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Weakly Jauch–Piron states in effect algebras

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Abstract

We study various types of weakly Jauch–Piron states and various types of state-space properties in effect algebras. We generalize several results of Tkadlec [4, 6] and of Matoušek and Pták [3]. In particular, we show when an effect algebra is an orthomodular poset, when a unital set of states is strongly order determining, and we present some state space characterizations of Boolean algebras.

Keywords effect algebra; orthomodular poset; Boolean algebra; maximality property, weakly Jauch–Piron state; unital set of states

1 Introduction

Effect algebras (and their equivalents D-posets) become a basic notion for quantum structures (originated in quantum physics) as "unsharp" generalizations of orthomodular lattices, orthomodular posets and orthoalgebras [1, 2].

Most of the results of this paper are stated for effect algebras with the maximality property. The maximality property was introduced by Tkadlec [4, 5] as a common generalization of lattices and of the orthocompleteness (and some other properties, too). It leads to generalizations of quite a few results stated originally for lattices or orthocomplete structures, see, e.g., [4, 5, 6].

We study various types of weakly Jauch–Piron states and various types of state-space properties in effect algebras, some of them introduced by Matoušek and Pták [3].

We show that every effect algebra with the maximality property and with a unital set of states is an orthomodular poset (this generalizes [6, Propositions 3.3]), that a unital set of weakly Jauch—Piron states in an effect algebra with the maximality property is strongly order determining (this generalizes [6, Theorem 3.4]) and present two characterizations of Boolean algebras generalizing results of [3, Theorem 2.3, Theorem 2.4].

2 Basic notions and properties

2.1 Definition. An *effect algebra* is an algebraic structure $(E, \oplus, \mathbf{0}, \mathbf{1})$ such that E is a set, $\mathbf{0}$ and $\mathbf{1}$ are different elements of E, and $\oplus$ is a partial binary operation on E such that for every $a, b, c \in E$ the following conditions hold:

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(1) $a \oplus b = b \oplus a$, if one side exists;
(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, if one side exists;
(3) there is a unique *orthosupplement* a' such that $a \oplus a' = \mathbf{1}$;
(4) $a = \mathbf{0}$ whenever $a \oplus \mathbf{1}$ is defined.

For simplicity, we will use the notation E for an effect algebra. A partial ordering on an effect algebra E is defined by $a \leq b$ if there is a $c \in E$ such that $b = a \oplus c$. Such an element c is unique (if it exists) and is denoted by $b \ominus a$. In particular, $1 \ominus a = a'$. With respect to this partial ordering, $\mathbf{0}$ ($\mathbf{1}$, resp.) is the least (the greatest, resp.) element of E. The orthosupplementation is an antitone involution, i.e., for every $a, b \in E$, $a'' = a$ and $b' \leq a'$ whenever $a \leq b$. An *orthogonality* relation on E is defined by $a \perp b$ if $a \oplus b$ exists (that is if and only if $a \leq b'$). It can be shown that $a \oplus \mathbf{0} = a$ for every $a \in E$ and that the *cancellation law* is valid: if $a \oplus c \leq b \oplus c$ then $a \leq b$ (in particular, if $a \oplus c = b \oplus c$ then $a = b$). See, e.g., [1, 2].

We will deal also with some special effect algebras.

2.2 Definition. Let E be an effect algebra. An element $a \in E$ is *principal* if $b \oplus c \leq a$ for every $b, c \in E$ such that $b, c \leq a$ and $b \perp c$.

An *orthomodular poset* is an effect algebra in which every element is principal.

An *orthomodular lattice* is an orthomodular poset that is a lattice.

The following notions were introduced by Tkadlec ([4] for orthomodular posets, [5] for effect algebras).

2.3 Definition. An effect algebra has the *maximality property* if every pair of its elements has a maximal lower bound.

An effect algebra E is *weakly Boolean* if for every $a, b \in E$ the condition $a \wedge b = a \wedge b' = \mathbf{0}$ implies $a = \mathbf{0}$.

Let us remark that since $'$ is an antitone involution, the maximality property is equivalent to the existence of a minimal upper bound for every pair of elements.

We will study several generalizations of Jauch–Piron states. The notions of weakly Jauch–Piron state and weakly positive state were introduced by Matoušek and Pták [3] for orthomodular lattices and are generalized here for effect algebras, the notion of weakly* positive state is a generalization of a weakly positive state. All these properties appeared (unnamed) in [4].

2.4 Definition. Let E be an effect algebra.

A *state* on E is a mapping $s\colon E \to [0, 1]$ such that (1) $s(\mathbf{1}) = 1$ and (2) $s(a \oplus b) = s(a) + s(b)$ whenever $a \perp b$.

A state s on E is *Jauch–Piron* if for every $a, b \in E$ with $s(a) = s(b) = 1$ there is a $c \in E$ such that $c \leq a, b$ and $s(c) = 1$.

A state s on E is *weakly Jauch–Piron* if for every $a, b \in E$ with $s(a) = s(b) = 1$ there is a nonzero lower bound of a, b (i.e., it is not true that $a \wedge b = \mathbf{0}$).

A state s on E is *weakly positive* if for every $a, b \in E$ with $s(a) = 1$ and $s(b) > 0$ there is a nonzero lower bound of a, b.

A state s on E is *weakly* positive* if for every $a, b \in E$ with $s(a) = 1$ and $s(b) \geq \frac{1}{2}$ there is a nonzero lower bound of a, b.

Obviously, a Jauch–Piron state is weakly Jauch–Piron, a weakly positive state is weakly* positive, a weakly* positive state is weakly Jauch–Piron.

We also use various notions of state-space properties. The notion of strongly unital set of states was introduced by Matoušek and Pták [3] for orthomodular latices.

2.5 Definition. Let E be an effect algebra.

A set S of states on E is *unital* if, for every $a \in E \setminus \{\mathbf{0}\}$, there is a state $s \in S$ such that $s(a) = 1$.

A set S of states on E is *strongly order determining* if, for every $a, b \in E$ with $a \not\leq b$, there is a state $s \in S$ such that $s(a) = 1 > s(b)$.

A set S of states on E is *strongly unital* if, for every $a, b \in E$ with $a \not\leq b$, there is a state $s \in S$ such that $s(a) = 1$ and $s(b) = 0$.

Obviously, a strongly unital set of states is strongly order determining and a strongly order determining set of states is unital.

3 Results

The following two statements generalize [6, Proposition 3.3, Theorem 3.4] stated for Jauch–Piron states.

3.1 Proposition. *Every effect algebra with the maximality property and with a unital set of weakly Jauch–Piron states is an orthomodular poset.*

Proof. Let E be an effect algebra with the maximality property and with a unital set S of Jauch–Piron states that is not is not an orthomodular poset and seek a contradiction. There are elements $a, b, c \in E$ such that $b, c \leq a$, $b \perp c$ and $b \oplus c \not\leq a$. Let us denote $d = b \oplus c$. Since E has the maximality property, there is a maximal lower bound e of a', d'. Since $d \not\leq a$, we obtain that $a' \not\leq d'$ and therefore $e < a'$ and $a' \ominus e \neq \mathbf{0}$. Since the set S is unital, there is a state $s \in S$ such that $s(a' \ominus e) = 1$. Hence $s(a') = 1$, $0 = s(e) = s(a) = s(b) = s(c) = s(d)$, $s(d') = 1$, $s(d' \ominus e) = 1$. Since the state s is weakly Jauch–Piron, there is an element $f \in E \setminus \{\mathbf{0}\}$ such that $f \leq (a' \ominus e), (d' \ominus e)$. Hence $e < e \oplus f \leq a', d'$—this contradicts to the maximality of e. □

3.2 Theorem. *A set of weakly Jauch–Piron states on an effect algebra with the maximality property is unital if and only if it is strongly order determining.*

Proof. $\Leftarrow$: Obvious.

$\Rightarrow$: Let E be an effect algebra with the maximality property and with a unital set S of weakly Jauch–Piron states. Let $a, b \in E$ such that $a \not\leq b$. Let $c \in E$ be a maximal lower bound of a, b. Then $c < a$ and therefore $a \ominus c \neq \mathbf{0}$. Since the set S is unital, there is a state $s \in S$ such that $s(a \ominus c) = 1$ and therefore $s(a) = 1$. Let us suppose that $s(b) = 1$ and seek a contradiction. Since s is weakly Jauch–Piron, there is an element $d \in E \setminus \{\mathbf{0}\}$ such that $d \leq a \ominus c$, $d \leq b$. Hence $c < c \oplus d \leq a$. According to Proposition 3.1, b is principal and therefore $c \oplus d \leq b$—this contradicts to the maximality of c. □

The following proposition is an analogue of [4, Proposition 2.2 (5)] or [5, Proposition 2.5 (2)] (“sufficiently enough” “sufficiently good” states force weakly Boolean structure).

3.3 Proposition. *Let E be an effect algebra, $x \in [0,1]$, S be a set of states on E such that the following conditions hold:*

(1) *for every $a, b \in E$ with $a \not\leq b$ there is a state $s \in S$ such that $s(a) = 1$ and $s(b) \leq x$ ($s(b) < x$, resp.);*

(2) *$s(b) < 1 - x$ ($s(b) \leq 1 - x$, resp.) whenever $a, b \in E$ and $s \in S$ with $s(a) = 1$ and $a \wedge b = \mathbf{0}$.*

Then E is weakly Boolean.

Proof. Let $a, b \in E$ with $a \wedge b = a \wedge b' = \mathbf{0}$. Let us suppose that $a \neq \mathbf{0}$ and seek a contradiction. Since $a \not\leq b$, we obtain, according to condition (1), that there is an $s \in S$ such that $s(a) = 1$ and $s(b) \leq x$ ($s(b) < x$, resp.). Hence $s(b') = 1 - s(b) \geq 1 - x$ ($s(b') > 1 - x$, resp.). This contradicts to condition (2). □

3.4 Proposition. *Every effect algebra with a strongly unital set of weakly Jauch–Piron states is weakly Boolean.*

Proof. It is a consequence of Proposition 3.3 for $x = 0$. □

The following theorem was proved in [4, Theorem 4.2].

3.5 Theorem. *Every weakly Boolean orthomodular poset with the maximality property is a Boolean algebra.*

The following statement generalizes the result of Matoušek and Pták [3, Theorem 2.3] stated for orthomodular lattices.

3.6 Theorem. *An effect algebra with the maximality property is a Boolean algebra if and only if it possesses a strongly unital set of weakly Jauch–Piron states.*

Proof. $\Rightarrow$: This is well-known.

$\Leftarrow$: Let E be an effect algebra with the maximality property and with a strongly unital set of weakly Jauch–Piron states. A strongly unital set of states is unital, hence, according to Proposition 3.1, E is an orthomodular poset. According to Proposition 3.4, E is weakly Boolean and, according to Theorem 3.5, E is a Boolean algebra. □

The following statement generalizes the result of Matoušek and Pták [3, Theorem 2.4] stated for orthomodular lattices with a unital set of weakly positive states.

3.7 Theorem. *An effect algebra with the maximality property is a Boolean algebra if and only if it possesses a unital set of weakly* positive states.*

Proof. $\Rightarrow$: Obvious.

$\Leftarrow$: Let E be an effect algebra with the maximality property and with a unital set of weakly* positive states. Every weakly* positive state is weakly Jauch–Piron, hence, according to Proposition 3.1, E is an orthomodular poset. According to [4, Proposition 2.2 (4'u)], E is weakly Boolean and, according to Theorem 3.5 E is a Boolean algebra. □

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