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CONDITIONS THAT FORCE AN ORTHOMODULAR POSET TO BE A BOOLEAN ALGEBRA

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ABSTRACT. We introduce two new classes of orthomodular posets—the class of weakly Boolean orthomodular posets and the class of orthomodular posets with the property of maximality. The main result of this paper is that the intersection of these classes is the class of Boolean algebras. Since the first class introduced here contains various classes of orthomodular posets with a given property of its state space and the second class contains, e.g., lattice (orthocomplete, resp.) orthomodular posets, the main theorem can be viewed as a generalization of various results concerning the question when an orthomodular poset has to be a Boolean algebra. Moreover, it gives alternative proofs to previous results and new results of this type.

1. Basic notions

Let us present some basic notions we will deal with in the sequel.

Definition 1.1. An *orthomodular poset* is a structure $(P, \leq, ', 0, 1)$ such that:
(1) $(P, \leq)$ is a partially ordered set such that $0 \leq a \leq 1$ for every $a \in P$;
(2) $'\colon P \to P$ is an *orthocomplementation*, i.e., (a) $a'' = a$, (b) $a \leq b \Rightarrow b' \leq a'$, (c) $a \vee a' = 1$ for every $a \in P$;
(3) $a \vee b \in P$ for every $a, b \in P$ with $a \leq b'$ (such a, b are called *orthogonal*, denoted by $a \perp b$);
(4) the *orthomodular law* is valid in $(P, \leq, ', 0, 1)$, i.e., $b = a \vee (b \wedge a')$ for every $a, b \in P$ with $a \leq b$.

We will write shortly P instead of $(P, \leq, ', 0, 1)$. For every $a, b \in P$ with $a \leq b$ let us denote $b - a = (b \wedge a') = (b' \vee a)' \in P$ (according to condition (3)). According to the orthomodular law, $b = a \vee (b - a)$ for every $a, b \in P$ with $a \leq b$ and, moreover, $a \perp (b - a)$.

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Definition 1.2. Let P be an orthomodular poset. A *state* on P is a mapping $s\colon P \to [0,1]$ such that:

(1) $s(1) = 1$;

(2) $s(a \vee b) = s(a) + s(b)$ whenever $a, b \in P$ with $a \perp b$.

A state s is called:

Jauch–Piron if for every $a, b \in P$ with $s(a) = s(b) = 1$ there is an element $c \in P$ such that $c \leq a, b$ and $s(c) = 1$;

subadditive if for every $a, b \in P$ there is an element $c \in P$ such that $c \geq a, b$ and $s(a) + s(b) \geq s(c)$.

Going to orthocomplements we obtain the following useful reformulation of subadditivity of states.

Lemma 1.3. *Let s be a state on an orthomodular poset P. Then s is subadditive iff for every $a, b \in P$ there is an element $c \in P$ such that $c \leq a, b$ and $s(a) + s(b) \leq 1 + s(c)$.*

Using Lemma 1.3 it is easy to see that every subadditive state is Jauch–Piron and that every two-valued (i.e., with values in $\{0, 1\}$) Jauch–Piron state is subadditive.

Definition 1.4. A set S of (not necessarily all) states on an orthomodular poset P is called:

weakly unital if for every $a \in P \setminus \{0\}$ there is a state $s \in S$ such that $s(a) > \frac{1}{2}$;

unital if for every $a \in P \setminus \{0\}$ there is a state $s \in S$ such that $s(a) = 1$;

full if for every $a, b \in P$ with $a \not\leq b$ there is a state $s \in S$ such that $s(a) \not\leq s(b)$.

It is easy to see that a unital (full, resp.) set of states is weakly unital ($a \neq 0 \Rightarrow a \not\leq a'$) and that a weakly unital set of two-valued states is unital.

Weak unitality is equivalent to fullness on all 4-element Boolean subalgebras (i.e., on all $\{0, a, a', 1\}$) and to fullness on all pairs of orthogonal elements. The next proposition shows when the notions of weak unitality and of fullness coincide.

Proposition 1.5. *Let P be an orthomodular poset such that $a \perp b$ whenever $a, b \in P$ with $a \wedge b = 0$ (such orthomodular poset is sometimes called Boolean—see, e.g., [10]). Then every weakly unital set of states on P is full.*

P r o o f. Let S be a weakly unital set of states on P and let $a, b \in P$ with $a \not\leq b$. Then $a \not\perp b'$, hence there is a $c \in P \setminus \{0\}$ such that $c \leq a, b'$ and a state $s \in S$ such that $s(c) > \frac{1}{2}$. Thus, $s(a) > \frac{1}{2} > 1 - s(b') = s(b)$. $\square$

2. Weakly Boolean orthomodular posets

Definition 2.1. An orthomodular poset P is called *weakly Boolean* if for every $a, b \in P$ the condition $a \wedge b = a \wedge b' = 0$ implies $a = 0$.

The above definition is a (very) weak form of distributivity. The distributivity equality $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ should be valid in the case that $c = b'$ and $a \wedge b = a \wedge c = 0$. It is an easy observation that any pair of distinct atoms (i.e., minimal elements of $P \setminus \{0\}$) in a weakly Boolean orthomodular poset is orthogonal.

Various results stating when an orthomodular poset has to be a Boolean algebra use conditions which imply that the orthomodular poset in question is weakly Boolean. Let us present some of them (and some new).

Proposition 2.2. *An orthomodular poset P is weakly Boolean if at least one of the following conditions is fulfilled (for every $a, b \in P$):*

(1f) *There is a full set of two-valued Jauch–Piron states on P.*

(2f) *There is a full set of subadditive states on P.*

(2u) *There is a unital set of subadditive states on P.*

(2w) *There is a weakly unital set of subadditive states on P.*

(2w') *There is a set S of states on P with the following properties:*

- (a) *for every 4-element set $Q \subset P \setminus \{0\}$ there is an element $a \in Q$ and a state $s \in S$ such that $s(a) > \frac{1}{2}$,*
- (b) *for every $a, b \in P$ there is a $c \in P$ such that $c \leq a, b$ and $s(a) + s(b) \leq 1 + s(c)$ for every $s \in S$ (S is a set of uniformly subadditive states).*

(3a) $a \perp b$ *whenever* $a \wedge b = 0$.

(3f) *There is a full set of states s on P with the property that $s(a)+s(b) \leq 1$ whenever $a \wedge b = 0$.*

(3u) *There is a unital set of states s on P with the property that $s(a) + s(b) \leq 1$ whenever $a \wedge b = 0$.*

(3w) *There is a weakly unital set of states s on P with the property that $s(a) + s(b) \leq 1$ whenever $a \wedge b = 0$.*

(3'u) *There is a unital set of states s on P with the property that $s(b) = 0$ whenever $a \wedge b = 0$ and $s(a) = 1$.*

(3'w) *There is a weakly unital set of states s on P with the property that $s(a) + s(b) \leq 1$ whenever $a \wedge b = 0$ and $s(a) > \frac{1}{2}$.*

(4u) *There is a unital set of states s on P with the property that $s(a) + s(b) < \frac{3}{2}$ whenever $a \wedge b = 0$.*

(4'u) *There is a unital set of states s on P with the property that $s(b) < \frac{1}{2}$ whenever $a \wedge b = 0$ and $s(a) = 1$.*

(5) *There is a $k \in [\frac{1}{2}, 1]$ and a set S of states on P with the following properties:*
(a) *for every $a \neq 0$ there is a state $s \in S$ such that $s(a) \geq k$ ($s(a) > k$, resp.),*
(b) *$s(a)+s(b) < \frac{1}{2}+k$ ($s(a)+s(b) \leq \frac{1}{2}+k$, resp.) whenever $a \wedge b = 0$ and $s \in S$ with $s(a) \geq k$ ($s(a) > k$, resp.).*

Proof. The following implications are obvious: (1f) $\Rightarrow$ (2f) $\Rightarrow$ (3f) $\Rightarrow$ (3w), (2u) $\Rightarrow$ (3u) $\Rightarrow$ (3'u) ((4u), resp.) $\Rightarrow$ (4'u) $\Rightarrow$ (5), (2w) $\Rightarrow$ (3w) $\Rightarrow$ (3'w) $\Rightarrow$ (5). It is known that an orthomodular poset fulfilling condition (3a) has a full set of two-valued states [5, 10] and every state on it obviously has the property in condition (3f), hence (3a) $\Rightarrow$ (3f). (On the other hand, it is easy to see that (3f) $\Rightarrow$ (3a)—see, e.g., [8, 11].)

(5) Let $a, b \in P$ such that $a \neq 0$ and $a \wedge b = a \wedge b' = 0$ and let us seek a contradiction. There is an $s \in S$ such that $s(a) \geq k$ ($s(a) > k$, resp.) and $s(a)+s(b) < \frac{1}{2}+k$, $s(a)+s(b') < \frac{1}{2}+k$ ($\leq$ in both inequalities, resp.). Adding these two inequalities and using the equality $s(b)+s(b') = 1$ we obtain $s(a) < k$ ($s(a) \leq k$, resp.)—a contradiction.

(2w') Let $a, b \in P$ such that $a \neq 0$ and $a \wedge b = a \wedge b' = 0$ and let us seek a contradiction. For every $s \in S$ we obtain $s(a) + s(b) \leq 1$ and $s(a) + s(b') \leq 1$, hence $s(a) \leq \frac{1}{2}\big(1 - s(b) + 1 - s(b')\big) = \frac{1}{2}$. Further, there are elements $c, d \in P$ such that $c \leq a', b$ and $d \leq a', b'$ and such that

$$s(a') + s(b) \leq 1 + s(c), \quad \text{i.e.,} \quad s(b - c) \leq s(a) \leq \frac{1}{2},$$
$$s(a') + s(b') \leq 1 + s(d), \quad \text{i.e.,} \quad s(b' - d) \leq s(a) \leq \frac{1}{2},$$

for every $s \in S$. Let us show now that the set $\big\{a, b - c, b' - d, a' - (c \vee d)\big\}$ is a 4-element set of nonzero elements. Indeed, $a \neq 0$ according to our assumption; since $a \not\leq b, b'$, we have $b - c, b' - d \neq 0$ and the elements $a, b - c, b' - d$ are mutually different; since $0 \neq b - c \leq b$, we have $b - c \not\leq a$, hence $a' \not\perp (b - c) \perp (c \vee d)$ and therefore $a' - (c \vee d) \neq 0$; since elements $a, a' - (c \vee d) \neq 0$ are orthogonal, they are different; since $a \not\leq b'$, we have $a' \neq b \vee d$ and therefore $a' - (c \vee d) \neq b - c$ (the inequality $a' - (c \vee d) \neq b' - d$ can be proved analogously). According to assumption (a), there is a state $s \in S$ such that $s\big(a' - (c \vee d)\big) > \frac{1}{2}$. Hence, $s(a) + s(c) + s(d) < \frac{1}{2}$ and $1 = s(b) + s(b') = s(b - c) + s(c) + s(b' - d) + s(d) \leq 2s(a) + s(c) + s(d) < 1$ — a contradiction. $\square$

Let us note that the subadditivity in assumption (b) of condition (2w') was used only for such pairs $a, b \in P$ that either $a \wedge b = 0$ or $a \vee b = 1$ and that in an orthomodular lattice this assumption can be given in the form "S is a set of subadditive states".

Further, the assumption $k \in [\frac{1}{2}, 1]$ in condition (5) of Proposition 2.2 is not necessary—this condition cannot be fulfilled for any $k \notin [\frac{1}{2}, 1]$.

As the following example shows, Proposition 2.2 is not valid if the inequalities in condition (5) ((3w), (4u), ..., resp.) are both unsharp.

Example 2.3. Let $P = \{0, a, a', b, b', 1\}$ be the horizontal sum of two 4-element Boolean algebras. Since $a \wedge b = a \wedge b' = 0$ and $a \neq 0$, P is not weakly Boolean. On the other hand, for every $k \in [\frac{1}{2}, 1]$ the set $S = \{s_x;\ x \in P \setminus \{0, 1\}\}$ defined by

$$s_x(y) = \begin{cases} 0 & \text{for } y = 0, \\ 1 & \text{for } y = 1, \\ k & \text{for } y = x, \\ 1-k & \text{for } y = x', \\ \frac{1}{2} & \text{for } y \in P \setminus \{0, x, x', 1\}, \end{cases} \qquad x \in \{a, a', b, b'\},$$

fulfills conditions (5a) and (5b) of Proposition 2.2 with unsharp inequalities.

3. Orthomodular posets with the property of maximality

Let us denote $[a, b] = \{e \in P;\ a \leq e \leq b\}$ for any orthomodular poset P.

Definition 3.1. An orthomodular poset P has the *property of maximality* if for every $a, b \in P$ the set $[0, a] \cap [0, b]$ has a maximal element.

The following observation will be useful in the next section.

Lemma 3.2. *Let P be an orthomodular poset and let $a, b \in P$. Then $c \in P$ is a maximal element of $[0, a] \cap [0, b]$ iff $(a - c) \wedge b = 0$.*

Proof. The following statements are equivalent: c is a maximal element of $[0, a] \cap [0, b]$; there is no $d \in [0, a] \cap [0, b]$ such that $d > c$; there is no $e \in [0, a] \cap [0, b] \setminus \{0\}$ such that $e \perp c$; $a \wedge b \wedge c' = 0$; $(a - c) \wedge b = 0$. □

Proposition 3.3. *An orthomodular poset P has the property of maximality if at least one of the following conditions is fulfilled:*

(A) *P is a lattice.*
(B) *P is orthocomplete, i.e., $\bigvee O$ exists for every set $O \subset P$ of mutually orthogonal elements.*
(C) *There is a countable unital set of states on P and every state on P is Jauch–Piron.*

P r o o f . (A) Obvious.

(B) Let $a, b \in P$. There is a maximal set O of mutually orthogonal elements in $[0, a] \cap [0, b]$. Then $c = \bigvee O \in P$ is a maximal element of $[0, a] \cap [0, b]$. Indeed, if $d \in [0, a] \cap [0, b]$ with $d \geq c$ then $(d - c) \in [0, a] \cap [0, b]$ and $(d - c) \perp c$. Due to the maximality of O, $d - c = 0$ and therefore $d = c$.

(C) Let $a, b \in P$. If $[0, a] \cap [0, b] = \{0\}$ then the maximal element of $[0, a] \cap [0, b]$ is 0. Let us suppose that $[0, a] \cap [0, b] \neq \{0\}$ and let us denote by S a countable unital set of states on P. Then the set $S_{a,b} = \{s \in S;\ s(a) = s(b) = 1\}$ is nonempty and countable. Let s_0 be a σ-convex combination (with nonzero coefficients) of all states in $S_{a,b}$. Then $s_0(a) = s_0(b) = 1$. Since the state s_0 is Jauch–Piron, there is a $c \in [0, a] \cap [0, b]$ such that $s_0(c) = 1$. It remains to prove that c is a maximal element of $[0, a] \cap [0, b]$. Indeed, if $d \in [0, a] \cap [0, b]$ with $d \geq c$ then $(d - c) \in [0, a] \cap [0, b]$ and $(d - c) \perp c$. Hence $s_0(d - c) = 0$ and therefore there is no state $s \in S$ such that $s(d - c) = 1$. Due to the unitality of S, $d - c = 0$ and therefore $d = c$. □

4. Weakly Boolean orthomodular posets with the property of maximality

Before we proceed to our main result, we need the following proposition. The proof can be found, e.g., in [6].

Proposition 4.1. *Let P be an orthomodular poset. Then P is a Boolean algebra iff every pair $a, b \in P$ is compatible, i.e., there is a $c \in [0, a] \cap [0, b]$ such that the elements $a - c, c, b - c$ are mutually orthogonal.*

Theorem 4.2. *Every weakly Boolean orthomodular poset with the property of maximality is a Boolean algebra.*

P r o o f . Let us denote by P a weakly Boolean orthomodular poset with the property of maximality and let $a, b \in P$. According to Proposition 4.1, it suffices to prove that a and b are compatible. Since P has the property of maximality, there is a maximal element c of $[0, a] \cap [0, b]$ and a maximal element

d of $[0, a-c] \cap [0, b']$. According to Lemma 3.2, $\big((a-c)-d\big) \wedge b' = 0$ and $(a-c) \wedge b = 0$, hence $\big((a-c)-d\big) \wedge b = 0$. Since P is weakly Boolean, $(a-c)-d = 0$. Thus, $a-c = d$ and therefore the elements $a-c, c, b-c$ are mutually orthogonal. □

Remarks 4.3. Let us present combinations of conditions in Propositions 2.2 and 3.3 that cover several previous results:

(A)+(3a)	See, e.g., [5].
(A)+(2f)	See [9, Theorem 4].
(A)+(2u)	See [7, Theorem 1].
(A)+(2w)	See [4].
(A)+(2w')	See [1].
(A)+(3f)	See [11, Corollary 5.5].
(A)+(3'u)	See [8, Theorem 13] (a stronger condition than (3'u) was assumed).
(B)+(1f)	See [2, Theorem 2] (complete additivity of states in condition (1f) was assumed).
(B)+(3a)	See [10, Theorem 3.6].
(C)+(1f)	See [5, Theorem 3.5] (a stronger condition than (C) was assumed) and [3, Theorem 4.1].
(C)+(3u)	See [11].

Obviously, not all theorems stating when an orthomodular poset is a Boolean algebra can be given in the form of Theorem 4.2. Nevertheless, the approach presented here may bring also generalizations of other results. Let us present a generalization of [8, Proposition 17].

Proposition 4.4. *Every weakly Boolean orthomodular poset with a finite weakly unital set of states is a finite Boolean algebra.*

Proof. Every orthomodular poset with a finite weakly unital set of states admits only finite sets of mutually orthogonal elements, hence it is orthocomplete (and atomic). According to Proposition 3.3 and Theorem 4.2, the orthomodular poset in question is a Boolean algebra. The rest follows from the observation that a Boolean algebra that admits only finite sets of mutually orthogonal elements is finite. □

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