6.2. Every orthoposet with a full set of two-valued Jauch–Piron states is Boolean.

Theorem 6.3. Every atomic Boolean orthoposet has a full set of two-valued Jauch–Piron states.

Theorem 6.4. There is a Boolean orthomodular poset that has no two-valued Jauch–Piron state.

As an example of a Boolean orthomodular poset without any two-valued Jauch–Piron state we can take an orthomodular poset generated by suitable triangles in a given square.

7. SET REPRESENTATION

Since the set of two-valued states on a Boolean orthoposet is full (Theorem 5.3) there is a Stone-like representation [9, 12, 13]:

Theorem 7.1. Every Boolean orthoposet has a set representation such that 0 corresponds to $\emptyset$, ordering corresponds to inclusion, orthocomplementation corresponds to set-theoretic complementation, and finite suprema of mutually orthogonal elements correspond to unions.

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