

## SET REPRESENTATIONS OF ORTHOPOSETS

Josef Tkadlec

Since the Stone representation of Boolean algebras (by means of clopen subsets of totally disconnected compact Hausdorff topological space) it has been natural to find connections between algebraical and topological structures. Here we study this question for orthoposets. Ideally, we would like to find set representation of an orthoposet such that the least element corresponds to the empty set, the partial ordering corresponds to the inclusion relation, the orthocomplementation corresponds to the set theoretical complementation and (finite) orthogonal suprema correspond to set theoretical unions. However, it was shown by Gudder [2] that such a representation exists exactly for orthoposets (quantum logics) with a full set of two-valued states. Thus it is necessary to give up the latter correspondence and look for a weaker one.

Previously, the investigation went in two directions. The first line is based on the concept of an M-base (Marlow [6], Mayet [7], Katrnoška [4, 5] and led to a set representation of an orthoposet. The second line is the effort to find “better” representation of orthomodular posets (quantum logics) in the sense that some (finite) orthogonal suprema correspond to unions. As a result, the representation then corresponds to the Stone representation for some Boolean subalgebras of orthomodular posets in question (Zierler, Schlessinger [10], Iturrioz [3], Binder, Pták [1] for the center, Tkadlec [9] for a given Boolean subalgebra).

In this paper we present a common generalization of all these results. The proofs will be published elsewhere.

### 1 Basic notions

**Definition.** A *closure space* is a pair  $(S, \bar{\phantom{x}})$  such that  $S \neq \emptyset$  and  $\bar{\phantom{x}} : S \rightarrow S$  is a *closure operation*, i.e.,

- (1)  $\bar{\emptyset} = \emptyset$ ,
- (2)  $A \subset \bar{A}$ ,
- (3)  $A \subset B \Rightarrow \bar{A} \subset \bar{B}$ ,
- (4)  $\overline{\bar{A}} = \bar{A}$ .

A set  $A \subset S$  is called *closed*, if  $\bar{A} = A$ , *open*, if  $S \setminus A$  is closed, *clopen* (denoted  $A \in CO(S)$ ), if  $A, S \setminus A$  are open.

A family  $\mathcal{B} \subset \exp S$  of open sets is called a *base of open sets* if for any open  $A \subset S$  there is a  $\mathcal{B}_1 \subset \mathcal{B}$  such that  $A = \bigcup \mathcal{B}_1$ .

A closure space  $(S, \bar{\phantom{x}})$  is called *Hausdorff* if any pair of points from  $S$  is separated by disjoint open sets, *compact* if any open covering of  $S$  has a finite subcovering, *0-dimensional* if  $CO(S)$  is a base of open sets.

The union of two closed sets (the intersection of two open sets, resp.) in closure space need not be closed (open). On the other hand, the intersection of any family of closed sets (the union of any family of open sets, resp.) has to be closed (open).

If we replace the condition (3) by the stronger condition

$$(3') \quad \overline{A \cup B} = \overline{A} \cup \overline{B},$$

we obtain the definition of a *topological space*.

Every 0-dimensional Hausdorff closure space is *totally disconnected*, i.e. any pair of points is separated by disjoint clopen sets.

Every family  $\mathcal{B} \subset \exp S$  such that  $\bigcup \mathcal{B} = S$  is as a base of open sets for some closure space  $(S, \bar{\phantom{x}})$  (we put  $\overline{A} = S \setminus \bigcup \{B \in \mathcal{B}; B \cap A = \emptyset\}$  for any  $A \subset S$ ).

**Definition.** An *orthoposet* is a triple  $(P, \leq, ')$  such that

- (1)  $(P, \leq)$  is a poset with the least element 0 and the greatest element 1,
- (2)  $\prime: P \rightarrow P$  is an *orthocomplementation*, i.e., for any  $a, b \in P$  we have
  - (a)  $a'' = a$ , (b)  $a \leq b \Rightarrow b' \leq a'$ , (c)  $a \vee a' = 1$ .

Further, elements  $a, b \in P$  are called *orthogonal* (denoted  $a \perp b$ ) if  $a \leq b'$ .

Let us write  $OS(P) = \{(a_1, \dots, a_n) \in P^n; a_1, \dots, a_n \text{ are pairwise orthogonal and } a_1 \vee \dots \vee a_n \in P\}$ .

## 2 Results

**Definition.** Let  $(P, \leq, ')$  be an orthoposet. Let us suppose that  $R \subset OS(P)$ . By a *partially additive state on  $P$  with respect to  $R$*  (abbr.  *$R$ -state*) we mean a mapping  $s: P \rightarrow [0, 1]$  such that

- (1)  $s(1) = 1$ ,
- (2)  $\forall a, b \in P: a \leq b \Rightarrow s(a) \leq s(b)$ ,
- (3a)  $\forall a \in P: s(a) + s(a') = 1$ ,
- (3b)  $\forall (a_1, \dots, a_n) \in R: s(a_1) + \dots + s(a_n) = s(a_1 \vee \dots \vee a_n)$ .

A set  $S$  of some partially additive states is called *full* if for any  $a, b \in P$ ,  $a \not\leq b$ , there is an  $s \in S$  such that  $s(a) \not\leq s(b)$ .

A partially additive state (i.e., with respect to  $\emptyset$ )  $s$  is called *Jauch–Piron* if for any  $a, b \in P$  such that  $s(a) = s(b) = 1$ , there is a  $c \in P$  such that  $c \leq a, b$  and  $s(c) = 1$ .

**Examples.** 1.  $R = \emptyset$ . Each  $R$ -state  $s$  corresponds to the M-base  $s^{-1}(1)$  [6, 7, 5, 4].

2.  $R = OS(C(P))$ , where  $C(P)$  is the center of  $P$  ([10, 3] for a logic  $P$ ).

3.  $R = \perp \cap (P \times C(P))$ ,  $C(P)$  as above ([8, 1] for a logic  $P$ ).

4.  $R = OS(B)$ , where  $B$  is a Boolean subalgebra of  $P$  ([9] for a logic  $P$ ).

5.  $R = OS(P)$  ([2] for a logic  $P$ ).

**Theorem.** Suppose that  $(P, \leq, ')$  is an orthoposet,  $R \subset OS(P)$ ,  $S$  is nonempty set of some two-valued  $R$ -states on  $P$ ,  $i : (P, \leq, ') \rightarrow (\exp S, \subset, ^c)$ , where  $i(a) = \{s \in S; s(a) = 1\}$  for any  $a \in P$  and  $i(P)$  is the base of open sets in  $(S, \neg)$ . Then

- (1)  $i$  is a homomorphism.
- (2)  $i(P) \subset CO(S)$ .
- (3)  $(S, \neg)$  is a 0-dimensional Hausdorff closure space.
- (4) If  $\mathcal{M} \subset i(P)$  and  $\bigvee \mathcal{M}$  exists in  $(i(P), \subset)$ , then  $\bigvee \mathcal{M} = \overline{\bigcup \mathcal{M}}$ .
- (5) For any  $(a_1, \dots, a_n) \in P$  we have  $i(a_1 \vee \dots \vee a_n) = i(a_1) \cup \dots \cup i(a_n) = i(a_1) \vee \dots \vee i(a_n)$ .
- (6)  $i$  is an embedding iff  $S$  is full.
- (7) If  $S$  is the set of all two-valued  $R$ -states, then  $(S, \neg)$  is a compact closure space.
- (8) If  $P$  is an ortholattice,  $S$  is the set of all two-valued  $R$ -states and is full, then  $i(P) = CO(S)$ .
- (9) If  $S$  is full then  $(S, \neg)$  is a topological space iff each  $s \in S$  is Jauch–Piron.

**Theorem.** Let  $P$  be an orthoposet and let  $B$  be a Boolean subalgebra of  $P$ ,  $R = OS(B)$ . Then the set of all two-valued  $R$ -states is full.

**Corollary.** Let  $P$  be an orthoposet and let  $B$  be a Boolean subalgebra of  $P$ ,  $R = OS(B)$ . Then there is a set representation of  $P$  by means of clopen sets in a 0-dimensional compact Hausdorff closure space such that the image of  $B$  is “almost” its Stone representation (i.e., the (finite) suprema in  $B$  correspond to set theoretical unions). Moreover, if  $P$  is an ortholattice, then we can ensure that the representation contains all clopen sets.

### 3 Open problems

**Problem.** Is the set of all two-valued  $R$ -states on  $P$  full for “better” (greater)  $R \subset OS(P)$ ?

**Problem.** When  $i(P) = CO(S)$ ?

The answer to the first problem is in general no for  $R = OS(B_1) \cup OS(B_2)$ ,  $B_1, B_2$  Boolean subalgebras of  $P$  [9] and it is not known for  $R = \perp \cap (B^2 \cup P \times C(P)$ ,  $B$  Boolean subalgebra of a logic  $P$ .

$i(P) = CO(S)$  also for some non-lattice logics  $P$ .

## References

- [1] J.Binder, P.Pták: *A representation of orthomodular lattices*. Acta Univ. Carolinae – Math. Phys. **31** (1990), 21–26
- [2] S.P.Gudder: *Stochastic Methods in Quantum Mechanics*. Elsevier North Holland, New York, 1979
- [3] L.Iturrioz: *A representation theory for orthomodular lattices by means of closure spaces*. Acta Math. Hung. **47** (1986), 145–151
- [4] F.Katrnoška: *On some topological results concerning the orthoposets*. Proc. Conf. Topology and Measure (Binz, 1987), Greifswald, 1988
- [5] F.Katrnoška: *On the representation of orthocomplemented posets*. Comm. Math. Univ. Carolinae **23** (1982), 489–498
- [6] A.R.Marlow: *Quantum theory and Hilbert space*. J. Math. Phys. **19** (1978), 1841–1846
- [7] R.Mayet: *Une dualité pour les ensembles ordonnés orthocomplémentés*. C. R. Acad. Sc. Paris Sér. I **294** (1982), 63–65
- [8] P.Pták: *Weak dispersion-free states and the hidden variables hypothesis*. J. Math. Phys. **24** (1983), 839–840
- [9] J.Tkadlec: *Partially additive states on orthomodular posets*. To appear in Colloquium Math.
- [10] N.Zierler, M.Schlessinger: *Boolean embeddings of orthomodular sets and quantum logic*. Duke Math. J. **32** (1965), 251–262

Author's address:

Department of Mathematics  
Faculty of Electrical Engineering  
Technical University of Prague  
166 27 Praha  
Czechoslovakia