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Vector states on orthoposets and extremality

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Vector states on orthoposets and quantum logics are analyzed. These states are characterized and the sphere-valued states are shown to be orthogonally scattered in the sense of Ref. 1 and Ref. 2. The relation between the sphere-valued states and extremal (= pure) states is investigated. It will be shown that for the strictly convex spaces the sphere-valued states are always extremal whereas the reverse inclusion is valid only for special types of orthoposets (Propositions 3.4 and 3.6). Finally, extensions of vector states from Boolean subalgebras are considered. A positive result for normed linear spaces, which have a predual (Theorem 5.1), is obtained.

I. INTRODUCTION

In this paper we shall consider states on orthoposets ranging in normed linear space. Let us first review basics on orthoposets.

1.1. Definition. An *orthoposet* is a triple $(P, \leq, ')$ such that the couple $(P, \leq)$ is a partially ordered set with a least element, 0, and a greatest element, 1, and with the operation of *orthocomplementation* $' : P \to P$. Thus, for any $a, b \in P$ we have (1) $a'' = a$, (2) $a \leq b$ implies $b' \leq a'$, (3) $a \vee a' = 1$.

Let us call elements $a, b \in P$ *orthogonal* (denote by $a \perp b$) if $a \leq b'$.

Further, let us call an orthoposet $(P, \leq, ')$ *orthocomplete poset* if $a \vee b$ exists for any pair $a, b \in P$ of orthogonal elements.

In the agreement with the standard terminology[3] let us call an orthocomplete poset $(P, \leq, ')$ an *orthomodular poset*, if $b = a \vee (b \wedge a')$ for every pair $a, b \in P$ such that $a \leq b$ (an orthomodular poset is sometimes called a quantum logic).

Typical example of an orthoposet is a Boolean algebra or the lattice of projectors on a Hilbert space (these are even orthomodular posets).

Dealing with an orthoposet $(P, \leq, ')$, we shall shortly denote it by P if there is no danger of misunderstanding.

1.2. Definition. A finite subset $D = \{d_1, \ldots, d_n\}$ of an orthoposet P is called a *partition of unity* in P if D consists of mutually orthogonal nonzero elements and if $d_1 \vee \cdots \vee d_n = 1$.

Let us now recall elements of normed linear spaces as we shall use them in the sequel.[4] We shall restrict ourselves to real normed linear spaces. We use the notion R for real numbers and $B(x,r)$ [$B'(x,r)$, resp.] for the closed ball (the sphere, resp.) with the center x and the radius r.

1.3. Definition. Let X be a normed linear space and let S be a subset of X. By the *convex hull* of S we mean the set

$$\operatorname{co} S = \{t_1x_1 + \cdots + t_nx_n;\ (x_1,\dots,x_n) \in S^n,\ (t_1,\dots,t_n) \in [0,1]^n,\ t_1+\cdots+t_n = 1\}.$$

If $\operatorname{co} S = S$, we call S *convex.*

Further, we call an element x of a convex set S an *extremal point of S* (denote by $x \in \operatorname{Ext} S$), if $x \notin \operatorname{co}(S \setminus \{x\})$.

Following the standard notion, let us call a normed linear space *strictly convex* if $\operatorname{Ext} B(0,1) = B'(0,1)$.

1.4. Lemma. Suppose that X is a normed linear space and that $(x_1,\dots,x_n) \in X^n$. Then

$$\{t_1x_1 + \cdots + t_nx_n;\ (t_1,\dots,t_n) \in [0,1]^n\} = \operatorname{co}\Big\{\sum_{k\in I} x_k;\ I \subset \{1,\dots,n\}\Big\}.$$

Proof. Without any loss of generality we may suppose that $t_1 \le \cdots \le t_n$. Then $t_1x_1+\cdots+t_nx_n = t_1(x_1+\cdots+x_n)+(t_2-t_1)\cdot(x_2+\cdots+x_n)+\cdots+(t_n-t_{n-1})\cdot x_n + (1-t_n)\cdot 0$ (we use the convention that $\sum_{k\in\emptyset} x_k = 0$).

For every normed linear space X, let us denote by X^* the *dual* of X. The *predual* of X is a normed linear space Y such that $Y^* = X$. (It is well-known that every Hilbert space is strictly convex and has a predual.)

1.5. Proposition. Suppose that X is a normed linear space. Then for every $e \in X$ such that $\|e\| = 1$ there is an $f \in X^*$ such that $f(e) = \|f\| = 1$. In particular, $f(B(e/2,1/2)) = [0,1]$. Moreover, if X is a Hilbert space, then $f(x) = (x,e)$ for any $x \in X$.

Proof. The first part follows from Hahn–Banach theorem, the second part follows from the differentiability of the ball in a Hilbert space.[4]

1.6. Lemma. Suppose that H is a Hilbert space. Then $\|x-e/2\|^2 + (x,e-x) = \|e\|^2/4$ for every $e \in H$.

Proof. $\|x-e/2\|^2 = (x-e/2,x-e/2) = (x,x-e)+(e/2,e/2) = -(x,e-x)+\|e\|^2/4$.

II. VECTOR STATES

Let us now state our basic definition.

2.1. Definition. Let P be an orthoposet and let X be a normed linear space. A mapping $s : P \to X$ is called

normed if $\|s(1)\| = 1$;

monotone if $(f \circ s)(a) \leq (f \circ s)(b)$ for every functional $f \in X^*$ with $(f \circ s)(1) = \|f\| = \|s(1)\|$ and for every $a, b \in P$ such that $a \leq b$;

additive if $s(1) = \sum_{d \in D} s(d)$ for every partition of unity D in P.

Finally, a normed monotone additive mapping $s : P \to X$ is called *vector state* [denoted by $s \in S(P, X)$] if

$$s(P) \subset B(s(1)/2, 1/2); \tag{1}$$

vector $'$*-state* [denoted by $s \in S'(P, X)$] if

$$s(P) \subset B'(s(1)/2, 1/2). \tag{2}$$

Let us now observe immediate consequences of the latter definition.

2.2. Remark. (1) If P is an orthomodular poset, then the additivity and the condition (1) implies monotonicity.

(2) Vector states $s : P \to R$ with $s(1) = 1$ are called *states*; the set of all states on P we denote by $S(P)$.

(3) $S(P, R) = S(P) \cup (-S(P))$.

(4) We have $f \circ s \in S(P)$ for every $s \in S(P, X)$ and for every $f \in X^*$ such that $f(s(1)) = \|f\| = 1$.

Let us first consider the important case of X being a Hilbert space. We find a characterization of vector states and show that vector $'$-states can be viewed as a generalization of orthogonal vector states.[1,2] Moreover, if P is an orthocomplete poset we obtain a new characterization of orthogonal vector states.

2.3. Theorem. Suppose that P is an orthoposet and H is a Hilbert space. Suppose further that $s : P \to H$ is an additive mapping. Consider the following three conditions:

(1) $s(a) \perp s(b)$ for every orthogonal pair $a, b \in P$;
(2) $s(a) \perp s(a')$ for every $a \in P$;
(3) $s(P) \subset B'(s(1)/2, \|s(1)\|/2)$.

Then (1)$\Rightarrow$(2)$\Leftrightarrow$(3) and, moreover, if P is an orthocomplete poset then (3)$\Rightarrow$(1).

Proof. (1)$\Rightarrow$(2). Evident.

(2)$\Leftrightarrow$(3). Suppose that $a \in P$. Then $s(a') = s(1) - s(a)$ and, according to Lemma 1.6, $\|s(a) - s(1)/2\| = \|s(1)\|/2$ if and only if $(s(a), s(a')) = 0$.

(3)⇒(1). Suppose that $a, b \in P$ and $a \perp b$. Then $s(a) + s(b) = s(a \vee b) \in s(P)$. According to Lemma 1.6 and the condition (3), $(x, x) = (s(1), x)$ for any $x \in s(P)$. Hence,

$$\begin{aligned}(s(a), s(b)) &= \Big((s(a) + s(b), s(a) + s(b)) - (s(a), s(a)) - (s(b), s(b))\Big)/2 \\ &= \Big((s(1), s(a) + s(b)) - (s(1), s(a)) - (s(1), s(b))\Big)/2 \;=\; 0.\end{aligned}$$

This completes the proof.

2.4. Proposition. Suppose that P is an orthoposet, H is a Hilbert space and $s : P \to H$ is an additive mapping. Then the following statements are equivalent:

(1) $(s(a), s(a')) \geq 0$ for any $a \in H$,
(2) $s(P) \subset B(s(1)/2, \|s(1)\|/2)$.

Proof. According to Lemma 1.6, $\|s(a) - s(1)/2\| \leq \|s(1)\|/2$ if and only if $(s(a), s(a'))$ ≥ 0.

As the following simple example shows, there is no formal analogy of this situation with Theorem 2.3. The verification is routine.

2.5. Example. Let B be the eight-element Boolean algebra and let a, b, c be its atoms. Then there is a vector state $s \in S(B, R^2)$ such that $s(a) = (1/4, \sqrt{3}/4)$, $s(b) = (1/4, -\sqrt{3}/4)$, $s(c) = (1/2, 0)$. Hence, for this vector state s, $(s(a), s(b)) < 0$.

In the Remark 2.2 we observed that composing an appropriate linear functional with a vector state leads to a state. We now show how to obtain a vector state from states. We shall need the following notion.

2.6. Definition. Let X be a normed linear space. A finite sequence $(x_1, \ldots, x_n) \in X^n$ is called a *partition of unity in* X if $\|x_1 + \cdots + x_n\| = 1$ and $\sum_{k \in I} x_k \in B((x_1 + \cdots + x_n)/2, 1/2)$ for every $I \subset \{1, \ldots, n\}$.

2.7. Proposition. Suppose that P is an orthoposet and X is a normed linear space. Then $x_1 s_1 + \cdots + x_n s_n$ is a vector state on X for every partition of unity $(x_1, \ldots, x_n)$ in X and every sequence $(s_1, \ldots, s_n)$ of states on P.

Proof. Put $e = x_1 + \cdots + x_n$. Obviously, the mapping $s = x_1 s_1 + \cdots + x_n s_n$ is normed and additive. According to Lemma 1.4, $s(P) \subset \mathrm{co}\{\sum_{k \in I} x_k;\ I \subset \{1, \ldots, n\}\} \subset B(e/2, 1/2)$. It remaines to be proved that the mapping s is monotone. Let $f \in X^*$ be such a functional that $f(e) = \|f\| = 1$ and let $a, b \in P$ be such that $a \leq b$. Then $s_k(a) \leq s_k(b)$ for any $k \in \{1, \ldots, n\}$. Hence, $(f \circ x_k s_k)(a) \leq (f \circ x_k s_k)(b)$. The linearity of f gives $(f \circ s)(a) \leq (f \circ s)(b)$ and this completes the proof.

For the potential applications in quantum axiomatics, it may be useful to know which orthoposets posesses "enough" vector states. As the following result says, this occurs exactly in case when the orthoposet posesses "enough" ordinary states [Recall[3] that an orthoposet P is called *unital* if for any $a \in P \setminus \{0\}$ there is a state s on P such that $s(a) = 1$].

2.8. Proposition. Suppose that P is an orthoposet and X is a normed linear space. Then the following statements are equivalent:

(1) P is unital.
(2) For any partition of unity $\{a_1, \dots, a_n\}$ in P and for any partition of unity $(x_1, \dots, x_n)$ in X there is a vector state $s : P \to X$ such that $s(a_k) = x_k$ for every $k \in \{1, \dots, n\}$.

Proof. (1)⇒(2). According to our assumption, there are states $s_1, \dots, s_n$ on P such that $s_1(a_1) = \cdots = s_n(a_n) = 1$. According to Lemma 2.7, $s = x_1 s_1 + \cdots + x_n s_n$ is a vector state on P. The rest is obvious.

(2)⇒(1). Suppose that $a \in P \setminus \{0\}$ and that $e \in X$ such that $\|e\| = 1$. Then either $\{a, a'\}$ or $\{a\}$ is a partition of unity in P. For appropriate partition of unity $(e, 0)$ or (e) in X there is a vector state $s : P \to X$ such that $s(a) = e$. According to Theorem 1.5, there is an $f \in X^*$ such that $f(e) = \|f\| = 1$ and, according to Remark 2.2, $f \circ s$ is a state on P. Moreover, $(f \circ s)(a) = 1$ and the proof is complete.

III. EXTREMAL VECTOR STATES

Among all states the significant role is usually played by the extremal states ("pure" states). In this paragraph we shall discuss the connection between extremal vector states and vector $'$-states. We restrict ourselves to strictly convex normed linear spaces. The reason for this restriction indicates the following example.

3.1. Example. Let $B = \{0, a, a', 1\}$ be a four-element Boolean algebra and let X be a normed linear space that is not strictly convex. Then

$$S'(B, X) \not\subset \operatorname{Ext} \operatorname{co} S'(B, X).$$

Proof. For any $x \in B'(0,1)$ there is a vector $'$-state $s_x : B \to X$ such that $s_x(a) = s_x(1) = x$. Since the space X is not strictly convex, there is an $x_0 \in B'(0,1)$ such that $x_0 \in \operatorname{co}(B'(0,1) \setminus \{0\})$. Hence, $s_{x_0} \in \operatorname{co}\{s_x;\ x \in B'(0,1) \setminus \{x_0\}\}$.

3.2. Proposition. Suppose that P is an orthoposet and X is a strictly convex normed linear space. Then

$$S'(P, X) \subset \operatorname{Ext} \operatorname{co} S(P, X).$$

Proof. Suppose that $s \in S'(P,X)$, $t_1,\ldots,t_n \in [0,1]$, $s_1,\ldots,s_2 \in S(P,X)$, $t_1 + \cdots + t_n = 1$ and $s = t_1 s_1 + \cdots + t_n s_n$. Then $s(1) = t_1 s_1(1) + \cdots + t_n s_n(1)$. Since $s(1) \in B'(0,1)$, $s_1(1),\ldots,s_n(1) \in B(0,1)$ and since X is strictly convex, we obtain $s_1(1) = \cdots = s_n(1) = s(1)$. Suppose that $a \in P$. Then $s(a) = t_1 s_1(a) + \cdots + t_n s_n(a)$. Since $s(a) \in B'(s(1)/2, 1/2)$, $s_1(a),\ldots,s_n(a) \in B(s(1)/2,1/2)$ and since X is strictly convex, we obtain $s_1(a) = \cdots = s_n(a) = s(a)$. It means that $s_1 = \cdots = s_n = s$ and therefore s is an extremal point of $\operatorname{co} S(P,X)$. The proof is complete.

For which P do we have the reverse inclusion, too? We shall see (Propositions 3.4 and 3.6) that in only restricted cases. Still, there are examples of orthoposets significant within quantum theories where we do have the reverse inclusion. Let us first recall a standard construction.[3]

3.3. Definition. Suppose that $(P_\alpha, \leq_\alpha, {}'_\alpha)$ are orthoposets $(\alpha \in I)$. Suppose further that $P_\alpha \cap P_\beta = \{0,1\}$ for distinct $\alpha, \beta \in I$. Then by *horizontal sum* of orthoposets P_α $(\alpha \in I)$ we mean the orthoposet $(\bigcup_{\alpha\in I} P_\alpha, \bigcup_{\alpha\in I} \leq_\alpha, \bigcup_{\alpha\in I} {}'_\alpha)$.

3.4. Proposition. Suppose that P is a horizontal sum of at most four-element Boolean algebras and that X is a normed linear space. Then

$$\operatorname{Ext} \operatorname{co} S(P,X) \subset S'(P,X).$$

Proof. Suppose that P is the horizontal sum of B_α $(\alpha \in I)$ and suppose that $s \in S(P,X) \setminus S'(P,X)$. Then there is an $\alpha \in I$ and an $a \in B_\alpha$ such that $s(a) \notin B'(s(1)/2, 1/2)$, i.e., there is an $x \in X \setminus \{0\}$ such that $s(a) \pm x \in B(s(1)/2, 1/2)$. Let us define mappings $s_k : P \to X$ $(k = 1,2)$ as follows:

$$\begin{aligned} s_k(a) &= s(a) + (-1)^k x, \\ s_k(a') &= s(a') - (-1)^k x, \\ s_k(b) &= s(b) \qquad \text{for any } b \in P \setminus \{a, a'\}. \end{aligned}$$

Then $s_1, s_2 \in S(P,X) \setminus \{s\}$ and $s = (s_1 + s_2)/2$. Hence, $s \notin \operatorname{Ext} \operatorname{co} S(P,X)$. This completes the proof.

Let us note that in the case $X = R$ we do not have to assume that the Boolean algebras in Proposition 3.4 are at most four-element.[5]

Let us now recall another standard notion.

3.5. Definition. Let P be an orthoposet. A state s on P is called *Jauch–Piron* if for every $a, b \in P$ such that $s(a) = s(b) = 1$ there is an element $c \in P$ such that $c \leq a, b$ and $s(c) = 1$.

The following result shows that the reverse of the inclusion of Proposition 3.2 is often false.

3.6. Proposition. Suppose that P is an orthomodular poset that has at least three two-valued states out of which at least two are Jauch–Piron. Suppose that X is an at least two-dimensional strictly convex normed linear space. Then

$$\operatorname{Ext}\operatorname{co} S(P,X) \not\subset S'(P,X).$$

Proof. Suppose that s_1, s_2, s_3 are distinct two-valued states on P and that s_1, s_2 are Jauch–Piron. There exists a partition of unity (x_1, x_2, x_3) in X such that for $e = x_1 + x_2 + x_3$ we have $x_1 + x_2 \in B'(e/2, 1/2) \setminus \{e\}$, $x_1 \notin B'(e/2, 1/2) \cup \{e/2\}$, $x_2 \in B'(e/2, 1/2)$. Then all the sums $\sum_{k\in I} x_k$, $I \subset \{1,2,3\}$ are distinct and, according to Lemma 2.7, the mapping $s = x_1 s_1 + x_2 s_2 + x_3 s_3$ is a vector state on P. Since $s_1 \neq s_2, s_3$, there are $a, b \in P$ such that $s_1(a) = s_1(b) = 1$ and $s_2(a) = s_3(b) = 0$. Since s_1 is Jauch–Piron state, there is a $c \in P$ such that $c \leq a, b$ and $s_1(c) = 1$. Then $s(c) = x_1$. Hence, s is not a vector $'$-state.

Let us show that s is an extremal point of $\operatorname{co} S(P,X)$. It suffices to prove that the only vector state $t : P \to X$ such that $t|s^{-1}\{x\} = s|s^{-1}\{x\}$ for any $x \in \operatorname{Ext} B(e/2, 1/2)$ is the vector state s. It follows if we show that $t|s^{-1}\{x_2 + x_3\} = s|s^{-1}\{x_2 + x_3\}$. Suppose that $a \in s^{-1}\{x_2 + x_3\}$. Then $s_1(a) = 0$, $s_2(a) = s_3(a) = 1$. Since $s_2 \neq s_3$, there is a $b \in P$ such that $s_2(b) = 1$, $s_3(b) = 0$. Since s_2 is a Jauch–Piron state, there is a $c \in P$ such that $c \leq a, b$ and $s_2(c) = 1$. Then $s(c) = x_2 \in \operatorname{Ext} B(e/2, 1/2)$ and therefore $t(c) = s(c)$. Since $a = c \vee (a \wedge c')$ and $c \perp (a \wedge c')$, we obtain that $s_2(a \wedge c') = s_1(a \wedge c') = 0$ and $s_3(a \wedge c') = 1$. Thus, $s(a \wedge c') = x_3 \in \operatorname{Ext} B(e/2, 1/2)$ and therefore $t(a \wedge c') = s(a \wedge c')$. We obtain that $t(a) = t(c) + t(a \wedge c') = s(a) + s(a \wedge c') = s(a)$ and the proof is complete.

It should be noted that the assumptions of Proposition 3.6 are indeed fulfilled for many orthoposets (e.g., for any Boolean algebra B with $\operatorname{card} B > 4$ and for much more[6]).

3.7. Corollary. Suppose that P is a horizontal sum of orthomodular posets P_α ($\alpha \in I$) such that every P_α has at least three two-valued states out of which at least two are Jauch–Piron. Suppose that X is at least two-dimensional strictly convex normed linear space. Then the equality

$$\operatorname{Ext}\operatorname{co} S(P,X) = S'(P,X)$$

is valid if and only if every summand of P is at most four-element Boolean algebra.

Proof. It follows from Propositions 3.2, 3.4, and 3.6 and from the fact that a vector state on a horizontal sum is extremal if and only if its restriction to every summand is extremal.

IV. CONTEMPLATING OTHER TYPES OF VECTOR STATES

If the set of all vector states $P \to X$ is nonempty then it is not convex (the convex combination of normed mappings $s, -s$ is not normed). This flaw may be removed by taking a fixed element $s(1) \in X$ in the definition of a normed mapping. The vector state defined in such a way is a strightforward generalization of a state [compare with Remark 2.2.(3)]. All results in this paper hold for such defined vector states, too.

It should be noted that the definition of a monotone mapping $s : P \to X$ might be simplified — we may require $(f \circ s)(a) \leq (f \circ s)(b)$ not for all appropriate $f \in X^*$ but only for one. Except for the above mentioned convexity property, all results remain valid.

Another possibility to introduce vector states is to define the normed mapping by the condition $\|s(1)\| \leq 1$. The convexity of the set of all such defined vector states is a consequence of a standard computation with the norm. Unfortunately, the corresponding vector $'$-state does not become a generalization of the orthogonal state. As the following simple example shows, we are not allowed to replace the radius $1/2$ of the ball in the definition of a vector state by $\|s(1)\|/2$.

4.1. Example. Let $B = \{0, a, a', 1\}$ be a four-element Boolean algebra. Let us define vector states $s_1, s_2 : B \to R$ as follows: $s_1(a) = s_1(1) = 1$, $s_2(a') = s_2(1) = -1$. Put $s = (s_1 + s_2)/2$. Then $\|s(1)\| = 0$ and $s(a) \in B'(0, 1/2)$.

The vector states are sometimes[7] defined as normed additive mappings $s : P \to X$ such that $s(P) \subset B(0, 1)$. In this case the connection between extremal vector states and the vector $'$-states does not seem to be obvious.

4.2. Proposition. Suppose that P is an orthoposet and H is a Hilbert space. Denote by $\mathrm{BNA}(P, H)$ the set of all normed additive mappings $s : P \to H$ such that $s(P) \subset B(0, 1)$. Then

$$S'(P, H) \cap \mathrm{Ext}\,\mathrm{co}\,\mathrm{BNA}(P, H) = \{s \in S'(P, H);\ s(P) = \{0, s(1)\}\}.$$

Proof. Suppose that $s : P \to H$ is a vector $'$-state such that $s(P) \neq \{0, s(1)\}$. Suppose further that $f \in H^*$ such that $f(x) = (x, s(1))$ for any $x \in H$. Define a linear mappings $T_t : H \to H$ such that $T_t(x) = f(x)s(1) + t(x - f(x)s(1))$. Then $T_t(s(1)) = s(1)$ and for any $t \in [-\sqrt{2}, \sqrt{2}]$ and for any $x \in B(s(1)/2, 1/2)$ we obtain that

$$\begin{aligned} \|T_t(x)\|^2 &= \|f(x)s(1)\|^2 + t^2\|x - f(x)s(1)\|^2 \\ &\leq f(x)^2 + 2 \cdot (1/4 - \|f(x)s(1) - s(1)/2\|^2) \\ &= f(x)^2 + 2 \cdot (1/4 - (f(x) - 1/2)^2) \\ &= -f(x)^2 + 2f(x) \ \leq\ 1. \end{aligned}$$

Thus, $T_t \circ s \in \mathrm{BNA}(P, H)$ for any $t \in [-\sqrt{2}, \sqrt{2}]$ and therefore $s = (T_{0.9} \circ s + T_{1.1} \circ s)/2 \notin \mathrm{Ext}\,\mathrm{co}\,\mathrm{BNA}(P, H)$. The rest follows from the strict convexity of Hilbert spaces.

V. EXTENSIONS OF VECTOR STATES

In this part we ask the question of the extension of a vector state from a Boolean subalgebra of an orthoposet over the entire orthoposet (the extension from general suborthoposets are usually impossible.[8] We obtain the following result which links vector states with the geometry of the range normed linear space (in some places, we have adapted the technique of Ref. 8).

5.1. Theorem. Suppose that P is an orthoposet and X is a normed linear space with predual. Then the following statements are equivalent:

(1) P is unital.
(2) Every vector state $s : B \to X$ on every Boolean subalgebra B of P can be extended to a vector state $\tilde{s} : P \to X$.

Proof. (1)⇒(2). The unit ball $B(0,1) \subset X$ is compact in the w^*-topology.[4] According to Tichonov's theorem, $B(0,1)^P$ is compact in the product topology. The set of all vector states $P \to X$ is closed in this topology, hence it is compact. Suppose that $s : B \to X$ is a vector state. Let us denote by $\mathcal{D}$ the set of all partitions of unity in B and for any $D \in \mathcal{D}$ put $F_D = \{\bar{s} \in S(P,X);\ \bar{s}|D = s|D\}$. We shall show that the set $\mathcal{F} = \{F_D;\ D \in \mathcal{D}\}$ is a base of a filter consisting of nonempty closed subsets of $S(P,X)$.

For any $D \in \mathcal{D}$ the set F_D is obviously closed and, according to Proposition 2.8, it is nonempty, too. Suppose that D and E are partitions of unity in B. Then $F_D \cap F_E \supset F_{D \wedge E \setminus \{0\}}$, where $D \wedge E \setminus \{0\} = \{d \wedge e;\ d \in D, e \in E\} \setminus \{0\}$ is a partition of unity in B.

We have obtained that $\mathcal{F}$ is a centered system in a compact topological space. Therefore, there is an $\tilde{s} \in \bigcap \mathcal{F}$. According to definition of $\mathcal{F}$, $\tilde{s}|B = s$.

(2)⇒(1). Suppose that $a \in P \setminus \{0\}$. Then there is a vector state $s : \{0, a, a', 1\} \to X$ such that $\|s(a)\| = 1$. According to Theorem 1.5, there is an $f \in X^*$ such that $\|f\| = 1$ and $(f \circ s)(a) = 1$. In view of Remark 2.2, $f \circ s$ is a state on P. The proof of Theorem 5.1 is complete.

VI. OPEN PROBLEMS

Let us conclude this paper with two open questions whose solution could be a considerable step forward in the study of vector states.

6.1. Problem. Characterize those orthomodular posets for which we have

$$\operatorname{Ext} \operatorname{co} S(P,X) = S'(P,X).$$

6.2. Problem. (the converse of Theorem 5.1) Suppose that X is a normed linear space. Suppose that for any Boolean subalgebra B of any unital orthoposet P we have the following statement valid: Every vector state $s : B \to X$ extends over the entire P. Does then X have to have a predual?

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