Czech Technical University in Prague
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Categorical Methods in Universal Algebra

Lecture Notes

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22 June 2017

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Contents

Preface

The following notes are intended to accompany the series of lectures of the *TACL Summer School* at the Palacký University in Olomouc in June 2017. They originated from the earlier notes for the undergraduate course at the Faculty of Arts at the Charles University in Prague in 2011.

The curriculum of the Summer School has affected the choice of the material presented in these notes: they contain the bare minimum of Category Theory needed for proving and understanding Beck's monadicity theorems.

I did my best to keep the text coherent and I hope that it may serve as a solid starting point for reader's further categorical adventures.

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Recommended further reading

Although most of the material in this text is standard, I have also included some material that, up to my knowledge, is written down only in research papers. These papers are referred to in the text, here I will only comment on textbooks that deal with the standard material.

Gentle Category Theory If you want to start with Category Theory at a rather pleasant and slow pace, you will find the book [4] by Michael Barr and Charles Wells invaluable. Monads form the climax of the book and you will get there through many interesting applications, mainly in Computer Science.

Standard Category Theory The book [15] by Saunders Mac Lane is a standard reference for Category Theory. Although it certainly covers the basics on monads, monads are not the book's main topic. However, Mac Lane's exposition is brilliant and the book is a very catchy read. Be prepared to solve a lot of exercises from standard algebra.

A lot on monads can be found in the book [19] by Horst Schubert. The English translation is quite different from the German original [18] — the German version contains hardly anything interesting on monads. Whereas the German version has been reprinted, the English translation is hopelesly sold out. Ask in your local library, if they have it, do not hesitate and borrow it.

Another great standard textbook is [1] by Jiří Adámek, Horst Herrlich and George Strecker. The authors introduce quite a lot of interesting notions, all based on the notion of a category equipped with an "underlying" functor. A big added value of the book is the plethora of examples and counterexamples. If you do not know, for example, whether swell epis coincide with extremal epis, this is the book where to find an answer. And, last but not least: the book is on the web for free!

Monads and Algebraic Theories A perhaps unsurpassed reference to monads and algebraic theories is a very thorough monograph [16] by Ernest G. Manes that contains wonderful material (and a huge amount of material, at that). I should mention that Manes' book is written in reverse Polish notation and that it may be harder to read if you are not fluent in this notation.

Not exactly a book, but if you happen to get a copy of an Århus preprint [22] by Gavin Wraith, you will find out that many notions in categorical algebraic theories stem naturally from module theory. The preprint is written in the language of algebraic theories, not monads, but still: it is a lovely read.

The recent book [2] focuses on a different aspect of algebraic theories: Lawvere theories (that we do not mention at all in this text). It is quite an advanced book, full of interesting facts and examples. Very up to date.

Advanced Category Theory If you are fine with working out a lot of interesting and not so easy exercises, go for another book [3] by Michael Barr and Charles Wells. Monads (called *triples* by the authors) are just one topic in the book, but you will find advanced material on monads there. Great news: the book is on the web for free.

I dare say that the beauty, strength and compactness of ideas in Max Kelly's book [11] is hard to beat. Although the book does not mention monads at all, it contains all the material one needs to start learning about tricks and treats of enriched Category Theory. Again, the book is freely available (having been typed by enthusiastic experts).

Monads can be used to define higher-dimensional categories. If you are curious what that could possibly be, see Tom Leinster's book [12]. Again, the book is freely available.

The good old sixties Finally, read the classics. There is a series of conference proceedings (Midwest Category Seminar and other meetings) in Springer Lecture Notes in Mathematics, mainly from the 1960's. In these proceedings you will find the revolutionary papers and see the development of the ideas, most of the material is solid gold. Do not worry about availability: Springer-Verlag prints these books on demand, or they are available through Springerlink.

There exist many more great books on Category Theory. Apologies: I could have mentioned only a few. Keep on looking around!

Chapter 1

Preliminaries

> *We will need to use some very simple notions of category theory, an esoteric subject noted for its difficulty and irrelevance.*
>
> Gregory Moore and Nathan Seiberg

In this chapter we gather the necessary definitions and results that we will need later in the text. This chapter is, therefore, not a comprehensive introduction to Category Theory. We refer the reader to the books [15], [19], or [1] for introductions at a slower pace.

1.1 Categories, functors, natural transformations

A category is, roughly speaking, a collection of objects and morphisms between the objects. The morphisms can be composed and the law of composition satisfies axioms known from composition of set-theoretical mappings: the composition is associative and there are identity morphisms serving as units for composition.

Before we give a formal definition of a category let us see few examples to get a feeling of what we will be dealing with.

1.1.1 Example The following are examples of categories:

(1) The category Set of all *sets* and all mappings. Notation $f : X \longrightarrow Y$ means that f is a set-theoretical mapping from X to Y. Given $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, by $g \cdot f : X \longrightarrow Y$ we denote the usual composition of functions. Observe that $h \cdot (g \cdot f) = (h \cdot g) \cdot f$ holds, whenever the composition makes sense and, if we denote by $1_X : X \longrightarrow X$ the identity mapping, the equations $1_Y \cdot f = f = f \cdot 1_X$ hold.

(2) The category Mon of all *monoids* and all *homomorphisms of monoids* looks formally like Set in that respect that $f : X \longrightarrow Y$ is a *particular* set-theoretical mapping. Namely, f is the mapping from the carrier set of the monoid X to the carrier set of the monoid Y and, moreover, f homomorphism of monoids, i.e., it respects the binary operation and sends the neutral element of X to the neutral element of Y. Since the composition of monoid homomorphisms as maps is a homomorphism of monoids and since the identity mapping is a monoid homomorphism, the axioms $h \cdot (g \cdot f) = (h \cdot g) \cdot f$ and $1_Y \cdot f = f = f \cdot 1_X$ hold, whenever the composition makes sense.

(3) To see a little more frivolous example of a category than the above two, consider a monoid $(X, i, \circ)$. We will identify it with a category $\mathscr{X}$ in the following manner: our category will have just one object that we denote by $*$. A morphism $f : * \longrightarrow *$ is an element of the monoid $(X, i, \circ)$. The composition makes sense always and we put $g \cdot f = g \circ f$ and $1_* = i$. Axioms of a monoid make sure that $\mathscr{X}$ is indeed a category: the axioms $h \cdot (g \cdot f) = (h \cdot g) \cdot f$ and $1_Y \cdot f = f = f \cdot 1_X$ hold.

The point of this example is that *a morphism in a category need not be a mapping*.

(4) A yet more frivoulous example is the category $\mathsf{Formulas}$ of all *formulas of classical propositional logic* as objects and *provability in the Hilbert-style axiomatics* as morphisms. That is, $f : X \longrightarrow Y$ means $X \vdash Y$,

for formulas X and Y. The identity morphism 1_X is $X \vdash X$ and $g \cdot f : X \longrightarrow Z$ means that $X \vdash Z$, whenever $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$. Again, the axioms $h \cdot (g \cdot f) = (h \cdot g) \cdot f$ and $1_Y \cdot f = f = f \cdot 1_X$ hold.

(5) Of course, provability in Hilbert-style axiomatics was only a *Schadenfreude* from the side of the author. The style of axiomatics had nothing to do with the properties of $\vdash$, what mattered was that $\vdash$ is a binary relation on the set of all formulas, that is reflexive and transitive, i.e., that formulas and the provability relation form a *preorder*. In fact, the category $\mathsf{Formulas}$ is an instance of the fact that *every preorder can be viewed as a category*: objects are the elements of the preorder and $f : X \longrightarrow Y$ means $X \leq Y$. Reflexivity of $\leq$ then takes care of the identity morphisms and transitivity of $\leq$ yields the notion of composition. The axioms $h \cdot (g \cdot f) = (h \cdot g) \cdot f$ and $1_Y \cdot f = f = f \cdot 1_X$ hold.

(6) And many others…

The above examples may have raised the feeling that, with a bit of effort, *almost everything forms a category*. That statement is true to the same extent as the statement that almost everything forms a poset, or a monoid, or a topological space, etc. Yes, the notion of a category is an extremely useful tool in *some* parts of mathematics but it is definitely *not* a tool with which one could tackle any problem. This text is devoted to the part where Category Theory can and does bring useful insights.

Our basic working tools will be *categories*, *functors* and *natural transformations*. Let us give first the definition of a category. We will give the definition in the form that will allow for massive and useful generalisations, see Exercise 1.4.1.

1.1.2 Definition A *category* $\mathscr{X}$ consists of a collection of objects, that will be denoted by X, Y, Z, etc. For each pair of objects X and Y there is given a *hom-set* $\mathscr{X}(X, Y)$ of arrows from X to Y. Moreover, for each X, Y, Z there are maps

$$\mathsf{unit}_X : 1 \longrightarrow \mathscr{X}(X, X) \quad \text{and} \quad \mathsf{comp}_{X,Y,Z} : \mathscr{X}(Y, Z) \times \mathscr{X}(X, Y) \longrightarrow \mathscr{X}(X, Z)$$

where 1 denotes a one-element set. The above mappings are subject to axioms making the following diagrams

$$\begin{tikzcd}
(\mathscr{X}(Z,W) \times \mathscr{X}(Y,Z)) \times \mathscr{X}(X,Y) \arrow[rr, "\mathsf{comp}_{Y,Z,W} \times \mathscr{X}(X,Y)"] \arrow[d, "\cong"'] & & \mathscr{X}(Y,W) \times \mathscr{X}(X,Y) \arrow[dd, "\mathsf{comp}_{X,Y,W}"] \\
\mathscr{X}(Z,W) \times (\mathscr{X}(Y,Z) \times \mathscr{X}(X,Y)) \arrow[d, "\mathscr{X}(Z,W) \times \mathsf{comp}_{X,Y,Z}"'] & & \\
\mathscr{X}(Z,W) \times \mathscr{X}(X,Z) \arrow[rr, "\mathsf{comp}_{X,Z,W}"'] & & \mathscr{X}(X,W)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathscr{X}(X,Y) \times 1 \arrow[r, "\mathscr{X}(X,Y) \times \mathsf{unit}_X"] \arrow[dr, "\cong"'] & \mathscr{X}(X,Y) \times \mathscr{X}(X,X) \arrow[d, "{X,X,Y}"] & \mathscr{X}(Y,Y) \times \mathscr{X}(X,Y) \arrow[d, "\mathsf{comp}_{X,Y,Y}"'] & 1 \times \mathscr{X}(X,Y) \arrow[l, "\mathsf{unit}_Y \times \mathscr{X}(X,Y)"'] \arrow[dl, "\cong"] \\
& \mathscr{X}(X,Y) & \mathscr{X}(X,Y) &
\end{tikzcd}$$

commutative.

1.1.3 Remark As said before, Definition 1.1.2 has the proper level of generality since it reveals what is essential for being a category, see Exercise 1.4.1. We will use, however, the usual conventions, since we will want to argue in analogous ways as we do with sets and mappings. Hence we will write

$1_X : X \longrightarrow X$ instead of $\mathsf{unit}_X(*)$, where $*$ is the unique element of 1

and

$g \cdot f : X \longrightarrow Z$ instead of $\mathsf{comp}_{X,Y,Z}(g, f)$.

Commutativity of the diagrams in Definition 1.1.2 then translates to

$$h \cdot (g \cdot f) = (h \cdot g) \cdot f : X \longrightarrow W, \text{ for every } f : X \longrightarrow Y,\ g : Y \longrightarrow Z,\ h : Z \longrightarrow W.$$

and

$$1_Y \cdot f = f = f \cdot 1_X : X \longrightarrow Y, \text{ for every } f : X \longrightarrow Y$$

1.1.4 Remark We will not be very precise about set-theoretical foundations, see Section 1.3 below. Most of the time we will work with *legitimate* categories, i.e., with categories $\mathscr{X}$ whose collection of objects form at most a *class* and such that $\mathscr{X}(X, Y)$ is a *set*, for any pair X, Y of objects.

Functors are "homomorphisms" of categories: they preserve the structure on the nose, i.e., a functor preserves composition and identity morphisms. We give the definition of a functor again in such form that allows for a massive generalisation, see Example 1.4.3. Let us see some examples of functors first.

1.1.5 Example The following are examples of functors:

(1) The underlying functor $U : \mathsf{Mon} \longrightarrow \mathsf{Set}$. For every monoid $\mathbb{M} = (M, i, \circ)$, $U\mathbb{M} = M$, and $Uf = f$, for every monoid homomorphism.

Clearly, the equalities $U(g \cdot f) = Ug \cdot Uf$ and $U1_{\mathbb{M}} = 1_{U\mathbb{M}}$ hold.

(2) Let $H : (M, i, \circ) \longrightarrow (N, j, *)$ be a homomorphism of monoids. If both monoids are considered as categories having one object, then H becomes a functor: the equalities $H(g \circ f) = Hg * Hf$ and $Hi = j$ hold.

(3) Let $\mathscr{X}$ and $\mathscr{Y}$ be preorders, considered as categories. Any monotone map $H : \mathscr{X} \longrightarrow \mathscr{Y}$ is a functor.

(4) Suppose $\mathsf{CommRings}$ denotes the category of commutative rings having a unit and ring homomorphisms. Let n be a postive natural number. The functor $\mathrm{Mat}_{n\times n} : \mathsf{CommRings} \longrightarrow \mathsf{Mon}$ assigns to each commutative ring A the monoid $\mathrm{Mat}_{n\times n}(A)$ of all $n \times n$ matrices over A.

To each ring homomorphism $f : A \longrightarrow A'$, the functor $\mathrm{Mat}_{n\times n}$ assigns the monoid homomorphism $\mathrm{Mat}_{n\times n}(f) : \mathrm{Mat}_{n\times n}(A) \longrightarrow \mathrm{Mat}_{n\times n}(A')$, that sends a matrix (a_{ij}) to the matrix $(f(a_{ij}))$. That $\mathrm{Mat}_{n\times n}(f)$ is a monoid homomorphism is ensured by the fact that f is a ring homomorphism.

Clearly, the identities $\mathrm{Mat}_{n\times n}(1_A) = 1_{\mathrm{Mat}_{n\times n}(A)}$ and $\mathrm{Mat}_{n\times n}(g \cdot f) = \mathrm{Mat}_{n\times n}(g) \cdot \mathrm{Mat}_{n\times n}(f)$ hold.

(5) For every category $\mathscr{A}$, we define the *representable functor* $\mathscr{A}(A_0, -) : \mathscr{A} \longrightarrow \mathsf{Set}$ as follows:

   (a) An object A gets sent to the set $\mathscr{A}(A_0, A)$ of all morphisms from A_0 to A.

   (b) Given $f : A \longrightarrow A'$, the mapping $\mathscr{A}(A_0, f) : \mathscr{A}(A_0, A) \longrightarrow \mathscr{A}(A_0, A')$ sends $h : A_0 \longrightarrow A$ to $f \cdot h : A_0 \longrightarrow A'$.

Clearly: the equalities $\mathscr{A}(A_0, 1_A) = 1_{\mathscr{A}(A_0, A)}$ and $\mathscr{A}(A_0, g \cdot f) = \mathscr{A}(A_0, g) \cdot \mathscr{A}(A_0, f)$ hold.

See Definition 1.2.7 for a slight generalisation.

1.1.6 Definition A *functor* U from the category $\mathscr{A}$ to the category $\mathscr{X}$ consists of an *object-assignment* $A \mapsto UA$ and an *action on hom-sets*

$$U_{A,A'} : \mathscr{A}(A, A') \longrightarrow \mathscr{X}(UA, UA')$$

such that the diagrams

$$\begin{tikzcd}
\mathscr{A}(A, A) \arrow[rr, "U_{A,A}"] & & \mathscr{X}(UA, UA) \\
& 1 \arrow[ul, "\mathsf{unit}^{\mathscr{A}}_A"] \arrow[ur, "\mathsf{unit}^{\mathscr{X}}_{UA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A}(A', A'') \times \mathscr{A}(A, A') \arrow[r, "U_{A',A''} \times U_{A,A'}"] \arrow[d, "\mathsf{comp}^{\mathscr{A}}_{A,A',A''}"'] & \mathscr{X}(UA', UA'') \times \mathscr{X}(UA, UA') \arrow[d, "\mathsf{comp}^{\mathscr{X}}_{UA,UA',UA''}"] \\
\mathscr{A}(A, A'') \arrow[r, "U_{A,A''}"'] & \mathscr{X}(UA, UA'')
\end{tikzcd}$$

commute.

1.1.7 Remark Definition 1.1.6 has the proper level of generality since it reveals what is essential for being a functor, see Exercise 1.4.3. We will use, however, the usual conventions, since we will want to argue in analogous ways as we do, e.g., with the underlying functor $U : \mathsf{Mon} \longrightarrow \mathsf{Set}$. Hence we will write, for $f : A \longrightarrow A'$, simply Uf instead of $U_{A,A'}f$. The diagrams of Definition 1.1.6 then translate to

$U1_A = 1_{UA}$, for every A in $\mathscr{A}$,

and

$U(g \cdot f) = Ug \cdot Uf$, for every $f : A \longrightarrow A'$, $g : A' \longrightarrow A''$.

Natural transformations are, alas, *not* "homomorphisms" between functors. But their name is well-chosen: they do transform one functor into another. To explain what we mean by that, consider a commutative triangle

$$\begin{tikzcd} & A' \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & A'' \end{tikzcd}$$

in a category $\mathscr{A}$. Given two functors $H : \mathscr{A} \longrightarrow \mathscr{X}$, $K : \mathscr{A} \longrightarrow \mathscr{X}$, we obtain two commutative triangles

$$\begin{tikzcd} & HA' \arrow[dr, "Hg"] & & & & KA' \arrow[dr, "Kg"] & \\ HA \arrow[ur, "Hf"] \arrow[rr, "Hh"'] & & HA'' & & KA \arrow[ur, "Kf"] \arrow[rr, "Kh"'] & & KA'' \end{tikzcd}$$

in the category $\mathscr{X}$. The existence of a natural transformation $\alpha : H \longrightarrow K$ ensures that the triangle on the left gets transformed to the triangle on the right in such a way that "nothing goes wrong". This means that all the faces of the diagram

$$\begin{tikzcd} & HA' \arrow[dr, "Hg"] \arrow[rrrr, dotted, "\alpha_{A'}"] & & & & KA' \arrow[dr, "Kg"] & \\ HA \arrow[ur, "Hf"] \arrow[rr, "Hh"'] \arrow[rrrr, dotted, "\alpha_A"] & & HA'' \arrow[rrrr, dotted, "\alpha_{A''}"'] & & KA \arrow[ur, "Kf"] \arrow[rr, "Kh"'] & & KA'' \end{tikzcd}$$

commute. The components of the natural transofrmations are the dotted arrows above.

Another useful slogan for a natural transformation is: a morphism $\alpha_A : HA \longrightarrow KA$ is natural in A, if it "does not really matter what A is", i.e., if the morphism α_A behaves "uniformly in A".

We give the definition of a natural transformation again in such form that allows for an immediate massive generalisation, see Example 1.4.4. Let us see examples first:

1.1.8 Example

(1) Suppose $\mathscr{X}$ and $\mathscr{Y}$ are preorders and $H : \mathscr{X} \longrightarrow \mathscr{Y}$, $K : \mathscr{X} \longrightarrow \mathscr{Y}$ are monotone maps. Considered as categories and functors, to give a natural transformation $\tau : H \longrightarrow K$ means that $HX \leq KX$, for every X.

(2) Let $\mathsf{Vec}_{\mathbb{F}}$ be the category of all vector spaces and all linear maps over a fixed field $\mathbb{F}$.

Denote by $(-)^{**} : \mathsf{Vec}_{\mathbb{F}} \longrightarrow \mathsf{Vec}_{\mathbb{F}}$ the functor that assigns to each vector space A its *double dual* space. That is: X^{**} is the vector space of linear forms on the vector space X^* of the linear forms on X. Then the mapping

$$\mathrm{ev}_A : A \longrightarrow A^{**}, \quad a \mapsto (f \mapsto f(a))$$

is a natural transformation from the identity functor on Vec_K to $(-)^{**}$.

For every linear map $f : A \longrightarrow A'$ the square

$$\begin{tikzcd}
A \arrow[r, "\mathrm{ev}_A"] \arrow[d, "f"'] & A^{**} \arrow[d, "f^{**}"] \\
A' \arrow[r, "\mathrm{ev}_{A'}"'] & A'^{**}
\end{tikzcd}$$

commutes.

(3) Recall the example of the matrix-formation functor $\mathrm{Mat}_{n\times n} : \mathsf{CommRings} \longrightarrow \mathsf{Mon}$ of Example 1.1.5. Further, let $|{-}| : \mathsf{CommRings} \longrightarrow \mathsf{Mon}$ be the functor that assigns the underlying multiplicative monoid to every commutative ring with a unit.

Then det (the formation of a determinant) is a natural transformation from $\mathrm{Mat}_{n\times n}$ to $|{-}|$.

The A-th component $\det_A : \mathrm{Mat}_{n\times n}(A) \longrightarrow |A|$ is the monoid homomorphism, computing the determinant $\det_A(a_{ij})$ of every matrix (a_{ij}).

The square

$$\begin{tikzcd}
\mathrm{Mat}_{n\times n}(A) \arrow[r, "\det_A"] \arrow[d, "\mathrm{Mat}_{n\times n}(f)"'] & {|A|} \arrow[d, "|f|"] \\
\mathrm{Mat}_{n\times n}(A') \arrow[r, "\det_{A'}"'] & {|A'|}
\end{tikzcd}$$

commutes for every ring homomorphism $f : A \longrightarrow A'$, since determinants of matrices are computed by the same formula over any commutative ring with a unit.

1.1.9 Definition A *natural transformation* from $H : \mathscr{A} \longrightarrow \mathscr{X}$ to $K : \mathscr{A} \longrightarrow \mathscr{X}$ is a collection

$$\alpha_A : 1 \longrightarrow \mathscr{X}(HA, KA)$$

indexed by objects of $\mathscr{A}$, such that the diagram

$$\begin{tikzcd}
& 1 \times \mathscr{A}(A, A') \arrow[r, "\alpha_{A'} \times H_{A,A'}"] & \mathscr{X}(HA', KA') \times \mathscr{X}(HA, HA') \arrow[dr, "\mathsf{comp}_{HA,HA',KA'}"] & \\
\mathscr{A}(A, A') \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & & & \mathscr{X}(HA, KA') \\
& \mathscr{A}(A, A') \times 1 \arrow[r, "K_{A,A'} \times \alpha_A"'] & \mathscr{X}(KA, KA') \times \mathscr{X}(HA, KA) \arrow[ur, "\mathsf{comp}_{HA,KA,KA'}"'] &
\end{tikzcd}$$

commutes.

We also write $\alpha : H \longrightarrow K$ and we will say that the collection $\alpha = (\alpha_A)$ is *natural in* A.

1.1.10 Remark Again, we will simplify the notation of Definition 1.1.9. Since 1 has precisely one element, to give $\alpha_A : 1 \longrightarrow \mathscr{X}(HA, KA)$ is to give a morphism $\alpha_A : HA \longrightarrow KA$ in $\mathscr{X}$, for each object A in $\mathscr{A}$. The diagram of the definition then translates to the requirement that the diagram

$$\begin{tikzcd}
HA \arrow[r, "\alpha_A"] \arrow[d, "Hf"'] & KA \arrow[d, "Kf"] \\
HA' \arrow[r, "\alpha_{A'}"'] & KA'
\end{tikzcd}$$

commutes in $\mathscr{X}$, for every $f : A \longrightarrow A'$ in $\mathscr{A}$.

Natural transformations can be composed in *two different* ways.

(1) The “obvious” way is to compose the morphism $\sigma_A : HA \longrightarrow KA$ with the morphism $\tau_A : KA \longrightarrow LA$ to obtain $\tau_A \cdot \sigma_A : HA \longrightarrow LA$ that is obviously the A-th component of a natural transformation denoted by $\tau \cdot \sigma : H \longrightarrow L$. Namely, the diagram

$$\begin{tikzcd}
HA \arrow[r, "\sigma_A"] \arrow[d, "Hf"'] \arrow[rr, bend left, "(\tau\cdot\sigma)_A"] & KA \arrow[r, "\tau_A"] \arrow[d, "Kf"] & LA \arrow[d, "Lf"] \\
HA' \arrow[r, "\sigma_{A'}"'] \arrow[rr, bend right, "(\tau\cdot\sigma)_{A'}"'] & KA' \arrow[r, "\tau_{A'}"'] & LA'
\end{tikzcd}$$

commutes in $\mathscr{X}$, for every $f : A \longrightarrow A'$.

One usually depicts the above situation by the diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=60, "H"] \arrow[r, "K" description] \arrow[r, bend right=60, "L"'] \arrow[r, phantom, shift left=3, "\downarrow\sigma"] \arrow[r, phantom, shift right=3, "\downarrow\tau"] & \mathscr{X}
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=40, "H"] \arrow[r, bend right=40, "L"'] \arrow[r, phantom, "\downarrow\tau\cdot\sigma"] & \mathscr{X}
\end{tikzcd}$$

and this is why this type of composition is often called *vertical*.

(2) Perhaps a less obvious way is to compose $\sigma : H \longrightarrow H'$ “in parallel” with $\tau : K \longrightarrow K'$, where $H, H' : \mathscr{B} \longrightarrow \mathscr{X}$ and $K, K' : \mathscr{A} \longrightarrow \mathscr{B}$. This results in the natural transformation, denoted by

$$\sigma * \tau : HK \longrightarrow H'K'$$

having as the A-th component the diagonal of the commutative square

$$\begin{tikzcd}
HKA \arrow[r, "\sigma_{KA}"] \arrow[d, "H\tau_A"'] & H'KA \arrow[d, "H'\tau_A"] \\
HK'A \arrow[r, "\sigma_{K'A}"'] & H'K'A
\end{tikzcd}$$

expressing naturality of σ. That $\sigma * \tau$ is indeed natural is witnessed, for every $f : A \longrightarrow A'$, by the square

$$\begin{tikzcd}
HKA \arrow[r, "\sigma_{KA}"] \arrow[d, "HKf"'] \arrow[rr, bend left, "(\sigma*\tau)_A"] & H'KA \arrow[r, "H'\tau_A"] \arrow[d, "H'Kf"] & H'K'A \arrow[d, "H'K'f"] \\
HKA' \arrow[r, "\sigma_{KA'}"'] \arrow[rr, bend right, "(\sigma*\tau)_{A'}"'] & H'KA' \arrow[r, "H'\tau_{A'}"'] & H'K'A'
\end{tikzcd}$$

where both squares commute by naturality.

One usually depicts the above situation by the diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=40, "K"] \arrow[r, bend right=40, "K'"'] \arrow[r, phantom, "\downarrow\tau"] & \mathscr{B} \arrow[r, bend left=40, "H"] \arrow[r, bend right=40, "H'"'] \arrow[r, phantom, "\downarrow\sigma"] & \mathscr{X}
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=40, "HK"] \arrow[r, bend right=40, "HK'"'] \arrow[r, phantom, "\downarrow\sigma*\tau"] & \mathscr{X}
\end{tikzcd}$$

and this is why this type of composition is often called *horizontal*.

The above two types of compositions of natural transformations are easily seen to give unambigous meaning to the picture

$$\begin{tikzcd}
\bullet \arrow[r, shift left=6] \arrow[r] \arrow[r, shift right=6] \arrow[r, phantom, shift left=3, "\downarrow"] \arrow[r, phantom, shift right=3, "\downarrow"] & \bullet \arrow[r, shift left=6] \arrow[r] \arrow[r, shift right=6] \arrow[r, phantom, shift left=3, "\downarrow"] \arrow[r, phantom, shift right=3, "\downarrow"] & \bullet
\end{tikzcd}$$

where $\bullet$ stands for various categories. Thus all the composites one can meet have unambigous meaning. This result is called the *Godement calculus for natural transformations*, since it was introduced in [7].

Denote by ι^H the identity natural transformation on H (with components $\iota^H_A = 1_{HA} : HA \longrightarrow HA$). Then we write

$H\tau$ instead of $\iota^H * \tau$,

τH instead of $\tau * \iota^H$,

to relax the notation.

1.2 Some useful basic notions and results

A very useful source of various examples of the notions in this section is the book [1].

Special properties of morphisms

Finding a proper generalisation of being "injective" and "surjective" is not an easy task. We will see later that, in a general category, there may be several candidates for notions that a morphism is "injective" or "surjective". We introduce the "weakest" and "strongest" notions:

1.2.1 Definition A morphism $m : X \longrightarrow Y$ is called

(1) a *monomorphism* (also *mono*), if $u = v$ for any pair of morphisms satisfying $m \cdot u = m \cdot v$.

(2) a *split monomorphism* (also *split mono*, if there exists a *splitting* $e : Y \longrightarrow X$ such that $e \cdot m = 1_X$ holds.

1.2.2 Definition A morphism $e : X \longrightarrow Y$ is called

(1) a *epimorphism* (also *epi*), if $u = v$ for any pair of morphisms satisfying $u \cdot e = v \cdot e$.

(2) a *split epimorphism* (also *split epi*, if there exists a *splitting* $m : Y \longrightarrow X$ such that $e \cdot m = 1_Y$ holds.

1.2.3 Definition A morphism $f : X \longrightarrow Y$ is called an *isomorphism* (also *iso*), if there exists a (necessarily unique) g such that $g \cdot f = 1_X$ and $f \cdot g = 1_Y$.

1.2.4 Example

(1) In the category Set: mono=injective map, epi=surjective map and iso=bijective map. Every epi is split epi (assuming the Axiom of Choice). A mono splits iff its domain is a nonempty set.

(2) In Top (topological spaces and continuous maps): mono=injective continuous maps, epi=surjective continuous map, where the topology on the codomain is final, iso=homeomorphism.

Mathematical notions are used to perform calculations. Sometimes, it is useful to use various "tricks" that help speeding up the calculations. We will emphasise such "tricks" and we will often use them. Here comes the first example.

1.2.5 Categorical Trick To prove that a square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
C \arrow[r, "k"'] & D
\end{tikzcd}$$

commutes, it suffices to *precompose* both legs of the above diagram with an epimorphism and prove that both composites are equal.

Indeed, if the diagram

$$\begin{tikzcd}
Z \arrow[rd, "e"] & & \\
& A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
& C \arrow[r, "k"'] & D
\end{tikzcd}$$

commutes, where e is an epi, then the equality $(h \cdot f) \cdot e = (k \cdot g) \cdot e$ holds. Since e is epi, we can infer the desired equality $h \cdot f = k \cdot g$.

Analogously, one can *postcompose* both legs of the above square with a mono. If both composites are equal, then the square commutes.

The following easy result give the basic relationship between the mono and epi notions:

1.2.6 Proposition *The following is true in any category.*

(1) *Every iso is split mono. Every split mono is mono.*

(2) *Every iso is split epi. Every split epi is epi.*

(3) *A morphism is an iso iff it is both split mono and epi iff it is both mono and split epi.*

The above notions will also be used for natural transformations. In fact, it is useful to introduce an (in general, *illegitimate*) category $[\mathscr{A}, \mathscr{X}]$ having functors $\mathscr{A} \longrightarrow \mathscr{X}$ as objects and natural transformations as morphisms. Hence, for example, a natural transformation $\alpha : H \longrightarrow K$ is called an *epi-transformation*, if it is epi in $[\mathscr{A}, \mathscr{X}]$. An iso in $[\mathscr{A}, \mathscr{X}]$ is called a *natural isomorphism*.

Special functors and special properties of functors

In Category Theory it is customary to work with notions up to isomorphism, since isomorphic objects are regarded as “abstractly the same”. The first example of this approach is the definition of a representable functor.

1.2.7 Definition A functor $H : \mathscr{A} \longrightarrow \mathsf{Set}$ such that H is naturally isomorphic to $\mathscr{A}(A_0, -) : \mathscr{A} \longrightarrow \mathsf{Set}$ is called a *representable functor*. The object A_0 is called the *representing object*.

We will frequently use special properties of functors that we introduce now.

1.2.8 Definition A functor $U : \mathscr{A} \longrightarrow \mathscr{X}$ is called

(1) *faithful*, if the action $U_{A,A'} : \mathscr{A}(A, A') \longrightarrow \mathscr{X}(UA, UA')$ is an injective map, for every A and A'.

(2) *full*, if the action $U_{A,A'} : \mathscr{A}(A, A') \longrightarrow \mathscr{X}(UA, UA')$ is an surjective map, for every A and A'.

(3) *fully faithful* (also *f.f.*), if it is both full and faithful.

(4) *essentially surjective on objects* (also *e.s.o.*), if for every X in $\mathscr{X}$ there is a A in $\mathscr{A}$ such that UA is isomorphic to X.

(5) *bijective on objects* (also *b.o.*), if the object assignment $A \mapsto UA$ is a bijection.

Yoneda Lemma

1.2.9 Lemma (Yoneda Lemma) *Suppose that $H : \mathscr{X} \longrightarrow \mathsf{Set}$ is a functor and let X be an object of $\mathscr{X}$. Then to give a natural transformation $\tau : \mathscr{X}(X,-) \longrightarrow H$ is to give an element $x_\tau \in HX$.*

Proof. Suppose τ is given. Define $x_\tau = \tau_X(1_X)$:

$$\begin{tikzcd}
\mathscr{X}(X,X) \arrow[r, "\tau_X"] & HX \\
1_X \arrow[r, mapsto] & x_\tau
\end{tikzcd}$$

Conversely, if $x \in HX$, then we can define, for every Z in $\mathscr{X}$, a mapping $\tau^x_Z : \mathscr{X}(X,Z) \longrightarrow HZ$ sending $f : X \longrightarrow Z$ to $Hf(x)$:

$$\begin{tikzcd}
\mathscr{X}(X,Z) \arrow[r, "\tau^x_Z"] & HZ \\
f \arrow[r, mapsto] & Hfx
\end{tikzcd}$$

We need to verify that τ^x_Z is natural in Z. To that end, consider $g : Z \longrightarrow Z'$ and the commutative square

$$\begin{tikzcd}
f \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & Hfx \arrow[ddd, mapsto] \\
& \mathscr{X}(X,Z) \arrow[r, "\tau^x_Z"] \arrow[d, "\mathscr{X}(X,g)"'] & HZ \arrow[d, "Hg"] & \\
& \mathscr{X}(X,Z') \arrow[r, "\tau^x_{Z'}"'] & HZ' & \\
g \cdot f \arrow[rrr, mapsto] & & & H(g \cdot f)x = Hg(Hf(x))
\end{tikzcd}$$

It remains to be proved that $x \mapsto \tau^x$ and $\tau \mapsto x_\tau$ are mutually inverse.

(1) Start with τ. We want to prove $\tau_Z = (\tau^{x_\tau})_Z$, for every Z.

For $f : X \longrightarrow Z$ we have $(\tau^{x_\tau})_Z(f) = Hf(x_\tau) = Hf(\tau_X(1_X))$. Since τ is natural, we can compute further: $Hf(\tau_X(1_X)) = \tau_Z \mathscr{X}(X,f)(1_X) = \tau_Z(f)$.

(2) Start with x. We want to prove $x_{\tau^x} = x$.

We have $x_{\tau^x} = \tau^x_X(1_X) = H1_X(x) = x$.

■

Yoneda Lemma is usually stated as an isomorphism

$$[\mathscr{X}, \mathsf{Set}](\mathscr{X}(X,-), H) \cong HX$$

of sets, where $[\mathscr{X}, \mathsf{Set}]$ is the *illegitimate* category having functors from $\mathscr{X}$ to Set as objects and natural transformations between them.

In fact, Yoneda Lemma can be proved to be a *natural isomorphism* of two functors

$$N : \mathscr{X} \times [\mathscr{X}, \mathsf{Set}] \longrightarrow \mathsf{Set} \quad \mathrm{ev} : \mathscr{X} \times [\mathscr{X}, \mathsf{Set}] \longrightarrow \mathsf{Set}$$

with object assignments

$$N : (X,H) \mapsto [\mathscr{X}, \mathsf{Set}](\mathscr{X}(X,-), H) \quad \text{and} \quad \mathrm{ev} : (X,H) \mapsto HX$$

It is worthwhile to work out the above in detail.

See Section 1.3 for explaining why and how we will work with objects like $[\mathscr{X}, \mathsf{Set}]$ and observe that the illegitimate form allows us to prove that there is an isomorphism

$$[\mathscr{X}, \mathsf{Set}](\mathscr{X}(X,-), \mathscr{X}(X',-)) \cong \mathscr{X}(X', X)$$

natural in both X and X'.

1.3 Set-theoretical comments

Set-theoretical comments seem to be almost inevitable when writing a longer text on Category Theory. We will use Set Theory as a useful tool and not as our master. That is:

> *We will work with objects that may not exist within ordinary Set Theory but we will do it very cautiously.*

Hence, when feeling that something fishy is going on, the reader is advised to analyse carefully the set-theoretical status of the object in question. She will usually find that the status of the object is rather harmless and it can be unravelled to a long complicated statement that is hard to remember.

1.4 Exercises

1.4.1 Exercise (Enriched categories) Read Definition 1.1.2 with glasses forcing you to replace the word ‘set’ by the word ‘poset’ and the word ‘map’ by ‘monotone map’ everywhere. The definition still makes sense and you have ended up with the definition of what it means to give a category $\mathscr{X}$ *enriched in the category* Pos of posets and monotone maps.

Try to convince yourself that the definition makes sense if you replace the word ‘set’ by the word ‘gadget’ and the word ‘map’ with the word ‘morphism of gadgets’. One should end up with the notion of a category $\mathscr{X}$ *enriched in the category* $\mathsf{Gadgets}$ *of all gadgets and their morphisms*. The only trouble may be to give a meaning to symbols 1 and $\times$.

Looking closer at Definition 1.1.2, convince yourself that the only properties you need from $\times$ and 1 that they make Set into a “commutative monoid”. More precisely:

(1) The assignment $(X, Y) \mapsto X \times Y$ is a *functor* of two variables that is “nearly associative”, i.e., there is, for all X, Y, Z, a bijection $X \times (Y \times Z) \cong (X \times Y) \times Z$.

(2) The one-element set 1 is “nearly a two-sided unit” for $\times$, i.e., there is, for every X, a bijection $1 \times X \cong X \cong X \times 1$.

(3) The assignment $(X, Y) \mapsto X \times Y$ is “nearly commutative”, i.e., there is, for all X and Y, a bijection $X \times Y \cong Y \times X$.

In high-level mode of speech one says that $(\mathsf{Set}, 1, \times)$ is a *symmetric monoidal category*.

In reality, slightly more has to be required: the above bijections should interact nicely with each other. Apart from this technicality that we do not want to discuss here, the above is all you need for starting enriched Category Theory. See the excellent book [11] if you are interested in this line of thoughts.

1.4.2 Exercise (The opposite of a category) Prove that given a category $\mathscr{X}$, the following data yield a category, denoted by $\mathscr{X}^{op}$ and called the *opposite of* $\mathscr{X}$:

(1) The objects of $\mathscr{X}^{op}$ are the same as the objects of $\mathscr{X}$.

(2) Morphisms from X to X' in $\mathscr{X}^{op}$ are the morphisms from X' to X in $\mathscr{X}$.

(3) The composition and identity morphisms in $\mathscr{X}$ are given by those of $\mathscr{X}$.

1.4.3 Exercise (Enriched functors) Reading Definition 1.1.6 with ‘map’ replaced by ‘monotone maps’ everywhere, you end up with the notion of an *enriched functor* (between the categories enriched in Pos). Of course, the most general notions is that of an enriched functor between categoires enriched in a symmetric monoidal category. See [11].

1.4.4 Exercise (Enriched natural transformations) Reading Definition 1.1.9 with ‘map’ replaced by ‘monotone maps’ everywhere, you end up with the notion of an *enriched natural transformation* (between the functors enriched in Pos). See [11] for the full generality.

1.4.5 Exercise (The category of elements of a functor) Suppose $H : \mathscr{A} \longrightarrow \mathsf{Set}$ is a functor. Construct the category $\mathsf{elts}(H)$ of *elements* of H as follows:

(1) Objects are pairs (x, A), where $x \in HA$.

(2) A morphism from (x, A) to (x', A') is $f : A \longrightarrow A'$ such that $Hf(x) = x'$.

(3) Composition and identity morphisms are defined as in $\mathscr{A}$.

Observe that there is a functor $\partial_H : \mathsf{elts}(H) \longrightarrow \mathscr{A}$ with the object assignment $(x, A) \mapsto A$. The action of ∂_H on hom-sets is identity: $\partial_H : f \mapsto f$.

1.4.6 Exercise (Split epis are exactly absolute epis) Prove, for $e : X \longrightarrow Y$ in $\mathscr{X}$, that the following conditions are equivalent:

(1) e is a split epi.

(2) He is epi, for every functor $H : \mathscr{X} \longrightarrow \mathscr{K}$. That is: e is an *absolute epi*.

(3) $\mathscr{X}(Y, e) : \mathscr{X}(Y, X) \longrightarrow \mathscr{X}(Y, Y)$ is epi in Set.

1.4.7 Exercise (A morphism is $(*)$ if it is representably so) In some literature you may find the following expression

> A morphism $f : X \longrightarrow X'$ is $(*)$ if it is representably so.

where $(*)$ is some property of morphisms. By this, the authors mean: the mapping $\mathscr{X}(X_0, f) : \mathscr{X}(X_0, X) \longrightarrow \mathscr{X}(X_0, X')$ has the property $(*)$, for every X_0 in $\mathscr{X}$.

Prove the following:

(1) $f : X \longrightarrow X'$ is an isomorphism iff it is representably so.

(2) $m : X \longrightarrow X'$ is a monomorphism iff it is representably so.

Being "representably so" often indicates that we are on the right track when defining some notion in an abstract category. Hence the above indicates that "mono" is a proper generalisation of being injective. We could have introduced the word "injective" instead of "mono" in an arbitrary category and, by the above, we could have written

> $m : X \longrightarrow X'$ is injective iff it is representably so.

We will not speak about "injective" morphisms, however. We will stick to the word "monomorphism".

Show that such an intuition fails badly for the notion of being surjective, i.e., describe what "representably surjective" should mean in a general category.

1.4.8 Exercise (coYoneda Lemma) Yoneda Lemma 1.2.9 speaks about natural transformations *from* a representable functor $\mathscr{X}(X, -)$ *to* any functor H. The following result (dubbed *coYoneda Lemma* in [15]) speaks about natural transformations in the opposite direction. Prove that to give

> a natural transformation $\tau : H \longrightarrow \mathscr{X}(X, -)$

is to give

> a natural transformation from const_X to ∂_H, where $\mathrm{const}_X : \mathsf{elts}(H) \longrightarrow \mathscr{X}$ is the constant-at-X functor and $\partial_H : \mathsf{elts}(H) \longrightarrow \mathscr{X}$ is the canonical projection from the category of elements of H (see Exercise 1.4.5).

Chapter 2

Adjunctions

Adjoint functors arise everywhere.

Saunders Mac Lane

If there would have to be chosen just one fundamental concept of category theory, the author of these notes would vote for an *adjunction*. Pairs of adjoint functors can be literally found almost everywhere in category theory. Since we are interested in applications of the theory in universal algebra, we will stress the following facets of adjunctions:

(1) Adjunctions describe *free objects*.

(2) Adjunctions give rise to *theories of terms*.

(3) Adjunctions of special kind characterise *equationally defined* objects.

Of course, in passing, we will discover other facets of an adjunction (for example, one can define limits and colimits using adjunctions).

2.1 Free and cofree objects

2.1.1 Definition Suppose $U : \mathscr{A} \longrightarrow \mathscr{X}$ is given. We say that an object F_0X, together with an arrow $\eta_X : X \longrightarrow UF_0X$, is a *free object on X* (w.r.t. U), provided that the following property is satisfied:

For every $f : X \longrightarrow UA$ there is a *unique* $f^\sharp : F_0X \longrightarrow A$ such that the triangle

$$\begin{tikzcd}
UF_0X \arrow[r, "Uf^\sharp"] & UA \\
X \arrow[u, "\eta_X"] \arrow[ur, "f"'] &
\end{tikzcd} \tag{2.1}$$

commutes.

The above definition captures exactly the notion of a free object, known from classical universal algebra. The free object on the "object X of generators" is F_0X and $\eta_X : X \longrightarrow UF_0X$ is the "insertion of generators" into the free object. Let us stress, however, that we do not require η_X to be an "embedding" in any sense.

2.1.2 Example Denote by $U : \mathsf{Mon} \longrightarrow \mathsf{Set}$ the underlying functor from the category Mon of monoids and their homomorphisms to the category Set of sets and mappings. We show that a free object exists for every set X.

In fact, this is well-known: denote by F_0X the monoid $(X^*, e, \cdot)$, where X^* is the set of all finite words in the alphabet X, e is the empty word and $\cdot$ denotes concatenation. Clearly, $UF_0X = X^*$. Define $\eta_X : X \longrightarrow X^*$ as the map sending $x \in X$ to x, considered as the word of length one.

The universal property of (F_0X, η_X) then says that for every monoid $(M, i, \circ)$ and every map $f : X \longrightarrow M$, there is a unique homomorphism of monoids $f^\sharp : (X^*, e, \cdot) \longrightarrow (M, i, \circ)$ such that $f^\sharp(x) = f(x)$, for every $x \in X$.

The definition of $f^\sharp$ is clear:

$$f^\sharp(e) = i, \quad f^\sharp(x_1 \ldots x_n) = f(x_1) \circ \ldots \circ f(x_n)$$

It is a monoid homomorphism by definition and it is clearly unique with the property $f^\sharp(x) = f(x)$, for every $x \in X$.

The dual of Definition 2.1.1 gives the notion of a *cofree object*. Let us spell it in detail.

2.1.3 Definition We say that G_0X in $\mathscr{A}$, together with a morphism $\gamma_X : UG_0X \longrightarrow X$, is a *cofree object* on X w.r.t. $U : \mathscr{A} \longrightarrow \mathscr{X}$, provided that the following couniversal property is satisfied:

For every $f : UA \longrightarrow X$ there is a unique $f^\flat : A \longrightarrow G_0X$ such that the triangle

$$\begin{tikzcd}
UA \arrow[r, "Uf^\flat"] \arrow[dr, "f"'] & UG_0X \arrow[d, "\gamma_X"] \\
& X
\end{tikzcd}$$

commutes.

As we will see, cofree objects abound. In fact, for example, the underlying set M of any monoid (M, i, m) is a cofree object on (M, i, m) w.r.t. $F : \mathsf{Set} \longrightarrow \mathsf{Mon}$. We will be primarily interested in free objects, though, since they naturally appear in universal-algebraic reasoning.

Functors U having both free and cofree objects are rare in universal algebra but they appear quite naturally in topology, see Exercise 2.6.7. We describe now a functor having both free and cofree objects and having a universal-algebraic flavour.

2.1.4 Example In this example, $\mathbb{M} = (M, i, \circ)$ denotes a fixed monoid. We define a category $\mathbb{M}$-Acts of $\mathbb{M}$-*actions* and $\mathbb{M}$-*equivariant maps* and prove that it has *both* free objects *and* cofree objects w.r.t. the obvious underlying functor $U : \mathbb{M}\text{-}\mathsf{Acts} \longrightarrow \mathsf{Set}$.

An *action* of $\mathbb{M}$ on a set X is a mapping $@ : M \times X \longrightarrow X$, satisfying the equations

$$i@x = x, \quad (m_1 \circ m_2)@x = m_1@(m_2@x)$$

An $\mathbb{M}$-*equivariant map* from $(X, @)$ to $X', @')$ is a map $f : X \longrightarrow X'$ such that the equation

$$f(m@x) = m@'f(x)$$

is satisfied.

It is clear that $\mathbb{M}$-actions and $\mathbb{M}$-equivariant maps organise themselves into a category $\mathbb{M}$-Acts.

(1) A *free* object on X is the action

$$f_X : M \times (M \times X) \longrightarrow M \times X, \quad (m_1, m_2, x) \mapsto (m_1 \circ m_2, x)$$

Then the mapping $\eta_X : X \longrightarrow M \times X$, sending x to (i, x) is easily seen to satisfy the desired universal property.

(2) A *cofree* object on X w.r.t. U is the action

$$c_X : M \times [M, X] \longrightarrow [M, X], \quad c_X(m, h) : m' \mapsto h(m' \circ m)$$

The mapping $\gamma_X : [M, X] \longrightarrow X$ that evaluates at i is the couniversal mapping.

Indeed, suppose $(X', @')$ is any action and $f : X' \longrightarrow X$ is a map. Define $f^\flat : X' \longrightarrow [M, X]$ by putting $f^\flat(x') : m \mapsto f(m@'x')$.

Then $\gamma_X \cdot f^\flat = f$ holds by definition and $f^\flat$ is an equivariant map:

$$f^\flat(m@'x')(m') = f((m' \circ m)@'x') = c_X(m, f^\flat(x'))(m')$$

It is easy to verify that $f^\flat$ is the unique map with the above two properties.

2.1.5 Proposition *Suppose* $U : \mathscr{A} \longrightarrow \mathscr{X}$ *is given and suppose a free object* $(F_0 X, \eta_X)$ *exists for every* X*. Then the assignment* $X \mapsto F_0 X$ *extends to a functor* $F : \mathscr{X} \longrightarrow \mathscr{A}$ *and the collection of* η_X*'s forms a natural trasformation* $\eta : \mathrm{Id}_{\mathscr{X}} \longrightarrow UF$*.*

Proof. To define $F : \mathscr{X} \longrightarrow \mathscr{A}$, we put $FX = F_0 X$ on objects. On morphisms, we define $Ff : FX \longrightarrow FX'$ as $(\eta_{X'} \cdot f)^\sharp$. By the definition, the square

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & UFX \arrow[d, "UFf"] \\
X' \arrow[r, "\eta_{X'}"'] & UFX'
\end{tikzcd}$$

commutes. The square would prove that the collection of η_X's constitutes a natural transformation, had we known that F we had defined is indeed a functor. But this is the case:

(1) F preserves identities. Since both squares

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "1_X"'] & UFX \arrow[d, "UF1_X"] \\
X \arrow[r, "\eta_X"'] & UFX
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "1_X"'] & UFX \arrow[d, "U1_{FX}"] \\
X \arrow[r, "\eta_X"'] & UFX
\end{tikzcd}$$

commute (the one on the left hand side by the definition of $F1_X$), we proved that $F1_X = 1_{FX}$.

(2) F preserves composition. Consider the diagrams

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & UFX \arrow[d, "UFf"] \\
X' \arrow[r, "\eta_{X'}"] \arrow[d, "g"'] & UFX' \arrow[d, "UFg"] \\
X'' \arrow[r, "\eta_{X''}"] & UFX''
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "g\cdot f"'] & UFX \arrow[d, "U(FU(g\cdot f)"] \\
X'' \arrow[r, "\eta_{X''}"'] & UFX''
\end{tikzcd}$$

The diagram on the left commutes by the definition of Ff and Fg, the diagram on the right commutes by the definition of $F(g \cdot f)$. Thus $Fg \cdot Ff = F(g \cdot f)$ holds.

■

2.2 Adjunctions

2.2.1 Definition Suppose $U : \mathscr{A} \longrightarrow \mathscr{X}$ and $F : \mathscr{X} \longrightarrow \mathscr{A}$ are functors. We say that U is a *right* adjoint of F (and F is a *left* adjoint of U), provided there is a bijection

$$b_{X,A} : \mathscr{A}(FX, A) \longrightarrow \mathscr{X}(X, UA)$$

of hom-sets, natural in X and A. We denote the adjunction by $F \dashv U$.

2.2.2 Remark It is very useful to introduce some more notation and terminology concerning Definition 2.2.1.

(1) Given $h : FX \longrightarrow A$ we denote the value $b_{X,A}(h)$ by $h^\flat : X \longrightarrow UA$ and call it a *transpose* of h. Analogously, for $f : X \longrightarrow UA$ we denote the value $b^{-1}_{X,A}(f)$ by $f^\sharp : FX \longrightarrow A$ and call it a *transpose* of f. As we will see later, the notation $f^\sharp$ is in accordance with Definition 2.1.1.

(2) Instead of writing the bijection $b_{X,A}$ "linearly", we will often use the "fraction" notation

$$\frac{FX \xrightarrow{h} A}{X \xrightarrow{f} UA} F \dashv U$$

to mean that $b_{X,A}(h) = f$ (or, equivalently, $b^{-1}_{X,A}(f) = h$). We will also omit writing $F \dashv U$ frequently.

(3) Naturality of $b_{X,A}$ in X can be spelt, using "fractions", as follows:

$$\frac{FX' \xrightarrow{Ff'} FX \xrightarrow{f^\sharp} A}{X' \xrightarrow{f'} X \xrightarrow{f} UA}$$

meaning $(f \cdot f')^\sharp = f^\sharp \cdot Ff'$ holds, for any $f' : X' \longrightarrow X$.

(4) Naturality of $b_{X,A}$ in A can be spelt, using "fractions", as follows:

$$\frac{FX \xrightarrow{h} A \xrightarrow{h'} A'}{X \xrightarrow{h^\flat} UA \xrightarrow{Uh'} UA'}$$

meaning $(h' \cdot h)^\flat = Uh' \cdot h^\flat$ holds, for any $h' : A \longrightarrow A'$.

The notation borrowed from music should help to remember the transposes: in an adjunction $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$, the category $\mathscr{A}$ should be thought of as "living upstairs of $\mathscr{X}$". Then in

$$\frac{FX \xrightarrow{f^\sharp} A}{X \xrightarrow{f} UX}$$

the passage $f \mapsto f^\sharp$ is the "transposition upwards", whereas in

$$\frac{FX \xrightarrow{h} A}{X \xrightarrow{h^\flat} UX}$$

the passage $h \mapsto h^\flat$ is the "transposition downwards".

2.2.3 Categorical Trick When we want to prove equality $h = k : FX \longrightarrow A$ in $\mathscr{A}$, it suffices to prove the equality $h^\flat = k^\flat : X \longrightarrow UA$ in $\mathscr{X}$. Analogously, when we want to prove equality $f = g : X \longrightarrow UA$ in $\mathscr{X}$, it suffices to prove the equality $f^\sharp = g^\sharp : FX \longrightarrow A$ in $\mathscr{A}$.

2.2.4 Theorem *Suppose $U : \mathscr{A} \longrightarrow \mathscr{X}$, $F : \mathscr{X} \longrightarrow \mathscr{A}$ are given. Then the following are equivalent:*

(1) *There is a bijection $b_{X,A} : \mathscr{A}(FX, A) \longrightarrow \mathscr{X}(X, UA)$ of hom-sets, natural in X and A.*

(2) *There are natural transformations $\eta : \mathrm{Id}_{\mathscr{X}} \longrightarrow UF$ (called the* unit*) and $\varepsilon : FU \longrightarrow \mathrm{Id}_{\mathscr{A}}$ (called the* counit*) such that the diagrams (called the* triangle identities*)*

$$\begin{tikzcd} U \arrow[r, "\eta U"] \arrow[dr, equal] & UFU \arrow[d, "U\varepsilon"] & & FUF \arrow[d, "\varepsilon F"'] & F \arrow[l, "F\eta"'] \arrow[dl, equal] \\ & U & & F & \end{tikzcd} \tag{2.2}$$

commute.

Proof. (1) implies (2): We define η_X and ε_A, using the bijections $b_{X,A}$, as follows:

$$\frac{FX \xrightarrow{1_{FX}} FX}{X \xrightarrow{\eta_X} UFX} \qquad \frac{FUA \xrightarrow{\varepsilon_A} A}{UA \xrightarrow{1_{UA}} UA}$$

Observe that

$$\frac{FX \xrightarrow{1_{FX}} FX \xrightarrow{Ff} FX'}{X \xrightarrow{\eta_X} UFX \xrightarrow{UFf} UFX'} \qquad \frac{FX \xrightarrow{Ff} FX' \xrightarrow{1_{FX'}} FX'}{X \xrightarrow{f} X' \xrightarrow{\eta_{X'}} UFX'}$$

prove that η_X's form a natural transformation. That the collection of ε_A's forms a natural transformation is proved in a similar way.

To verify the commutativity of the triangle on the left of (2.2), consider

$$\frac{FUA \xrightarrow{1_{FUA}} FUA \xrightarrow{\varepsilon_A} A}{UA \xrightarrow{\eta_{UA}} UFUA \xrightarrow{U\varepsilon_A} UA} \qquad \frac{FUA \xrightarrow{\varepsilon_A} A}{UA \xrightarrow{1_{UA}} UA}$$

The commutativity of the triangle on the right of (2.2) is verified in a similar manner.

(2) implies (1): Given $h : FX \longrightarrow A$, define $b_{X,A}(h) : X \longrightarrow UA$ to be the composite

$$X \xrightarrow{\eta_X} UFX \xrightarrow{Uh} UA$$

We prove that $b_{X,A}$ is a bijection.

(i) $b_{X,A}$ is an injection. Suppose

$$X \xrightarrow{\eta_X} UFX \xrightarrow{Uh_1} UA \;=\; X \xrightarrow{\eta_X} UFX \xrightarrow{Uh_2} UA$$

Then (apply F to both sides)

$$FX \xrightarrow{F\eta_X} FUFX \xrightarrow{FUh_1} FUA \;=\; FX \xrightarrow{F\eta_X} FUFX \xrightarrow{FUh_2} FUA$$

and (postcompose with ε_A)

$$FX \xrightarrow{F\eta_X} FUFX \xrightarrow{FUh_1} FUA \xrightarrow{\varepsilon_A} A \;=\; FX \xrightarrow{F\eta_X} FUFX \xrightarrow{FUh_2} FUA \xrightarrow{\varepsilon_A} A$$

Using naturality of ε, we obtain

$$FX \xrightarrow{F\eta_X} FUFX \xrightarrow{\varepsilon_{FX}} FX \xrightarrow{h_1} A \;=\; FX \xrightarrow{F\eta_X} FUFX \xrightarrow{\varepsilon_{FX}} FX \xrightarrow{h_2} A$$

and finally, by the triangle on the right of (2.2), we obtain

$$FX \xrightarrow{h_1} A \;=\; FX \xrightarrow{h_2} A$$

as desired.

(ii) $b_{X,A}$ is a surjection. Given $f : X \longrightarrow UA$, define $h : FX \longrightarrow A$ as the composite

$$FX \xrightarrow{Ff} FUA \xrightarrow{\varepsilon_A} A$$

We prove $b_{X,A}(h) = f$. To that end, recall that $b_{X,A}(h)$ is the composite

$$X \xrightarrow{\eta_X} UFX \xrightarrow{UFf} UFUA \xrightarrow{U\varepsilon_A} UA$$

or, equivalently, the composite

$$X \xrightarrow{f} UA \xrightarrow{\eta_{UA}} UFUA \xrightarrow{U\varepsilon_A} UA$$

since η is natural. The last composite is equal to f due to the triangle on the left of (2.2).

So far we have proved that, for fixed X and A, there is a bijection

$$\frac{FX \xrightarrow{h} A}{X \xrightarrow{\eta_X} UFX \xrightarrow{Uh} UA} \qquad \frac{FX \xrightarrow{Ff} FUA \xrightarrow{\varepsilon_A} A}{X \xrightarrow{f} UA} \tag{2.3}$$

We prove naturality in X. This follows immediately from the "fraction"

$$\frac{FX' \xrightarrow{Ff'} FX \xrightarrow{Ff} FUA \xrightarrow{\varepsilon_A} A}{X' \xrightarrow{f'} X \xrightarrow{f} UA}$$

Naturality in A is proved analogously. ■

2.2.5 Remark (Left adjoints are essentially unique) Suppose $F_1 \dashv U$ and $F_2 \dashv U$ hold. Then there is a unique *natural isomorphism* $\tau : F_1 \longrightarrow F_2$.

This is immediate from the Yoneda Lemma: $F_1 \dashv U$ means that $\mathscr{A}(F_1-, A) \cong \mathscr{X}(-, UA)$ holds, and $F_2 \dashv U$ means that $\mathscr{A}(F_2-, A) \cong \mathscr{X}(-, UA)$ holds. Therefore we have an isomorphism $\mathscr{A}(F_1X, A) \cong \mathscr{A}(F_2X, A)$, natural in X and A. This means that the objects F_1X and F_2X are isomorphic (here we use Yoneda Lemma) and the isomorphism is natural in X.

2.2.6 Theorem (Universal-algebraic Adjoint Functor Theorem) *For a functor $U : \mathscr{A} \longrightarrow \mathscr{X}$, the following are equivalent:*

(1) *U has a left adjoint.*

(2) *There exists, for every X, a free object on X w.r.t. U.*

Proof. (1) implies (2). Denote the left adjoint of U by F and and by η the unit of $F \dashv U$. Then $\eta_X : X \longrightarrow UFX$ exhibits FX as a free object on X.

(2) implies (1). By Proposition 2.1.5 we know that there is a functor $F : \mathscr{X} \longrightarrow \mathscr{A}$ and a natural transformation $\eta : \mathrm{Id}_X \longrightarrow UF$ such that $\eta_X : X \longrightarrow UFX$ exhibits FX as a free object on X. Therefore the assignment

$$(f : X \longrightarrow UA) \mapsto (f^\sharp : FX \longrightarrow A)$$

provides us with a bijection $\mathscr{X}(X, UA) \longrightarrow \mathscr{A}(FX, A)$. Naturality of this bijection in X follows immediately from the universal property of free objects.

The inverse of the above bijection is given by

$$(h : FX \longrightarrow A) \mapsto (Uh \cdot \eta_X : X \longrightarrow UA)$$

from which naturality in A follows immediately. ■

2.3 Properties of adjoints in terms of the unit and the counit

The "fractions" (2.3) state that the diagrams

$$\begin{tikzcd}
\mathscr{X}(X', X) \arrow[r, "F_{X',X}"] \arrow[rr, bend right, "\mathscr{X}(X',\eta_X)"'] & \mathscr{A}(FX', FX) \arrow[r, "b_{X',FX}"] & \mathscr{X}(X', UFX)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathscr{A}(A, A') \arrow[r, "U_{A,A'}"] \arrow[rr, bend right, "\mathscr{A}(\varepsilon_A,A')"'] & \mathscr{X}(UA, UA') \arrow[r, "b^{-1}_{UA,A'}"] & \mathscr{A}(FUA, A')
\end{tikzcd}$$

commute. Therefore the two above diagrams yield the following two propositions stating properties of U in terms of the counit and properties of F in terms of the unit.

2.3.1 Proposition *Suppose $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$. Let ε denote the counit of $F \dashv U$. Then:*

(1) *The functor U is faithful iff every ε_A is an epimorphism.*

(2) *The functor U is full iff every ε_A is a split monomorphism.*

(3) *The functor U is fully faithful iff every ε_A is an isomorphism.*

PROOF. (1) U is faithful iff every $U_{A,A'}$ is an injective map. The latter is equivalent to the map

$$\mathscr{A}(\varepsilon_A, A') : (h : A \longrightarrow A') \mapsto (h \cdot \varepsilon_A : FUA \longrightarrow A')$$

being injective, for every A and A'. But this is to say that every ε_A is an epimorphism.

(2) U is full iff every $U_{A,A'}$ is a surjective map. The latter is equivalent to the map

$$\mathscr{A}(\varepsilon_A, A') : (h : A \longrightarrow A') \mapsto (h \cdot \varepsilon_A : FUA \longrightarrow A')$$

being surjective, for every A and A'. In particular, the map $\mathscr{A}(\varepsilon_A, FUA) : \mathscr{A}(A, FUA) \longrightarrow \mathscr{A}(FUA, FUA)$ is surjective. Therefore there exists $e : A \longrightarrow FUA$ such that $e \cdot \varepsilon_A = 1_{FUA}$ and we have proved that ε_A is split mono.

Conversely: if ε_A is split mono, there exists $e : A \longrightarrow FUA$ such that $e \cdot \varepsilon_A = 1_{FUA}$. Then, for every A', the map

$$(h : A \longrightarrow A') \mapsto (h \cdot \varepsilon_A : FUA \longrightarrow A')$$

is surjective: given $k : FUA \longrightarrow A'$, define $h = k \cdot e : A \longrightarrow A'$. Then

$$h \cdot \varepsilon_A = k \cdot e \cdot \varepsilon_A = k$$

and the mapping $\mathscr{A}(\varepsilon_A, A')$ is surjective.

(3) By the above, U is fully faithful iff every ε_A is both epi and split mono. The latter is equivalent to ε_A being an isomorphism. ■

2.3.2 Proposition *Suppose $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$. Let η denote the unit of $F \dashv U$. Then:*

(1) *The functor F is faithful iff every η_X is a monomorphism.*

(2) *The functor F is full iff every η_X is a split epimorphism.*

(3) *The functor F is fully faithful iff every η_X is an isomorphism.*

PROOF. This is analogous to the proof of Proposition 2.3.1. ■

2.4 Equivalences of categories

2.4.1 Definition Suppose $U : \mathscr{A} \longrightarrow \mathscr{X}$ and $F : \mathscr{X} \longrightarrow \mathscr{A}$ are functors and $\alpha : \mathrm{Id}_{\mathscr{X}} \longrightarrow UF$ and $\beta : FU \longrightarrow \mathrm{Id}_{\mathscr{A}}$ are natural transformations. The quadruple (U, F, α, β) is called

(1) An *isomorphism of categories*, if both α and β are identities.

(2) An *equivalence of categories*, if both α and β are isomorphisms.

2.4.2 Remark As we will see later, an isomorphism of categories is too strong a requirement of “being abstractly the same”. As we will see, in practice all one really needs when speaking about two categories “being abstractly the same” is the notion of an equivalence of categories.

Of course, an adjunction $F \dashv U$ such that both the unit η and the counit ε are isomorphisms, is an equivalence of categories. Such an adjunction is called an *adjoint equivalence*. The next result proves that there are no other equivalences of categories.

2.4.3 Proposition *For $U : \mathscr{A} \longrightarrow \mathscr{X}$, the following are equivalent:*

(1) *There exist F, α and β such that (U, F, α, β) is an equivalence of categories.*

(2) *There exists F such that $F \dashv U$ is an adjoint equivalence.*

(3) *The functor U is fully faithful and e.s.o.*

Proof. (2) clearly implies (1). To prove that (3) implies (2), we are going to construct a free object on every X. Since U is e.s.o., there is an object F_0X in $\mathscr{A}$ and an isomorphism $\eta_X : X \longrightarrow UF_0X$. Consider $f : X \longrightarrow UA$ and define $f^\sharp : FX \longrightarrow A$ as the unique morphism with $Uf^\sharp = f \cdot (\eta_X)^{-1}$ (here we have used that U is fully faithful). Hence the assignment $X \mapsto F_0X$ can be extended to a functor such that $F \dashv U$ wiht η as a unit. Since U is fully faithful, the counit is an isomorphism by Proposition 2.3.1. We have proved that U is part of an adjoint equivalence.

It remains to prove that (1) implies (3): We prove first that *both U and F* are faithful:

(i) U is faithful. Consider the diagram

$$\begin{tikzcd}
FUA \arrow[r, "\beta_A"] \arrow[d, "FUh"'] & A \arrow[d, "h"] \\
FUA' \arrow[r, "\beta_{A'}"'] & A'
\end{tikzcd}$$

Hence $h = \beta_{A'} \cdot FUh \cdot \beta_A^{-1}$. The last equality proves that U is faithful: if $Uh_1 = Uh_2$, then

$$h_1 = \beta_{A'} \cdot FUh_1 \cdot \beta_A^{-1} = \beta_{A'} \cdot FUh_2 \cdot \beta_A^{-1} = h_2$$

(ii) F is faithful. Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[d, "f"'] & UFX \arrow[d, "UFf"] \\
X' \arrow[r, "\alpha_{X'}"'] & UFX'
\end{tikzcd}$$

Hence $f = \alpha_{X'}^{-1} \cdot UFf \cdot \alpha_X$. The last equality proves that F is faithful: if $Ff_1 = Ff_2$, then

$$f_1 = \alpha_{X'}^{-1} \cdot UFf_1 \cdot \alpha_X = \alpha_{X'}^{-1} \cdot UFf_2 \cdot \alpha_X = f_2$$

To prove that U is full, suppose $f : UA \longrightarrow UA'$ is given and define h as $\beta_{A'} \cdot Ff \cdot \beta_A^{-1}$. Then both squares

$$\begin{tikzcd}
FUA \arrow[r, "\beta_A"] \arrow[d, "Ff"'] & A \arrow[d, "h"] \\
FUA' \arrow[r, "\beta_{A'}"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
FUA \arrow[r, "\beta_A"] \arrow[d, "FUh"'] & A \arrow[d, "h"] \\
FUA' \arrow[r, "\beta_{A'}"'] & A'
\end{tikzcd}$$

commute and this proves that $Ff = FUh$. Using faithfulness of F, the equality $f = Uh$ follows.

That U is e.s.o. is trivial, use the isomorphism $\alpha_X : X \longrightarrow UFX$. ■

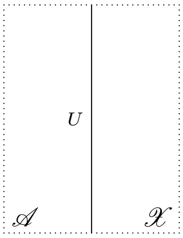
2.4.4 Remark The proof of Proposition 2.4.3 can be easily modified to obtain a useful characterisation of *isomorphisms* of categories. Namely, $U : \mathscr{A} \longrightarrow \mathscr{X}$ is an isomorphism of categories iff it is fully faithful and *bijective* on objects.

2.5 String diagrams for adjunctions

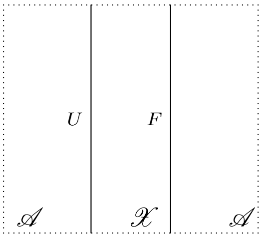
In various parts of category theory, a different notation for expressing commutative diagrams is used. The so-called *string diagrams* allow us to do computations with functors and natural transformations in a rather elegant way. Moreover, some results gain a very descriptive "graphic" meaning.

We will not introduce string diagrams formally, we just give examples. A string diagram consists of *areas*, *boxes* and *wires*. An area represents a category, a box represents a natural transfomation, and a wire represents a functor.

For example, a functor $U : \mathscr{A} \longrightarrow \mathscr{X}$ is represented as follows:

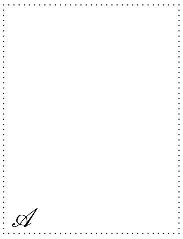


The string diagrams concerning functors are to be read from left to right and they can be composed together by horizontal pasting, hence



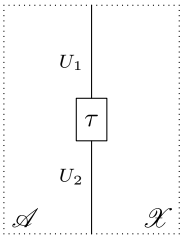
represents the composite $F \cdot U : \mathscr{A} \longrightarrow \mathscr{A}$.

The wire labelled by the identity functor as, e.g., $\mathrm{Id}_{\mathscr{X}}$ may be omitted from any picture. Hence



represents $\mathrm{Id}_{\mathscr{A}}$.

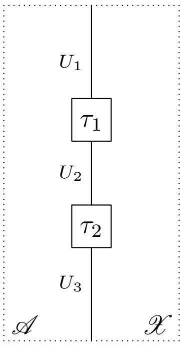
Adding boxes to wires allows us to represent natural transformation. For example



represents a natural transformation $\tau : U_1 \longrightarrow U_2$.

Pasting the string diagrams vertically corresponds to the *vertical* composition of natural transformations:

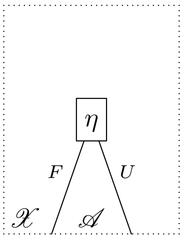
the diagram



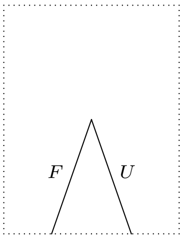
represents the composite of $\tau_1 : U_1 \longrightarrow U_2$ and $\tau_2 : U_2 \longrightarrow U_3$.

Analogously, pasting the string diagrams horizontally corresponds to the *horizontal* composition of natural transformations.

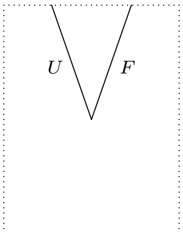
Therefore $\eta : \mathrm{Id}_{\mathscr{X}} \longrightarrow UF$ is represented by



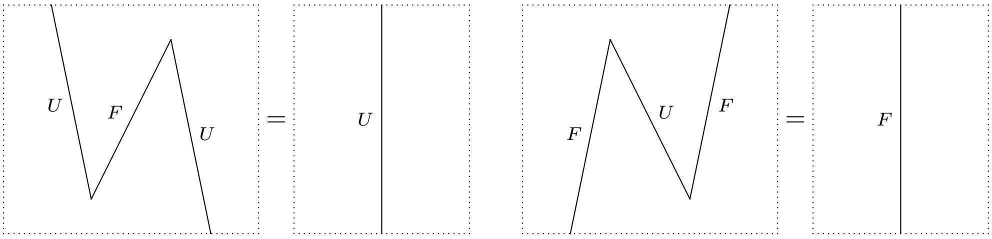
or even by



Analogously, $\varepsilon : FU \longrightarrow \mathrm{Id}_{\mathscr{A}}$ is represented by



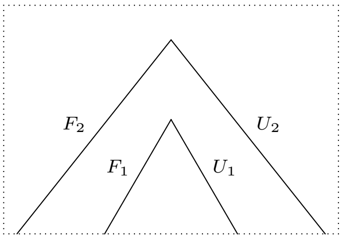
The triangle identities (2.2) then take the form of “yanking” axioms



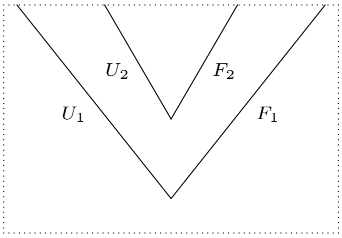
As an example of usefulness of string diagrams, we prove the following result.

2.5.1 Proposition (Adjunctions compose) *Suppose* $F_1 \dashv U_1 : \mathscr{A} \longrightarrow \mathscr{B}$ and $F_2 \dashv U_2 : \mathscr{B} \longrightarrow \mathscr{C}$ *hold. Then* $F_1F_2 \dashv U_2U_1 : \mathscr{A} \longrightarrow \mathscr{C}$ *holds.*

PROOF. Define a natural transformation $\mathrm{Id}_{\mathscr{C}} \longrightarrow U_2U_1F_1F_2$ by the diagram



and a natural transformation $F_1F_2U_2U_1 \longrightarrow \mathrm{Id}_{\mathscr{A}}$ by the diagram



The "yanking" axioms for the above natural transformations are clearly satisfied. ■

2.6 Exercises

2.6.1 Exercise (Adjunctions between preorders) Let $\mathscr{A}$ and $\mathscr{X}$ be preorders, considered as categories. Prove that an adjunction $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ amounts to

$$\frac{FX \leq A}{X \leq UA}$$

for every A in $\mathscr{A}$ and X in $\mathscr{X}$.

Prove that every adjunction $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ between preorders is a *Galois adjunction*, i.e., prove that equalities $FUFX = FX$ and $UA = UFUA$ hold, for all X and A.

2.6.2 Exercise (Galois connections on powersets) Let $R \subseteq A \times B$ be a binary relation. Denote by $\mathscr{A} = (PA, \subseteq)$ and $\mathscr{B} = (PB, \subseteq)$ the posets of subsets, regarded as categories. Prove that R defines an adjunction

$$(-)^u \dashv (-)^\ell : \mathscr{B}^{op} \longrightarrow \mathscr{A}$$

where, for $X \subseteq B$, X^ℓ is the set of all lower bounds of X, i.e.,

$$X^\ell = \{a \in A \mid a \mathrel{R} b \text{ for all } b \in X\}$$

and, for $Y \subseteq A$, Y^u is the set of all upper bounds of Y, i.e.,

$$Y^u = \{b \in B \mid a \mathrel{R} b \text{ for all } a \in Y\}$$

Decipher the above, in connection with Exercise 2.6.1, to derive that

$$\frac{X \subseteq Y^u}{Y \subseteq X^\ell}$$

holds for every $X \subseteq B$ and $Y \subseteq A$.

Prove that every adjunction $F \dashv U : \mathscr{B}^{op} \longrightarrow \mathscr{A}$ has the above form, i.e., find a binary relation $R \subseteq A \times B$ such that U computes the lower bounds and F computes the upper bounds. Hint: think of, e.g., the lower bounds of singleton sets.

An adjunction of the form $(-)^u \dashv (-)^\ell : \mathscr{B}^{op} \longrightarrow \mathscr{A}$ is often called a *Galois connection*.

2.6.3 Exercise (Dedekind cuts on rational numbers) Consider the set $\mathbb{Q}$ of rational numbers and let R be the relation $\{(r,s) \mid r \leq s\} \subseteq \mathbb{Q} \times \mathbb{Q}$. Consider the induced Galois connection

$$(-)^u \dashv (-)^\ell : \mathscr{Q}^{op} \longrightarrow \mathscr{Q}$$

on the category $\mathscr{Q} = (P\mathbb{Q}, \subseteq)$ and prove that to give a pair (L, U) with $L^u = U$ and $U^\ell = L$ is to give a *Dedekind cut* on $\mathbb{Q}$, i.e., prove that the pair (L, U) encodes an *extended real number* (that is, it encodes either an honest real number or $+\infty$ or $-\infty$).

2.6.4 Exercise (The Lindenbaum-Tarski algebra) Let BA denote the category of Boolean algebras and their homomorphisms. Denote by $U : \mathsf{BA} \longrightarrow \mathsf{Set}$ the obvious underlying functor.

Recall that a free Boolean algebra on a set X is usually called the *Lindenbaum-Tarski* algebra of formulas of classical propositional logic on the set X of atomic propositions.

Hence U has a left adjoint, denote it by F. Prove that $F \dashv U : \mathsf{BA} \longrightarrow \mathsf{Set}$ is equivalent to the fact that every *valuation* $\mathsf{val} : X \longrightarrow UA$ of atomic propositions in the (underlying set of) Boolean algebra A can be uniquely extended to a homomorphism $\|-\|_{\mathsf{val}} : FX \longrightarrow A$ from the Lindenbaum-Tarski algebra of all formulas.

2.6.5 Exercise (Heyting implication as a right adjoint) Recall the notion of a *Heyting algebra*. Prove that a distributive lattice $(H, \wedge, \vee)$ is a Heyting algebra iff the monotone map $- \wedge a : H \longrightarrow H$ (considered as a functor) has a right adjoint $a \Rightarrow -$. Write down the corresponding bijection and give it an interpretation known from logic.

2.6.6 Exercise (Residuated lattices) Recall that a *residuated lattice* is a lattice $(L, \wedge, \vee)$ equipped with a constant e, and three binary operations $\otimes$, $\rightarrow$, $\leftarrow$, satisfying certain axioms.

Prove that the axioms can be stated in a compact way as follows: a residuated lattice is a lattice together with an associative and unitary monotone binary operation $\otimes$, such that all monotone maps $a \otimes (-)$ and all monotone maps $(-) \otimes b$ have right adjoints.

2.6.7 Exercise (Discrete and indiscrete topological spaces) Let Top be the category of topological spaces and continuous maps. Denote by $U : \mathsf{Top} \longrightarrow \mathsf{Set}$ the obvious underlying functor. Prove that there exist free and cofree objects w.r.t. U. Hint: think of discrete and indiscrete topological spaces.

2.6.8 Exercise (The calculus of mates) Suppose $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ and $F' \dashv U' : \mathscr{A} \longrightarrow \mathscr{X}$ are adjunctions. Prove that there is a bijection

$$[\mathscr{A}, \mathscr{X}](U, U') \cong [\mathscr{X}, \mathscr{A}](F', F)$$

Given $\tau : U \longrightarrow U'$, the corresponding $\overline{\tau} : F' \longrightarrow F$ is called a *mate* of τ. Vice versa, τ is called a mate of $\overline{\tau}$.

Hint: given $\tau : U \longrightarrow U'$, define $\overline{\tau}$ as in

$$\begin{tikzcd}
F' \arrow[r, dotted, "\overline{\tau}"] \arrow[d, "F'\eta"'] & F \\
F'UF \arrow[r, "F\tau F'"'] & F'U'F \arrow[u, "\varepsilon' F"']
\end{tikzcd}$$

where η is the unit of $F \dashv U$ and ε' the counit of $F' \dashv U'$.

2.6.9 Exercise (U is faithful, if it reflects epis) Suppose $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$. Prove that U is faithful iff it *reflects epimorphisms*. The latter statement means: e is epi, whenever Ue is epi.

2.6.10 Exercise (Properties of U in terms of transposes) Suppose $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$. Prove that U is faithful iff the following condition holds: $f^\sharp : FX \longrightarrow A$ is an epimorphism, whenever $f : X \longrightarrow UA$ is an epimorphism.

2.6.11 Exercise (Not every general adjunction is of Galois type) Recall Exercise 2.6.1. Find an adjunction $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ such that at least one of the natural transformations $\varepsilon F : FUF \longrightarrow F$, $\eta U : U \longrightarrow UFU$ is not an isomorphism. Hint: most of the adjunctions you know from Universal Algebra will do.

Conclude that a general adjunction $F \dashv U$ need not be of *Galois type* (i.e., one where *both* ηU *and* εF are isomorphisms). For more on Galois adjunctions, see [10].

2.6.12 Exercise (Actions of a monoid, diagrammatically) Prove that the requirements on $\mathbb{M}$-actions and $\mathbb{M}$-equivariant maps of Example 2.1.4 can be stated in diagrammatical form as follows: the diagrams

$$\begin{tikzcd}
1 \times X \arrow[r, "i \times M"] \arrow[dr, equal] & M \times X \arrow[d, "@"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times M \times X \arrow[r, "M \times @"] \arrow[d, "\circ \times X"'] & M \times X \arrow[d, "@"] \\
M \times X \arrow[r, "@"'] & X
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \times X \arrow[r, "M \times f"] \arrow[d, "@"'] & M \times X' \arrow[d, "@'"] \\
X \arrow[r, "f"'] & X'
\end{tikzcd}$$

commute. Above, 1 denotes the one-element set and we have identified the cartesian product $1 \times X$ with X.

2.6.13 Exercise (The opposite adjunction) Use string diagrams to prove the following: suppose an adjunction $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ is given, with the unit η and the counit ε. Prove that $U^{op} \dashv F^{op} : \mathscr{X}^{op} \longrightarrow U^{op}$. Hint: given the string diagram called D, think about the string diagram D turned upside down, decorate all the items in D by writing ^{op} to them, and call the resulting diagram D^{op}. Then use the string diagrams for η and ε to give the unit and the counit of $U^{op} \dashv F^{op}$.

Chapter 3

Limits and colimits

> *A category theorist believes that a category without equalisers is "incomplete" and regards with suspicion statements such as "all sets will be assumed nonempty" which preface many books and papers; to her, it is like assuming that all complex numbers are nonzero.*
>
> Ernest G. Manes

Notions of a limit and colimit will generalise constructions we know from universal algebra: for example, a product of two lattices, a quotient of a group modulo a group congruence, etc. In fact, certain colimits and their interaction with an underlying functor $U : \mathscr{A} \longrightarrow \mathscr{X}$ will play a fundamental rôle in recognising $\mathscr{A}$ as a "variety" over $\mathscr{X}$.

3.1 Limits by universal cones

Limits of diagrams in categories generalise the notion of a *greatest lower bound* (aka *infimum*) of a set of elements of a poset. For posets, to say that we have a greatest lower bound comprises of two facts: that we have a lower bound and that we have a greatest of lower bounds. Lower bounds are generalised to cones, having a greatest lower bound is generalised to having a cone satisfying a universal property.

3.1.1 Definition Suppose $\mathscr{D}$ is a category. A functor $D : \mathscr{D} \longrightarrow \mathscr{X}$ is called a *diagram* of scheme $\mathscr{D}$ in $\mathscr{X}$. The diagram D is called *small*, if the scheme $\mathscr{D}$ is a small category.[1]

3.1.2 Definition A *cone* for $D : \mathscr{D} \longrightarrow \mathscr{X}$ is a tuple (X, f_d), where $f_d : X \longrightarrow Dd$ is a collection indexed by objects of $\mathscr{D}$, such that the triangle

$$\begin{tikzcd} & Dd \arrow[dd, "D\delta"] \\ X \arrow[ur, "f_d"] \arrow[dr, "f_{d'}"'] & \\ & Dd' \end{tikzcd}$$

commutes, for every $\delta : d \longrightarrow d'$ in $\mathscr{D}$.

A cone (L, proj_d) for D is called a *limit* of D, provided it has the following universal property:

For every cone (X, f_d) for D there is a unique $f : X \longrightarrow L$ such that the triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f_d"'] & L \arrow[d, "\mathrm{proj}_d"] \\ & Dd \end{tikzcd}$$

commutes, for every d in $\mathscr{D}$.

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[1] That is, the objects of $\mathscr{D}$ form a set.

A category $\mathscr{X}$ is called *complete*, if it has limits of all small diagrams.

As we expect, limits are *essentially unique*.

3.1.3 Lemma *Suppose* (L_1, proj^1_d) *and* (L_2, proj^2_d) *are limits of* $D : \mathscr{D} \longrightarrow \mathscr{X}$*. Then there is a canonical isomorphism* $c : L_1 \longrightarrow L_2$*. Conversely, if* (L, proj_d) *is a limit of* D *and* $f : L \longrightarrow L'$ *an isomorphism, then* $(L', f \cdot \mathrm{proj}_d)$ *is a limit of* D*.*

Proof. Define $c : L_1 \longrightarrow L_2$ using the universal property of (L_2, proj^2_d) and define $d : L_2 \longrightarrow L_1$ using the universal property of (L_1, proj^1_d). Then $d \cdot c = 1_{L_1}$ follows by the universal property of (L_1, proj^1_d) and $c \cdot d = 1_{L_2}$ follows by the universal property of (L_2, proj^2_d). ■

3.1.4 Example Let $\mathscr{X}$ be a preorder, considered as a category. A limit of a diagram D having two-object discrete category $\mathscr{D}$ as a scheme is the "greatest lower bound" of two points in $\mathscr{X}$. Unless $\mathscr{X}$ is actually a poset, such an "infimum" need not be determined uniquely.

To see an example, let $\mathscr{D}$ have two objects denoted by 0 and 1 and let the identity morphisms be the only morphisms in $\mathscr{D}$. To give a diagram $D : \mathscr{D} \longrightarrow \mathscr{X}$ is to give two elements $D0$ and $D1$ of the preorder $\mathscr{X}$. Suppose $\mathscr{X}$ is the following preorder on the set $\{a, b, c, d\}$ with $a \le c$, $a \le d$, $b \le c$, $b \le d$, $a \le b$, $b \le a$. If $D0 = c$ and $D1 = d$, then both a and b are limits of D.

3.1.5 Example It is quite easy to verify that the following formulas describe various notable limits in the category Set.

(1) A *product* and a *terminal object*. A set I can be regarded as a small discrete category. A limit of $D : I \longrightarrow \mathsf{Set}$ is the *cartesian product*

$$\prod_{i \in I} Di$$

and proj_i is the projection on the i-th cordinate. This construction includes the case of empty I: the product of an empty family is any one-element set. The product of an empty family is called a *terminal object*.

Observe that there exists an isomorphism

$$\mathsf{Set}(X', \prod_{i \in I} Di) \cong \prod_{i \in I} \mathsf{Set}(X', Di)$$

natural in X'.

(2) An *S-th power* of an object X. A special case of the product is the S-th power of X, denoted by $S \pitchfork X$. It is thus the limit of the diagram $D : S \longrightarrow \mathsf{Set}$, where S is discrete and $Ds = X$ for all s. The set $S \pitchfork X$ has functions $f : S \longrightarrow X$ as elements, the projection $\mathrm{proj}_s : S \pitchfork X \longrightarrow X$ is the evaluation-at-s, i.e., $\mathrm{proj}_s(f) = f(s)$.

Observe that there is an isomorphism

$$\mathsf{Set}(X', S \pitchfork X) \cong \mathsf{Set}(S, \mathsf{Set}(X', X))$$

natural in X'.

(3) An *equaliser* is a limit of a diagram $D : \mathscr{D} \longrightarrow \mathsf{Set}$, where $\mathscr{D}$ has the following shape

$$\begin{tikzcd} d \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] & d' \end{tikzcd}$$

Hence a cone for D can be expressed by giving $f : X \longrightarrow Dd$ such that $D\delta_0 \cdot f = D\delta_1 \cdot f$, or, in the language of diagrams, by giving a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Dd \arrow[r, shift left, "D\delta_0"] \arrow[r, shift right, "D\delta_1"'] & Dd' \end{tikzcd}$$

We say that f *equalises* $D\delta_0$ and $D\delta_1$.

It is easy to see that a limit, i.e., an *equaliser* of $D\delta_0$ and $D\delta_1$ is given by the set

$$E = \{x \in Dd \mid D\delta_0(x) = D\delta_1(x)\}$$

and the inclusion map $e : E \longrightarrow Dd$.

In our definition of a limit there was no restriction on the size of the scheme of a diagram. However, possession of limits that have a *large* category as a scheme is rare (although very useful in some applications). This is why completeness of a category was defined using *small* diagrams. A small complete category, however, trivialises to a preorder.

3.1.6 Example A *small* category having all small powers is necessarily a preorder.

Suppose $\mathscr{X}$ is small having all small powers. We want to prove that every hom-set $\mathscr{X}(X', X)$ contains at most one element. Suppose this is not the case: fix two objects X, X', with $\mathscr{X}(X, X')$ having at least two elements. Form the power $I \pitchfork X'$, where I is a set having cardinality λ greater than the cardinality of the set of all arrows in $\mathscr{X}$. Then the isomorphism

$$\mathscr{X}(X, I \pitchfork X') \cong \mathsf{Set}(I, \mathscr{X}(X, X'))$$

proves that $\mathscr{X}(X, I \pitchfork X')$ contains at least 2^λ elements. This is a contradiction.

Dualising the notion of a limit yields the notion of a colimit. We spell out the definition.

3.1.7 Definition A *cocone* for $D : \mathscr{D} \longrightarrow \mathscr{X}$ is a tuple (X, f_d), where $f_d : Dd \longrightarrow X$ is a collection indexed by objects of $\mathscr{D}$, such that the triangle

$$\begin{tikzcd} & Dd \arrow[dl, "f_d"'] \arrow[d, "D\delta"] \\ X & \\ & Dd' \arrow[ul, "f_{d'}"] \end{tikzcd}$$

commutes, for every $\delta : d \longrightarrow d'$ in $\mathscr{D}$.

A cocone (C, inj_d) for D is called a *colimit* of D, provided it has the following universal property:

For every cocone (X, f_d) for D there is a unique $f : C \longrightarrow X$ such that the triangle

$$\begin{tikzcd} X & C \arrow[l, "f"'] \\ & Dd \arrow[ul, "f_d"] \arrow[u, "\mathrm{inj}_d"'] \end{tikzcd}$$

commutes, for every d in $\mathscr{D}$.

A category $\mathscr{X}$ is called *cocomplete*, if it has colimits of all small diagrams.

Observe that a colimit in $\mathscr{X}$ is a limit in $\mathscr{X}^{op}$. Thus, colimits are essentially unique. However, the description of colimits is typically more difficult than that of limits. Let us see an example.

3.1.8 Example Dualising Example 3.1.5, we obtain the corresponding colimit concepts in the category Set.

(1) A *coproduct* an an *initial object*. A set I can be regarded as a small discrete category. A colimit of $D : I \longrightarrow \mathsf{Set}$ is the *disjoint union*

$$\coprod_{i \in I} Di$$

and inj_i is the injection to the i-th cordinate. This construction includes the case of empty I: the coproduct of an empty family is the empty set. The coproduct of an empty family is called an *initial object*.

Observe that there exists an isomorphism

$$\mathsf{Set}(\coprod_{i \in I} Di, X') \cong \prod_{i \in I} \mathsf{Set}(Di, X')$$

natural in X'.

(2) An *S-th copower* of an object X. A special case of the product is the S-th copower of X, denoted by $S \bullet X$. It is thus the colimit of the diagram $D : S \longrightarrow \mathsf{Set}$, where S is discrete and $Ds = X$ for all s. The set $S \bullet X$ has pairs (s, x) as elements, the injection $\mathrm{inj}_s : X \longrightarrow S \bullet X$ sends x to (s, x).

Observe that there is an isomorphism

$$\mathsf{Set}(S \bullet X, X') \cong \mathsf{Set}(S, \mathsf{Set}(X, X'))$$

natural in X'.

(3) A *coequaliser* is a colimit of a diagram $D : \mathscr{D} \longrightarrow \mathsf{Set}$, where $\mathscr{D}$ has the following shape

$$\begin{tikzcd} d \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] & d' \end{tikzcd}$$

Hence a cocone for D can be expressed by giving $f : Dd' \longrightarrow X$ such that $f \cdot D\delta_0 = f \cdot D\delta_1$, or, in the language of diagrams, by giving a commutative diagram

$$\begin{tikzcd} Dd \arrow[r, shift left, "D\delta_0"] \arrow[r, shift right, "D\delta_1"'] & Dd' \arrow[r, "f"] & X \end{tikzcd}$$

We say that f *coequalises* $D\delta_0$ and $D\delta_1$.

It is easy to see that a colimit, i.e., a *coequaliser* of $D\delta_0$ and $D\delta_1$ is given by the quotient map

$$e : Dd' \longrightarrow Dd'/E$$

where E is the equivalence relation generated by the set $\{(D\delta_0(x), D\delta_1(x)) \mid x \in Dd\}$ and e is the canonical mapping.

3.1.9 Remark In what follows we will speak of products, coproducts, equalisers and coequalisers in a general category. Their defining schemata are as in Examples 3.1.5 and 3.1.8, but their concrete descriptions will depend on the target category $\mathscr{X}$.

3.1.10 Categorical Trick Universal properties of limits and colimits will often be used in the following way:

(1) To define a morphism $X \longrightarrow L$, where L is a limit of some diagram, is to give a cone with vertex X for that diagram.

(2) To define a morphism $C \longrightarrow X$, where C is a colimit of some diagram, is to give a cocone with vertex X for that diagram.

3.1.11 Remark The limit and colimit concepts that we introduced are often called *conical*. There exists another very useful concept of *weighted* limits and colimits. See the monograph [11] for the full-fledged development of weighted limits and colimits in the setting of *enriched* categories.

3.2 Maranda's Theorem

We state and prove a result that allows us to compute any (co)limit, using just (co)products and (co)equalisers. In fact, we will only need to compute (co)equalisers of certain pairs, called reflexive.

3.2.1 Definition A parallel pair

$$\begin{tikzcd} X \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X' \end{tikzcd}$$

is called *reflexive* if there is a *common splitting* $s : X' \longrightarrow X$, i.e., s is such that the equalities $d_0 \cdot s = d_1 \cdot s = 1_{X'}$ hold.

We will formulate Maranda's Theorem for the case of colimits, the case of limits is dual (it deals with products and equalisers of (reflexive) pairs).

3.2.2 Theorem (Maranda's Theorem) *For a category $\mathscr{X}$, the following are equivalent:*

(1) *$\mathscr{X}$ has all small colimits.*

(2) *$\mathscr{X}$ has all small coproducts and all coequalisers.*

(3) *$\mathscr{X}$ has all small coproducts and all coequalisers of reflexive pairs.*

Proof. Clearly (1) implies (2) and (2) implies (3). To prove (3) implies (1), suppose that $D : \mathscr{D} \longrightarrow \mathscr{X}$ is a small diagram, and construct the following reflexive pair

$$\begin{tikzcd}\coprod_{\delta \in M} D\mathrm{dom}(\delta) \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & \coprod_{d \in O} Dd\end{tikzcd}$$

where O denotes the set of objects of $\mathscr{D}$ and M denotes the set of morphisms of $\mathscr{D}$, and $\mathrm{dom}(\delta)$ is the domain of δ.

The morphisms u and v are constructed using the universal property of coproducts as follows: the diagrams

$$\begin{tikzcd}\coprod_{\delta \in M} D\mathrm{dom}(\delta) \arrow[r, "u"] & \coprod_{d \in O} Dd \\ D\mathrm{dom}(\delta) \arrow[u, "\mathrm{inj}_\delta"] \arrow[r, "D\delta"'] & D\mathrm{cod}(\delta) \arrow[u, "\mathrm{inj}_{\mathrm{cod}(\delta)}"']\end{tikzcd}$$

and

$$\begin{tikzcd}\coprod_{\delta \in M} D\mathrm{dom}(\delta) \arrow[r, "v"] & \coprod_{d \in O} Dd \\ D\mathrm{dom}(\delta) \arrow[u, "\mathrm{inj}_\delta"] \arrow[ur, "\mathrm{inj}_{\mathrm{dom}(\delta)}"'] & \end{tikzcd}$$

are required to commute, for every $\delta \in M$.

The pair u, v is reflexive: the common splitting s is given by

$$\begin{tikzcd}\coprod_{d \in O} Dd \arrow[r, "s"] & \coprod_{\delta \in M} D\mathrm{dom}(\delta) \\ Dd \arrow[u, "\mathrm{inj}_d"] \arrow[ur, "\mathrm{inj}_{\mathrm{dom}(1_d)}"'] & \end{tikzcd}$$

Observe that $f : \coprod_{d \in O} Dd \longrightarrow X$ coequalises u, v iff the collection $f \cdot \mathrm{inj}_d : Dd \longrightarrow X$ forms a cocone for D.

Hence a coequaliser of u and v (that is assumed to exist) gives a colimit cocone for D. ■

Notice that from Examples 3.1.5 and 3.1.8 and from the above theorem, we can infer that the category Set has limits and colimits of *all* small diagrams, hence Set is *both* complete *and* cocomplete. Moreover, Maranda's Theorem gives us a concrete desciption of limits and colimits in Set.

3.2.3 Example Description of limits and colimits in the category Set.

(1) How to construct a limit of a diagram $D : \mathscr{D} \longrightarrow \mathsf{Set}$.

First form a cartesian product $P = \prod_d Dd$ of all objects Dd, denote by $\pi_d : P \longrightarrow Dd$ the projection onto the d-th coordinate. Then form the set

$$L = \{(x_d) \in P \mid D\delta(x_d) = x_{d'},\ \delta : d \longrightarrow d'\}$$

Denote the composite of the inclusion map $i : L \longrightarrow P$ with π_d by $\mathrm{proj}_d : L \longrightarrow Dd$. Then (L, proj_d) is a limit of D.

If you draw a picture you realise why the elements of L are often called *compatible threads*.

(2) How to construct a colimit of a diagram $D : \mathscr{D} \longrightarrow \mathsf{Set}$.

First form a disjoint union $U = \coprod_d Dd$ of all objects Dd, denote by $\iota_d : Dd \longrightarrow U$ the injection of the d-th coordinate. Then define an equivalence relation E on U that is generated by pairs

$$(x_d, x_{d'}), \quad \text{where } D\delta(x_d) = x_{d'},\ \delta : d \longrightarrow d'$$

and form the quotient map $e : U \longrightarrow U/E$. Denote the composite of ι_d with e by $\mathrm{inj}_d : Dd \longrightarrow U/E$. Then $(U/E, \mathrm{inj}_d)$ is a colimit of D.

3.3 Interaction of functors and limits

3.3.1 Definition We say that $U : \mathscr{A} \longrightarrow \mathscr{X}$ *preserves* a limit (L, proj_d) of a diagram $D : \mathscr{D} \longrightarrow \mathscr{A}$, if $(UL, U\mathrm{proj}_d)$ is a limit of $U \cdot D : \mathscr{A} \longrightarrow \mathscr{X}$. The preservation of a colimit is defined dually.

3.3.2 Theorem *Suppose $U : \mathscr{A} \longrightarrow \mathscr{X}$ has a left adjoint. Then U preserves any limit existing in $\mathscr{A}$.*

Proof. Denote by F the left adjoint of U. Suppose (L, proj_d) is a limit of $D : \mathscr{D} \longrightarrow \mathscr{A}$. We need to prove that $(UL, U\mathrm{proj}_d)$ is a limit of $U \cdot D : \mathscr{D} \longrightarrow \mathscr{X}$. To that end, consider a cone (X, f_d) for $U \cdot D$.

Since $f_d : X \longrightarrow UDd$, we can consider its transpose $f_d^\sharp : FX \longrightarrow Dd$. Observe that $(FX, f_d^\sharp)$ is a cone for D. This is seen as follows: the diagram

$$\begin{tikzcd}
 & & Dd \arrow[dddd, "D\delta"] \\
 & FUDd \arrow[ur, "\varepsilon_{Dd}"] \arrow[dd, "FUD\delta"] & \\
FX \arrow[ur, "Ff_d"] \arrow[dr, "Ff_{d'}"'] \arrow[uurr, bend left, "f_d^\sharp"] \arrow[ddrr, bend right, "f_{d'}^\sharp"'] & & \\
 & FUDd' \arrow[dr, "\varepsilon_{Dd'}"'] & \\
 & & Dd'
\end{tikzcd}$$

commutes, since the triangle does ((FX, Ff_d) is a cocone for FUD) and the trapezoid does commute (naturality of the counit ε).

Since (L, proj_d) is a limit, there is a unique $f : FX \longrightarrow L$ such that $\mathrm{proj}_d \cdot f = f_d^\sharp$ holds for every d. Consider now the transpose $f^\flat : X \longrightarrow UL$ of f. Then $U\mathrm{proj}_d \cdot f^\flat = f_d$ holds by the uniqueness of transposes. ■

Theorem 3.3.2 does not have a converse in general. For example, consider the category cBA of complete Boolean algebras and all Boolean homomorphisms preserving all suprema and all infima. Then cBA has all small limits and the obvious underlying functor $U : \mathsf{cBA} \longrightarrow \mathsf{Set}$ preserves them. However, for a given countable set X, there is a proper class of complete Boolean algebras that are generated by X, see [20]. Thus, the solution set at X does not exist.

There is a converse to the statement of Theorem 3.3.2 provided that U satisfies a certain side condition that is reminiscent of the existence of free objects.

3.3.3 Definition We say that $U : \mathscr{A} \longrightarrow \mathscr{X}$ *satisfies the Solution Set Condition at* X, provided there exists a set $S_X = \{f_i : X \longrightarrow UA_i \mid i \in I\}$ such that for every $f : X \longrightarrow UA$ there is (not necessarily unique) $h : A_i \longrightarrow A$ such that the triangle

$$\begin{tikzcd}
UA_i \arrow[r, "Uh"] & UA \\
X \arrow[u, "f_i"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes.

The set S_X is called the *solution set* for X.

3.3.4 Remark The notion of a solution set for X generalises the notion of a free object (F_0X, η_X) on X in two ways:

(1) One universal arrow $\eta_X : X \longrightarrow UF_0X$ is replaced by a set of arrows $f_i : X \longrightarrow UA_i$ in the solution set. As we will see, under certain circumstances, the arrows f_i can serve as a "germ" of the universal arrow.

(2) The universal property of $\eta_X : X \longrightarrow UF_0X$ is weakened: one requires the existence of some (not necessarily unique) extension of $f : X \longrightarrow UA$ along some f_i in the solution set.

3.3.5 Theorem (Freyd's General Adjoint Functor Theorem aka GAFT) *Suppose $\mathscr{A}$ is a category having all small limits, suppose $U : \mathscr{A} \longrightarrow \mathscr{X}$ is a functor. Then the following are equivalent:*

(1) *U has a left adjoint.*

(2) *U preserves all small limits and satisfies the following Solution Set Condition at every X.*

Proof. (1) implies (2): The functor U preserves limits by Theorem 3.3.2. Moreover, the one-element set

$$\{\eta_X : X \longrightarrow UFX\}$$

forms the solution set at X.

To prove that (2) implies (1), we need to construct a free object $\eta : X \longrightarrow UF_0X$ on every X. We divide the proof into two steps:

(i) We prove that a *weakly* free object on X exists. That is, we will construct $\eta'_X : X \longrightarrow UA$ having the universal property with the uniqueness condition removed.

(ii) We will "reduce" A to an honest free object F_0X.

We fix X, a solution set $S_X = \{f_i : X \longrightarrow UA_i \mid i \in I\}$, and we proceed as follows:

(i) Define $A = \prod_{i \in I} A_i$ in $\mathscr{A}$, denote the product projections by π_i. Since U preserves (A, π_i), the cone $(UA, U\pi_i)$ is a product in $\mathscr{X}$ and we can define $\eta'_X : X \longrightarrow UA$ as the unique morphism making the triangles

$$\begin{tikzcd}
X \arrow[r, "\eta'_X"] \arrow[dr, "f_i"'] & UA \arrow[d, "U\pi_i"] \\
& UA_i
\end{tikzcd} \tag{3.1}$$

commutative.

To see that η'_X exhibits A as *weakly free* on X, we need to show that for every $f : X \longrightarrow UB$ there is a (not necessarily unique) $h : A \longrightarrow B$ such that $Uh \cdot \eta'_X = f$ holds.

Given f, observe that there exists $f_i : X \longrightarrow UA_i$ in the solution set S_X and $h_i : A_i \longrightarrow B$ such that $Uh_i \cdot f_i = f$. Hence we may extend diagram (3.1) and define $h : A \longrightarrow B$ as indicated:

$$\begin{tikzcd}
X \arrow[r, "\eta'_X"] \arrow[dr, "f_i"] \arrow[ddr, "f"'] & UA \arrow[d, "U\pi_i"] \arrow[dd, bend left=60, "Uh"] \\
& UA_i \arrow[d, "Uh_i"] \\
& UB
\end{tikzcd}$$

(ii) The "reduction" of A will be done by a "collective equaliser". More precisely, define F_0X as the (vertex) of a limit

$$\begin{tikzcd}
F_0X \arrow[r, "e"] & A \arrow[r, shift left=2, "h"] \arrow[r, shift right=2, "h'"'] \arrow[r, phantom, "\vdots"] & A
\end{tikzcd} \tag{3.2}$$

where h, h' range over *all* morphisms from A to A such that $Uh \cdot \eta'_X = \eta'_X$ (and $Uh' \cdot \eta'_X = \eta'_X$).

This limit exists, since $\mathscr{A}(A, A)$ is a small set and h, h' are picked from that set. Moreover, U preserves the above limit and we can use the universal property to define $\eta_X : X \longrightarrow UF_0X$

$$\begin{tikzcd}
UF_0X \arrow[r, "Ue"] & UA \arrow[r, shift left=2, "Uh"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "Uh'"'] & UA \\
X \arrow[u, "\eta_X"] \arrow[ur, "\eta'_X"'] \arrow[urr, bend right, "\eta'_X"'] & &
\end{tikzcd} \tag{3.3}$$

We prove that $X \mapsto F_0X$, $\eta_X : X \longrightarrow UF_0X$ is a *free* object on X.

We know that $X \mapsto F_0X$, $\eta_X : X \longrightarrow UF_0X$ is certainly a *weakly free* object on X, since $X \mapsto A$, $\eta'_X : X \longrightarrow UA$ is.

Suppose that for $f : X \longrightarrow UA$ there are $h_1, h_2 : F_0X \longrightarrow A$ such that $Uh_1 \cdot \eta_X = Uh_2 \cdot \eta_X = f$. We need to prove $h_1 = h_2$. To that end, form the equaliser

$$\begin{tikzcd}
E \arrow[r, "j"] & F_0X \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & A
\end{tikzcd} \tag{3.4}$$

and observe that *to prove* $h_1 = h_2$ *it suffices to prove that* j *is an isomorphism.*

The equaliser (3.4) is preserved by U, hence there is a factorisation

$$\begin{tikzcd}
UE \arrow[r, "Uj"] & UF_0X \arrow[r, shift left, "Uh_1"] \arrow[r, shift right, "Uh_2"'] & UA \\
X \arrow[u, "f"] \arrow[ur, "\eta_X"'] & &
\end{tikzcd} \tag{3.5}$$

since $Uh_1 \cdot \eta_X = Uh_2 \cdot \eta_X$ holds by assumption.

Now we use weak freeness of $\eta'_X : X \longrightarrow UA$ to define (not necessarily in a unique way) $k : A \longrightarrow E$, such that

$$\begin{tikzcd}
UA \arrow[r, "Uk"] & UE \\
X \arrow[u, "\eta'_X"] \arrow[ur, "f"'] &
\end{tikzcd} \tag{3.6}$$

commutes.

Putting (3.6), (3.5) and (3.3) together yields a diagram

$$\begin{tikzcd}
UA \arrow[r, "Uk"] \arrow[rrr, bend left, "U1_A"] & UE \arrow[r, "Uj"] & UF_0X \arrow[r, "Ue"] & UA \\
& X \arrow[ul, "\eta'_X"] \arrow[u, "f"'] \arrow[ur, "\eta_X"] \arrow[urr, "\eta'_X"'] & &
\end{tikzcd}$$

showing that e equalises 1_A and $e \cdot j \cdot k$. Moreover, both 1_A and $e \cdot j \cdot k$ belong to the set of which e is an equaliser.

Therefore the diagram

$$\begin{tikzcd}
F_0X \arrow[r, "e"] \arrow[rrr, bend right, "1_{F_0X}"'] & A \arrow[r, "k"] \arrow[r, phantom, "(*)"', shift right=3] \arrow[rrr, bend left, "1_A"] & E \arrow[r, "j"] & F_0X \arrow[r, "e"] & A
\end{tikzcd}$$

commutes. But the area $(*)$ commutes, since e is a monomorphism, being an equaliser.

We proved that j is split epi. Since j is an equaliser, it is a monomorphism. Therefore j is an isomorphism.

The proof is finished. ■

3.3.6 Example The Solution Set Condition is, of course, a void requirement in the case of preorders. More precisely, the following are equivalent, for a complete preorder $\mathscr{A}$ and a monotone map $U : \mathscr{A} \longrightarrow \mathscr{X}$:

(1) U has a left adjoint.

(2) U preserves infima.

The reason is that the set $\{X \leq UA_i \mid i \in I\}$, where I is the set of *all* objects of $\mathscr{A}$, clearly satisfies the requirements of Theorem 3.3.5.

3.4 Exercises

3.4.1 Exercise (Equalisers and coequalisers in preorders) Prove that *any* preorder has equalisers and coequalisers.

3.4.2 Exercise (Completeness does not imply cocompleteness) Find a complete category that is not cocomplete. Hint: think, for example, of the large poset of all ordinals, ordered by reversed inclusion. Were it cocomplete, the largest ordinal would exist.

3.4.3 Exercise (Pullbacks and kernel pairs) A limit of the diagram

$$\begin{tikzcd} & B \arrow[d, "d_1"] \\ A \arrow[r, "d_0"'] & C \end{tikzcd}$$

is called a *pullback* (of d_0 along d_1). Describe pullbacks in Set.

Pay special attention to the pullback of d_0 along itself (the corresponding cone is called the *kernel pair* of d_0). Prove that, in the category of Set, this construction gives rise to an equivalence relation on A.

3.4.4 Exercise (A factorisation using kernel pairs and coequalisers of reflexive pairs) Let $\mathscr{X}$ be a category having kernel pairs and coequalisers of reflexive pairs. Perform, for $f : X \longrightarrow X'$, the following constructions:

(1) Form the kernel pair of f:

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_0"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

(2) Using the universal property of pullbacks, prove that the pair

$$\begin{tikzcd} P \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \end{tikzcd}$$

is reflexive.

(3) Form a coequaliser

$$\begin{tikzcd} P \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[r, "e"] & Z \end{tikzcd}$$

(4) Use the universal property of coequalisers to define $m : Z \longrightarrow X'$ as the unique morphism in the diagram

$$\begin{tikzcd}
P \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[r, "e"] \arrow[dr, "f"'] & Z \arrow[d, "m"] \\
& & X'
\end{tikzcd}$$

Analyse the above construction in Set and prove that $f = m \cdot e$ is the usual factorisation of a map through its image. That is, prove that one can put $Z = f[X]$ and m is the inclusion.

3.4.5 Exercise (Limits and colimits in sets) Work out in detail Examples 3.1.5 and 3.1.8.

3.4.6 Exercise (Filtered colimits) A colimit of a small diagram $D : \mathscr{D} \longrightarrow \mathscr{X}$ is called *filtered*, if $\mathscr{D}$ is a *filtered category*. That $\mathscr{D}$ is filtered means: every *finite* diagram $S : \mathscr{C} \longrightarrow \mathscr{D}$ has a cocone in $\mathscr{D}$.

Prove the following:

(1) A category $\mathscr{D}$ is filtered iff it is non-empty, it contains a cocone for every pair of objects, and it contains a cocone for every pair of parallel morphisms.

Conclude that a preorder $\mathscr{D}$ is filtered iff it is non-empty and *upwards-directed*, i.e., every pair d_0, d_1 has an upper bound in $\mathscr{D}$.

(2) Prove that every set X can be expressed as a filtered colimit of its finite subsets.

(3) A functor $\mathsf{Set}(X, -) : \mathsf{Set} \longrightarrow \mathsf{Set}$ preserves filtered colimits iff the set X is finite. Hint: you will use Example 3.2.3.

3.4.7 Exercise (Natural numbers as an initial object) Define the category $\mathscr{A}$ as follows:

(1) Objects of $\mathscr{A}$ are diagrams of the form

$$\begin{tikzcd}
1 \arrow[r, "z"] & X \arrow[r, "s"] & X
\end{tikzcd}$$

where 1 is a one-element set, X is a set, and $z : 1 \longrightarrow X$, $X \longrightarrow X$ are mappings. We will write (X, z, s) for short.

(2) A morphism from (X, z, s) to (X', s', f') is a map $h : X \longrightarrow X'$, making both squares in the diagram

$$\begin{tikzcd}
1 \arrow[r, "z"] \arrow[d, equal] & X \arrow[r, "s"] \arrow[d, "h"] & X \arrow[d, "h"] \\
1 \arrow[r, "z'"'] & X' \arrow[r, "s'"'] & X'
\end{tikzcd}$$

commutative.

Prove that $(\mathbb{N}, \mathsf{zero}, \mathsf{succ})$, where $\mathbb{N}$ is the set of natural numbers, the function zero picks up number 0, and the function succ is the successor function, is an initial object of $\mathscr{A}$.

Hint: think of induction principles and definition by recursion.

3.4.8 Exercise (Natural numbers as a free algebra for a functor) Rewrite the category $\mathscr{A}$ from Exercise 3.4.7 as follows:

(1) Prove that the assignment $X \mapsto X + 1$ can be extended to a functor $L : \mathsf{Set} \longrightarrow \mathsf{Set}$. Hint: use the universal property of a coproduct.

(2) Prove that to give an object of $\mathscr{A}$ is to give a map $a : LX \longrightarrow X$. Prove that to give a morphism in $\mathscr{A}$ is to give a mapping $h : X \longrightarrow X'$ such that the square

$$\begin{tikzcd} LX \arrow[r, "Lh"] \arrow[d, "a"'] & LX' \arrow[d, "a'"] \\ X \arrow[r, "h"'] & X' \end{tikzcd}$$

commutes. Hint: use the universal property of a coproduct again.

(3) Denote the category, having mappings $a : LX \longrightarrow X$ as objects and mappings $h : X \longrightarrow X'$ making the above squares commutative, by Set^L. The resulting category is called the *category of algebras for the functor* L.

(4) Prove that the obvious assignment $(X, a) \mapsto X$ extends to a functor $U^L : \mathsf{Set}^L \longrightarrow \mathsf{Set}$.

(5) Prove that U^L has a left adjoint. If you denote the left adjoint by F^L, show that natural numbers with zero function and successor operation form a free L-algebra on the empty set. What is $F^L(X)$ for a general set X?

3.4.9 Exercise (Natural number objects in a general category) Generalise Exercise 3.4.8 by replacing Set with any category $\mathscr{X}$ having a terminal object 1 and binary coproducts. More in detail:

(1) Prove that the assignment $X \mapsto X + 1$ extends to a functor $L : \mathscr{X} \longrightarrow \mathscr{X}$.

(2) Define the category $\mathscr{X}^L$ and the functor $U^L : \mathscr{X}^L \longrightarrow \mathscr{X}$ in the obvious way.

(3) Suppose U^L has a left adjoint. Denote the adjoint by F^L. If $\mathscr{X}$ has an initial object 0, think of $F^L 0$ as of the "object of natural numbers" in $\mathscr{X}$. Try to describe the concept in some categories other than Set. What happens when $\mathscr{X}$ is a preorder?

3.4.10 Exercise (Algebras for a signature) We generalise Exercise 3.4.9 and prove that a finitary syntax can be encoded into functors of a special kind. Denote by N the discrete category having finite ordinals as objects and fix a category $\mathscr{X}$ having all limits and colimits that are needed in the following constructions:

(1) Think of a functor $S : N \longrightarrow \mathsf{Set}$ as of a *finitary signature*. More precisely, think of each value Sn as of the *set of* n*-ary operations* of the signature.

(2) Given a finitary signature S, prove that the assignment $X \mapsto \coprod_n Sn \bullet X^n$, where the coproduct is taken over all finite ordinals and X^n denotes the power $n \pitchfork X$, can be extended to a functor $L_S : \mathscr{X} \longrightarrow \mathscr{X}$.

(3) Analyse a morphism $a : L_S X \longrightarrow X$ as follows:

   (a) To give a is to give $a_n : Sn \bullet X^n \longrightarrow X$, for every finite ordinal n. Hint: use the universal property of coproducts.

   (b) To give $a_n : Sn \bullet X^n \longrightarrow X$ is to give a map $\breve{a}_n : Sn \longrightarrow \mathscr{X}(X^n, X)$. Hint: use the universal property of copowers.

   (c) To give $\breve{a}_n : Sn \longrightarrow \mathscr{X}(X^n, X)$ is to give, for each n-ary operation symbol σ, a morphism $[\![\sigma]\!] : X^n \longrightarrow X$ in $\mathscr{X}$. The morphism $[\![\sigma]\!] : X^n \longrightarrow X$ is the *interpretation* of the operation symbol σ in the algebra.

Think of $L_S X$ as of the "object of terms in variables X of depth ≤ 1. Such terms are commonly called *flat terms*. The above analysis shows that to give $a : L_S X \longrightarrow X$ is to give interpretations in X for all operations in the signature.

(4) Analyse the commutative square

$$\begin{tikzcd} L_S X \arrow[r, "L_S h"] \arrow[d, "a"'] & L_S X' \arrow[d, "a'"] \\ X \arrow[r, "h"'] & X' \end{tikzcd}$$

in an analogous way as you did analyse the morphism $a : L_S X \longrightarrow X$ and conclude that the commutativity of the above square is equivalent to commutativity of the squares

$$\begin{tikzcd} X^n \arrow[r, "h^n"] \arrow[d, "[\![\sigma]\!]"'] & X'^n \arrow[d, "[\![\sigma]\!]"] \\ X \arrow[r, "h"'] & X' \end{tikzcd}$$

for every n and every σ in Sn. Shortly: homomorphisms of are exactly those morphisms that preserve all the specified operations.

The category $\mathscr{X}^{L_S}$ is called the category of *algebras for the signature* S. Give various instances of $\mathscr{X}^{L_S}$ when $\mathscr{X} = \mathsf{Set}$.

3.4.11 Exercise (Free algebras for a signature) Let $S : N \longrightarrow \mathsf{Set}$ be a finitary signature in the sense of Exercise 3.4.10. Prove, using Theorem 3.3.5, that $U^{L_S} : \mathsf{Set}^{L_S} \longrightarrow \mathsf{Set}$ has a left adjoint.

Try to analyse in detail what you use when finding a solution set S_X for a set X and try to generalise the result for other categories than Set.

3.4.12 Exercise (Algebras and coalgebras for a general endofunctor) Generalise Exercise 3.4.10 for an *arbitrary* functor $L : \mathscr{X} \longrightarrow \mathscr{X}$. That is, define the category $\mathscr{X}^L$ of *algebras for* L and the functor $U^L : \mathscr{X}^L \longrightarrow \mathscr{X}$.

Consider the category $(\mathscr{X}^{op})^{L^{op}}$ and give an explicit description of its objects and morphisms. Write $\mathscr{X}_L$ instead of $(\mathscr{X}^{op})^{L^{op}}$ and call it the category of *coalgebras for* L.

3.4.13 Exercise (Algebras as prefixed points and Lambek's Lemma) Let $\mathscr{X}$ be a poset, let $L : \mathscr{X} \longrightarrow \mathscr{X}$ be a monotone map. Prove that $\mathscr{X}^L$ is exactly the *poset of prefixed points* of L, where X is a prefixed point for L, iff the inequality $LX \leq X$ holds.

Prove:

(1) A least prefixed point of L is a fixed point of L.

(2) Generalise the above to obtain *Lambek's Lemma*: if $\mathscr{X}$ is a category, $L : \mathscr{X} \longrightarrow \mathscr{X}$ a functor, and $a : LX \longrightarrow X$ is an initial object of $\mathscr{X}^L$, then a is an isomorphism. Hint: consider the L-algebra $La : LLX \longrightarrow LX$ and use initiality of $a : LX \longrightarrow X$ to conclude that the square

$$\begin{tikzcd} LX \arrow[r, "Lh"] \arrow[d, "a"'] & LLX \arrow[d, "La"] \\ X \arrow[r, "h"'] & LX \end{tikzcd}$$

commutes for a unique $h : X \longrightarrow LX$. Conclude that h is the inverse of a, using initiality again.

(3) Conclude that Set^P, where $P : \mathsf{Set} \longrightarrow \mathsf{Set}$ is the powerset functor, is not a cocomplete category. Hint: a cocomplete category has to have an initial object.

3.4.14 Exercise (Kripke frames as coalgebras) For those who know some basic modal logic. Denote by $P : \mathsf{Set} \longrightarrow \mathsf{Set}$ the *covariant powerset functor*, i.e., let PX be the set of subsets of X and, for a mapping $f : X \longrightarrow X'$, let Pf send $a \subseteq X$ to its image $f[a] \subseteq X'$. Prove that Set_P (notation as in Exercise 3.4.12) is exactly the category of *Kripke frames* and *bounded morphisms* that you know from modal logic.

Use Lambek's Lemma in a clever way to conclude that Set_P does not have a terminal object (= a *final* Kripke frame does not exist).

3.4.15 Exercise (Weak limits) A cone (L, proj_d) is a *weak limit* of a diagram $D : \mathscr{D} \longrightarrow \mathscr{X}$, provided that it has the universal property with the uniqueness requirement removed.

Prove that any functor that preserves weak limits, preserves honest limits. Hint: proceed as follows:

(1) Prove that a weak limit (L, proj_d) is an honest limit iff the cone $\mathrm{proj}_d : L \longrightarrow Dd$ is *collectively mono*, i.e., if $u = v$, whenever $\mathrm{proj}_d \cdot u = \mathrm{proj}_d \cdot v$ holds for every d.

(2) Prove that the cone $H\mathrm{proj}_d : HL \longrightarrow HDd$ is collectively mono, whenever (L, proj_d) is a limit and H preserves weak limits.

3.4.16 Exercise (Truncated GAFT) In this exercise, $\lambda \geq 1$ denotes a regular cardinal. A set is λ-small if it has fewer than λ elements. A λ-small limit is a limit of a diagram where the scheme has a λ-small set of morphisms.

Prove the *Truncated General Adjoint Functor Theorem*:

> *Suppose $\mathscr{A}$ has λ-small limits. For $U : \mathscr{A} \longrightarrow \mathscr{X}$ the following are equivalent:*
>
> (1) *U has a left adjoint.*
>
> (2) *U preserves λ-small limits, for every X there exists a λ-small solution set S_X and, moreover, for every $f : X \longrightarrow UA$ in S_X, the set of all $h : A \longrightarrow A$ such that $Uh \cdot f = f$ is λ-small.*

Hint: go carefully through the proof of Theorem 3.3.5.

3.4.17 Exercise (Heyting implication and the distributive law) Recall from Exercise 2.6.5 the characterisation of Heyting algebras. Prove, using Theorem 3.3.5, that a *complete* lattice is a Heyting algebra iff the *infinite distributive law*

$$\bigvee_{i \in I} (x_i \wedge a) = \left(\bigvee_{i \in I} x_i \right) \wedge a$$

holds for any a and any set I.

3.4.18 Exercise (Right adjoints to Set are representable) Prove the following: Suppose $F \dashv U : \mathscr{A} \longrightarrow \mathsf{Set}$ is given. Then U is representable (see Definition 1.2.7).

Hint: define the representing object as $F1$, and use $F \dashv U$ and Yoneda Lemma.

3.4.19 Exercise (Left adjoints to representable functors) Prove the following: Suppose $U : \mathscr{A} \longrightarrow \mathsf{Set}$ is representable with representing object A_0. Then U has a left adjoint iff, for every set X, the copower $X \bullet A_0$ exists in $\mathscr{A}$.

Hint: use the universal property of copowers.

3.4.20 Exercise (Left adjoint to the ultrafilter functor) Let BA denote the category of Boolean algebras and their homomorphisms. Let 2 denote the two-element Boolean algebra. Use Exercise 3.4.19 in a clever way to establish the existence and description of a left adjoint of the functor $\mathsf{BA}(-, 2) : \mathsf{BA}^{op} \longrightarrow \mathsf{Set}$. Observe that $\mathsf{BA}(A, 2)$ is the set of all *ultrafilters* on the Boolean algebra A.

3.4.21 Exercise (Left adjoints to contravariant representable functors) Use Exercise 3.4.19 in a clever way to establish the necessary and sufficient conditions on $\mathscr{A}$ such that the *contravariant representable functor* $\mathscr{A}(-, A_0) : \mathscr{A}^{op} \longrightarrow \mathsf{Set}$ has a left adjoint.

Chapter 4

Monads

> *Each monad perceives all the other monads more or less clearly, but only God perceives all monads with utter clarity.*
>
> Baron Gottfried Wilhelm von Leibniz

We are going to introduce the main concept of this text — a *monad* on a category. As we will see, a monad encodes what it means to be an algebraic theory. A monad can be viewed in various ways. We will stress the following two aspects:

(1) A monad is a monoid in a certain sense.

(2) A monad is an abstract formation of terms.

Both views will be useful. Namely, we will introduce the category of "actions of a monad" in an analogous way as the actions of a monoid are introduced. On the other hand, we will introduce a "category of substitutions" for a monad. Both concepts are of great interest: actions of a monad (the *Eilenberg-Moore category*) encode varieties in the sense of Category Theory, substitutions (the *Kleisli category*) encode the minimal information of an algebraic theory that is needed to reconstruct its variety of algebras.

4.1 Monads as monoids

Consider an adjunction $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$. Recall that the diagrams

$$\begin{tikzcd}
U \arrow[r, "\eta U"] \arrow[dr, equal] & UFU \arrow[d, "U\varepsilon"] \\
& U
\end{tikzcd}
\qquad
\begin{tikzcd}
FUF \arrow[d, "\varepsilon F"'] & F \arrow[l, "F\eta"'] \arrow[dl, equal] \\
F &
\end{tikzcd}
\qquad
\begin{tikzcd}
FUFUF \arrow[r, "FU\varepsilon F"] \arrow[d, "\varepsilon FUF"'] & FUF \arrow[d, "\varepsilon F"] \\
FUF \arrow[r, "\varepsilon F"'] & F
\end{tikzcd}$$

commute. In fact, the triangles are the triangle identities (2.2) for $F \dashv U$ and the square is the naturality square for ε.

Precomposing the triangle on the left with F and postcomposing the remaining two diagrams with U, we therefore obtain commutative diagrams

$$\begin{tikzcd}
UF \arrow[r, "\eta UF"] \arrow[dr, equal] & UFUF \arrow[d, "U\varepsilon F"] \\
& UF
\end{tikzcd}
\qquad
\begin{tikzcd}
UFUF \arrow[d, "U\varepsilon F"'] & UF \arrow[l, "UF\eta"'] \arrow[dl, equal] \\
UF &
\end{tikzcd}
\qquad
\begin{tikzcd}
UFUFUF \arrow[r, "UFU\varepsilon F"] \arrow[d, "U\varepsilon FUF"'] & UFUF \arrow[d, "U\varepsilon F"] \\
UFUF \arrow[r, "U\varepsilon F"'] & UF
\end{tikzcd}$$

or, writing $T = UF$, $\mu = U\varepsilon F$, the diagrams

$$\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, equal] & TT \arrow[d, "\mu"] \\ & T \end{tikzcd} \qquad \begin{tikzcd} TT \arrow[d, "\mu"'] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\ T & \end{tikzcd} \qquad \begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"'] & T \end{tikzcd} \tag{4.1}$$

Compare the above with the diagrammatic description of a monoid $\mathbb{M} = (M, i, \circ)$:

$$\begin{tikzcd} 1 \times M \arrow[r, "i \times M"] \arrow[dr, equal] & M \times M \arrow[d, "\circ"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} M \times M \arrow[d, "\circ"'] & M \times 1 \arrow[l, "M \times i"'] \arrow[dl, equal] \\ M & \end{tikzcd} \qquad \begin{tikzcd} M \times M \times M \arrow[r, "M \times \circ"] \arrow[d, "\circ \times M"'] & M \times M \arrow[d, "\circ"] \\ M \times M \arrow[r, "\circ"'] & M \end{tikzcd}$$

4.1.1 Definition A triple $\mathbb{T} = (T, \eta, \mu)$ consisting of a functor $T : \mathscr{X} \longrightarrow \mathscr{X}$, natural transformations $\eta : \mathrm{Id}_{\mathscr{X}} \longrightarrow T$, $\mu : TT \longrightarrow T$, such that the diagrams (4.1) commute is called a *monad on* $\mathscr{X}$. The transformation η is called the *unit* of $\mathbb{T}$, the transformation μ is called the *multiplication* of $\mathbb{T}$. The two triangles in (4.1) are said to express that η is a *two-sided unit* for μ and the rectangle in (4.1) is said to express the *associativity* of μ.

4.1.2 Remark In some literature, monads are called *triples* or *standard constructions*. Both names seem to be unfortunate and they become obsolete.

The following result is almost a tautology.

4.1.3 Lemma *Every adjunction $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ gives rise to a monad $\mathbb{T}$ on $\mathscr{X}$.*

To give examples of monads that (at the first sight) do not come from an adjunctions, consider the following two examples.

4.1.4 Example Suppose $\mathscr{X}$ is a poset. To give a monad $\mathbb{T} = (T, \eta, \mu)$ on $\mathscr{X}$ is to give a monotone map $T : \mathscr{X} \longrightarrow \mathscr{X}$ satisfying $X \leq TX$ and $TTX \leq TX$, for every X. Such data are commonly called a *closure operator* on $\mathscr{X}$.

4.1.5 Example Let $P : \mathsf{Set} \longrightarrow \mathsf{Set}$ denote the *powerset* functor. That is: PX is the set of subsets of X, $Pf : PX \longrightarrow PX'$ sends a set $a \subseteq X$ to its image $f[a] \subseteq X'$. Denote further by $\{.\}_X : X \longrightarrow PX$ the map sending x to $\{x\}$ and denote by $\bigcup_X : PPX \longrightarrow PX$ sending an element $\{a_i \mid i \in I\}$ to $\bigcup_{i \in I} a_i$.

Then the triple $(P, \{.\}, \bigcup)$ is a monad on Set.

As we know, looks can be deceiving: both monads above *are* given by adjunctions. In fact, we prove that *every* monad is given by an adjunction. Moreover, every monad can be "resolved" into an adjunction in at least two ways, see Section 4.2 and Section 4.3.

4.2 The Eilenberg-Moore category

Bearing in mind the monad-monoid analogy, we define the category of Eilenberg-Moore algebras for a monad $\mathbb{T}$ as the category of "$\mathbb{T}$-actions". More precisely, the category $\mathscr{X}^{\mathbb{T}}$ is formed in the following manner:

(1) A pair (X, a), where X is an object of $\mathscr{X}$ and $a : TX \longrightarrow X$ is a morphism, such that the following two diagrams

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & TX \arrow[d, "a"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} TTX \arrow[r, "Ta"] \arrow[d, "\mu_X"'] & TX \arrow[d, "a"] \\ TX \arrow[r, "a"'] & X \end{tikzcd} \tag{4.2}$$

commute, is called an *Eilenberg-Moore algebra* for $\mathbb{T}$. We will often say that (X, a) is a $\mathbb{T}$-*algebra*.

(2) Given two $\mathbb{T}$-algebras (X, a), (X, a'), a *morphism of algebras* is an arrow $h : X \longrightarrow X'$ such that the square

$$\begin{tikzcd} TX \arrow[r, "Th"] \arrow[d, "a"'] & TX' \arrow[d, "a'"] \\ X \arrow[r, "h"'] & X' \end{tikzcd} \tag{4.3}$$

commutes. Morphisms in $\mathscr{X}^{\mathbb{T}}$ compose the way they do in $\mathscr{X}$.

4.2.1 Remark Recall the notion of an algebra for a *functor* from Exercise 3.4.12. For every monad $\mathbb{T} = (T, \eta, \mu)$, there is an obvious *fully faithful* functor

$$E : \mathscr{X}^{\mathbb{T}} \longrightarrow \mathscr{X}^{T}$$

that is *almost never* an equivalence.

4.2.2 Example Recall from Example 4.1.4 that a monad $\mathbb{T} = (T, \eta, \mu)$ on a poset $\mathscr{X}$ is a closure operator. A $\mathbb{T}$-algebra (X, a) is a *closed element*: the morphism $a : TX \longrightarrow X$ witnesses the inequality $TX \leq X$ and the inequality $X \leq TX$ is witnessed by η_X. Hence $X = TX$ by antisymmetry.

Observe that, in this case, the categories $\mathscr{X}^{\mathbb{T}}$ and $\mathscr{X}^{T}$ are the same.

4.2.3 Example An algebra for the powerset monad $\mathbb{P} = (P, \{.\}, \bigcup)$ of Example 4.1.5 is exactly a *complete join-semilattice* and an algebra homomorphism is exactly a *join-preserving map*.

(1) Suppose $(X, \bigvee)$ is a complete join-semilattice. Define $a : PX \longrightarrow X$ by putting $a(A) = \bigvee A$.

Then $\bigvee\{x\} = x$ holds, establishing the triangle in (4.2).

Furthemore, the equality $\bigvee(\bigcup\{A_i \mid i \in I\}) = \bigvee\{\bigvee A_i \mid i \in I\}$ establishes the square in (4.2).

Hence, every join-semilattice is a $\mathbb{P}$-algebra.

(2) Given a $\mathbb{P}$-algebra map $a : PX \longrightarrow X$, define $x \leq y$ iff $a(\{x, y\}) = y$. Then $\leq$ is a partial order:

  (a) For reflexivity, use $a(\{x\}) = x$.

  (b) Suppose $x \leq y$ and $y \leq z$. Then $a(\{x, z\}) = a(\{x, a(\{y, z\})\}) = a(\{x\} \cup \{y, z\})$ by (4.2). Using the axioms again, we proceed $a(\{x\} \cup \{y, z\}) = a(\{x, y, z\}) = a(\{x, y\} \cup \{z\}) = a(\{a(\{x, y\}), z\}) = a(\{y, z\}) = z$. Hence $\leq$ is transitive.

  (c) If $x \leq y$ and $y \leq x$, then $y = a(\{x, y\}) = x$, hence $\leq$ is antisymmetric.

(3) To prove that $a(A) = \sup_{\leq} A$, observe that for every $x \in A$, $a(\{x, a(A)\}) = a(\{x\} \cup A) = a(A)$. Therefore $a(A)$ is an upper bound of A. Suppose u is an upper bound of A, then $a(\{a(A), u\}) = a(A \cup \{u\}) = a(\bigcup_A \{a, u\}) = a(a(\{u\})) = u$. Hence $a(A) \leq u$.

(4) The square (4.3) clearly says that $\mathbb{T}$-algebra morphisms are exactly the join-preserving maps.

4.2.4 Proposition (The Eilenberg-Moore adjunction) *Suppose $\mathbb{T}$ is a monad on $\mathscr{X}$. The assignments $(X, a) \mapsto X$, $f \mapsto f$ define a functor $U^{\mathbb{T}} : \mathscr{X}^{\mathbb{T}} \longrightarrow \mathscr{X}$ that has a left adjoint $F^{\mathbb{T}}$. The monad of $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ is $\mathbb{T}$.*

Proof. Define $F^{\mathbb{T}}X = (TX, \mu_X)$. The axioms for a monad guarantee that (TX, μ_X) is an Eilenberg-Moore algebra for $\mathbb{T}$:

$$\begin{tikzcd} TX \arrow[r, "\eta_{TX}"] \arrow[dr, equal] & TTX \arrow[d, "\mu_X"] & TTTX \arrow[r, "T\mu_X"] \arrow[d, "\mu_{TX}"'] & TTX \arrow[d, "\mu_X"] \\ & TX & TTX \arrow[r, "\mu_X"'] & TX \end{tikzcd}$$

Given $f : X \longrightarrow X'$, define $F^{\mathbb{T}} f = Tf$. That Tf is a morphism of algebras follows from naturality of μ:

$$\begin{tikzcd}
TTX \arrow[r, "TTf"] \arrow[d, "\mu_X"'] & TTX' \arrow[d, "\mu_{X'}"] \\
TX \arrow[r, "Tf"'] & TX'
\end{tikzcd}$$

We prove that, for every X, the pair $(F^{\mathbb{T}} X, \eta_X)$ is a free object on X, w.r.t. $U^{\mathbb{T}}$.

Indeed, suppose (X', a') is any algebra and suppose $f : X \longrightarrow X'$ is any morphism. Define $f^\sharp : TX \longrightarrow X'$ as the composite $a' \cdot Tf$. The following diagram proves that $f^\sharp$ is a morphism of algebras:

$$\begin{tikzcd}
TTX \arrow[r, "TTf"] \arrow[d, "\mu_X"'] \arrow[rr, bend left, "Tf^\sharp"] & TTX' \arrow[r, "Ta'"] \arrow[d, "\mu_{X'}"] & TX' \arrow[d, "a'"] \\
TX \arrow[r, "Tf"'] \arrow[rr, bend right, "f^\sharp"'] & TX' \arrow[r, "a'"'] & X'
\end{tikzcd}$$

Above, the square on the left commutes due to naturality of μ and the square on the right commutes due to the fact that (X', a') is an algebra for $\mathbb{T}$.

To prove that $f^\sharp \cdot \eta_X = f$, consider the diagram

$$\begin{tikzcd}
TX \arrow[r, "Tf"] & TX' \arrow[r, "a'"] & X' \\
X \arrow[u, "\eta_X"] \arrow[r, "f"'] & X' \arrow[u, "\eta_{X'}"'] \arrow[ur, equal] &
\end{tikzcd}$$

where the square is naturality of η and the triangle is an axiom for algebra (X, a').

To prove that $f^\sharp$ is uniquely determined, consider any $h : (TX, \mu_X) \longrightarrow (X', a')$ such that $h \cdot \eta_X = f$. Then the diagram

$$\begin{tikzcd}
TX \arrow[r, "T\eta_X"] \arrow[dr, equal] \arrow[rr, bend left, "Tf"] & TTX \arrow[r, "Th"] \arrow[d, "\mu_X"] & TX' \arrow[d, "a'"] \\
& TX \arrow[r, "h"'] & X'
\end{tikzcd}$$

commutes, proving that $h = a' \cdot Tf$. ■

The next proposition will tell us that the formation of limits in the Eilenberg-Moore category is very easy: one computes the limit in the underlying category and endowes the resulting object with the structure of an algebra. In fact, the algebraic structure on the limit is determined uniquely, hence we actually compute a *limit* in the Eilenberg-Moore catgeory. Recall that this process is well-known from Universal Algebra.

4.2.5 Proposition (Limits in the Eilenberg-Moore category) *Let $D : \mathscr{D} \longrightarrow \mathscr{X}^{\mathbb{T}}$ be a diagram such that a limit of $U^{\mathbb{T}} \cdot D : \mathscr{D} \longrightarrow \mathscr{X}$ exists. Then a limit of D exists in $\mathscr{X}^{\mathbb{T}}$.*

Proof. Denote Dd by $a_d : TX_d \longrightarrow X_d$. Denote further by (L, proj_d) a limit of $U^{\mathbb{T}} \cdot D$ in $\mathscr{X}$. Therefore $\mathrm{proj}_d : L \longrightarrow X_d$.

We will construct a $\mathbb{T}$-algebra $a : TL \longrightarrow L$. The morphism a is defined as a unique one such that the triangle

$$\begin{tikzcd}
TL \arrow[rr, "a"] \arrow[dr, "T\mathrm{proj}_d"'] & & L \arrow[dd, "\mathrm{proj}_d"] \\
& TX_d \arrow[dr, "a_d"'] & \\
& & X_d
\end{tikzcd}$$

commutes, for every d.

To prove that (L, a) is a $\mathbb{T}$-algebra, we have to verify that equations $a \cdot \eta_L = 1_L$ and $a \cdot Ta = a \cdot \mu_L$ hold.

(1) The equation $a \cdot \eta_L = 1_L$ is derived from the commutative diagram

$$\begin{tikzcd}
L \arrow[r, "\eta_L"] \arrow[dr, "\mathrm{proj}_d"'] & TL \arrow[rr, "a"] \arrow[dr, "T\mathrm{proj}_d"] & & L \arrow[dd, "\mathrm{proj}_d"] \\
& X_d \arrow[r, "\eta_{X_d}"] \arrow[drr, equal] & TX_d \arrow[dr, "a_d"] & \\
& & & X_d
\end{tikzcd}$$

using the universal property of limits. Above, the square commutes due to naturality of η and the lower triangle commutes since (X_d, a_d) is a $\mathbb{T}$-algebra.

(2) The equation $a \cdot Ta = a \cdot \mu_L$ is verified using the universal property of limits. Consider the following diagram

$$\begin{tikzcd}
TTL \arrow[r, "Ta"] \arrow[dr, "TT\mathrm{proj}_d"'] & TL \arrow[rr, "a"] \arrow[dr, "T\mathrm{proj}_d"] & & L \arrow[dd, "\mathrm{proj}_d"] \\
& TTX_d \arrow[r, "Ta_d"'] & TX_d \arrow[dr, "a_d"'] & \\
& & & X_d
\end{tikzcd}$$

where the trapezoid commutes by the definition of a.

Consider further the following commutative diagram

$$\begin{tikzcd}
TTL \arrow[r, "\mu_L"] \arrow[dr, "TT\mathrm{proj}_d"'] & TL \arrow[rr, "a"] \arrow[dr, "T\mathrm{proj}_d"] & & L \arrow[dd, "\mathrm{proj}_d"] \\
& TTX_d \arrow[r, "\mu_{X_d}"'] & TX_d \arrow[dr, "a_d"'] & \\
& & & X_d
\end{tikzcd}$$

■

4.2.6 Corollary *Suppose $\mathscr{X}$ is complete. Then $\mathscr{X}^{\mathbb{T}}$ is complete.*

4.2.7 Remark The fact that $U^{\mathbb{T}}$ preserves limits is not surprising, since $U^{\mathbb{T}}$ has a left adjoint. In fact, the behaviour of $U^{\mathbb{T}}$ is a lot stronger — it *creates limits*.

A general functor $U : \mathscr{A} \longrightarrow \mathscr{X}$ is said to *create a limit* of $D : \mathscr{D} \longrightarrow \mathscr{A}$, provided that for a limit (L, proj_d) of $U \cdot D$, there is a unique cone $(\widehat{L}, \widehat{\mathrm{proj}}_d)$ for D such that $U\widehat{L} = L$ and $U\widehat{\mathrm{proj}}_d = \mathrm{proj}_d$ and, moreover, $(\widehat{L}, \widehat{\mathrm{proj}}_d)$ is a limit of D.

The existence of colimits in $\mathscr{X}^{\mathbb{T}}$ is more subtle and we postpone it to Proposition 5.1.7. See however Exercise 4.5.8.

4.3 The Kleisli category

Since a monad $\mathbb{T} = (T, \eta, \mu)$ can be considered as an abstract way of manipulating algebraic terms, it is natural to think of a morphism of the form $f : X \longrightarrow TY$ as of a *substitution*. We now introduce a *category of substitutions* $\mathsf{Kl}(\mathbb{T})$, called the *Kleisli category* of $\mathbb{T}$.

(1) Objects of $\mathsf{Kl}(\mathbb{T})$ are the same as the objects of $\mathscr{X}$.

(2) An arrow $f : X \longrightarrow Y$ in $\mathsf{Kl}(\mathbb{T})$ is a substitution from X to Y, i.e., a morphism of the form $f : X \longrightarrow TY$ in $\mathscr{X}$.

(3) Given substitutions $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ in $\mathsf{Kl}(\mathbb{T})$ we define their *composition* to be the arrow

$$X \xrightarrow{f} TY \xrightarrow{Tg} TTZ \xrightarrow{\mu_Z} TZ$$

4.3.1 Example Suppose that $\mathbb{T} = (T, \eta, \mu)$ is a monad on Set. Then TX is the set of terms of an algebraic theory, having X as the set of variables. A morphism $f : X \longrightarrow Y$ in $\mathsf{Kl}(\mathbb{T})$ is indeed a substitution: the mapping $f : X \longrightarrow TY$ assings to each $x \in X$ a term $t_x \in TY$ in variables Y.

4.3.2 Lemma *The above data indeed constitute a category* $\mathsf{Kl}(\mathbb{T})$.

PROOF. To see that composition is associative, consider arrows $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$, $h : Z \longrightarrow W$ in $\mathsf{Kl}(\mathbb{T})$.

Then $h \cdot (g \cdot f)$ in $\mathsf{Kl}(\mathbb{T})$ is the composite

$$X \xrightarrow{f} TY \xrightarrow{Tg} TTZ \xrightarrow{\mu_Z} TZ \xrightarrow{Th} TTW \xrightarrow{\mu_W} TW$$

and $(h \cdot g) \cdot f$ in $\mathsf{Kl}(\mathbb{T})$ is the composite

$$X \xrightarrow{f} TY \xrightarrow{Tg} TTZ \xrightarrow{TTh} TTTW \xrightarrow{T\mu_W} TTW \xrightarrow{\mu_W} W$$

and we want to prove that the above two composites are equal. They are indeed, consider the following diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & TY \arrow[r, "Tg"] & TTZ \arrow[r, "TTh"] \arrow[d, "\mu_Z"] & TTTW \arrow[d, "\mu_{TW}"] \arrow[dr, "T\mu_W"] & \\
 & & TZ \arrow[r, "Th"] & TTW \arrow[d, "\mu_W"] & TTW \arrow[dl, "\mu_W"] \\
 & & & TW &
\end{tikzcd}$$

where the square commutes by naturality of μ and the diamond commutes by the associative law for μ.

We prove now that $\eta_X : X \longrightarrow TX$ is a unit for the composition.

(1) $\eta_X \cdot f = f$ holds in $\mathsf{Kl}(\mathbb{T})$, for any $f : X' \longrightarrow TX$. The composite $\eta_X \cdot f$ in $\mathsf{Kl}(\mathbb{T})$ is the composite

$$X' \xrightarrow{f} TX \xrightarrow{T\eta_X} TTX \xrightarrow{\mu_X} TX$$

in $\mathscr{X}$. Now use the unit law for a monad to conclude that the above composite is f.

(2) $f \cdot \eta_X = f$ holds in $\mathsf{Kl}(\mathbb{T})$, for any $f : X \longrightarrow TX'$. The composite $f \cdot \eta_X$ in $\mathsf{Kl}(\mathbb{T})$ is the composite

$$X \xrightarrow{\eta_X} TX \xrightarrow{Tf} TTX' \xrightarrow{\mu_{X'}} TX'$$

in $\mathscr{X}$. Use naturality of η and the unit law for a monad to conclude that the above composite is f:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & TX \arrow[r, "Tf"] & TTX' \arrow[r, "\mu_{X'}"] & TX' \\
& TX' \arrow[ur, "\eta_{TX'}"] \arrow[urr, equal] & &
\end{tikzcd}$$

■

4.3.3 Proposition (The Kleisli adjunction) *Suppose $\mathbb{T}$ is a monad on $\mathscr{X}$. The assignments $X \mapsto TX$, $(f : X \longrightarrow X') \mapsto (\mu_{X'} \cdot Tf : TX \longrightarrow TX')$ define a functor $U_{\mathbb{T}} : \mathsf{Kl}(\mathbb{T}) \longrightarrow \mathscr{X}$ that has a left adjoint $F_{\mathbb{T}}$. The monad of $F_{\mathbb{T}} \dashv U_{\mathbb{T}}$ is $\mathbb{T}$.*

Proof. That $U_{\mathbb{T}}$ is a functor is easy.

We prove that $X \mapsto X$, $\eta_X : X \longrightarrow TX$ exhibits X in $\mathsf{Kl}(\mathbb{T})$ as a free object on X in $\mathscr{X}$.

To that end, consider any $f : X \longrightarrow U_{\mathbb{T}}(X')$. Since $U_{\mathbb{T}}(X') = TX$, we have $f : X \longrightarrow TX'$, i.e., we have defined $f^\sharp : X \longrightarrow X'$ in $\mathsf{Kl}(\mathbb{T})$.

Then $U_{\mathbb{T}}(f^\sharp) : TX \longrightarrow TX'$ is defined as the composite $\mu_{X'} \cdot Tf$. To prove that $U_{\mathbb{T}}(f^\sharp) \cdot \eta_X = f$, consider the following diagram

$$\begin{tikzcd}
TX \arrow[r, "Tf"] & TTX' \arrow[r, "\mu_{X'}"] & TX' \\
X \arrow[u, "\eta_X"] \arrow[r, "f"'] & TX' \arrow[u, "\eta_{TX'}"'] \arrow[ur, equal] &
\end{tikzcd}$$

in $\mathscr{X}$, where the square is naturality of η and the triangle commutes by axioms of a monad.

Suppose $h : X \longrightarrow X'$ in $\mathsf{Kl}(\mathbb{T})$ is such that $U_{\mathbb{T}}(h) \cdot \eta_X = f$. We need to prove that $h = f$. The diagram above (written with h in the above line) proves that. ■

4.4 The Eilenberg-Moore and Kleisli comparison functors

We will want to determine "how far" the category $\mathscr{A}$ is from $\mathsf{Kl}(\mathbb{T})$ and $\mathscr{X}^{\mathbb{T}}$ and therefore we will introduce two prominent functors

$$K_{\mathbb{T}} : \mathsf{Kl}(\mathbb{T}) \longrightarrow \mathscr{A}, \quad K^{\mathbb{T}} : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$$

called *comparison functors*.

(1) The *Eilenberg-Moore comparison functor* $K^{\mathbb{T}}$ is defined by putting

$$K^{\mathbb{T}}A = (UA, U\varepsilon_A), \quad K^{\mathbb{T}}h = Uh$$

This definition is correct: the pair $(UA, U\varepsilon_A)$ is a $\mathbb{T}$-algebra, since the diagrams

$$\begin{tikzcd}
UA \arrow[r, "\eta_{UA}"] \arrow[dr, equal] & UFUA \arrow[d, "U\varepsilon_A"] \\
& UA
\end{tikzcd}
\qquad
\begin{tikzcd}
UFUFUA \arrow[r, "UFU\varepsilon_A"] \arrow[d, "U\varepsilon_{FUA}"'] & UFUA \arrow[d, "U\varepsilon_A"] \\
UFUA \arrow[r, "U\varepsilon_A"'] & UA
\end{tikzcd}$$

commute (the triangle by (2.2) and the square by naturality of ε).

For every $h : A \longrightarrow A'$, the square

$$\begin{tikzcd}
UFUA \arrow[r, "UFUh"] \arrow[d, "U\varepsilon_A"'] & UFUA' \arrow[d, "U\varepsilon_{A'}"] \\
UA \arrow[r, "Uh"'] & UA'
\end{tikzcd}$$

commutes by naturality of ε, hence $K^{\mathbb{T}}$ is well-defined on morphisms.

(2) The *Kleisli comparison functor* $K_{\mathbb{T}}$ is defined by putting

$$K_{\mathbb{T}}X = FX, \quad K_{\mathbb{T}}f = f^{\sharp}$$

where $f^{\sharp} : FX \longrightarrow FX'$ is the transpose of $f : X \longrightarrow UFX'$.

4.4.1 Theorem (Uniqueness of comparisons)

(1) *The Eilenberg-Moore comparison functor is the unique one making the triangles*

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, "K^{\mathbb{T}}"] \arrow[dr, "U"'] & & \mathscr{X}^{\mathbb{T}} \arrow[dl, "U^{\mathbb{T}}"] \\
& \mathscr{X} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A} \arrow[rr, "K^{\mathbb{T}}"] & & \mathscr{X}^{\mathbb{T}} \\
& \mathscr{X} \arrow[ul, "F"] \arrow[ur, "F^{\mathbb{T}}"'] &
\end{tikzcd}$$

commutative.

(2) *The Kleisli comparison functor is the unique one making the triangles*

$$\begin{tikzcd}
\mathsf{Kl}(\mathbb{T}) \arrow[rr, "K_{\mathbb{T}}"] \arrow[dr, "U_{\mathbb{T}}"'] & & \mathscr{A} \arrow[dl, "U"] \\
& \mathscr{X} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Kl}(\mathbb{T}) \arrow[rr, "K_{\mathbb{T}}"] & & \mathscr{A} \\
& \mathscr{X} \arrow[ul, "F_{\mathbb{T}}"] \arrow[ur, "F"'] &
\end{tikzcd}$$

commutative.

Proof. (1) It is clear that equalities $U^{\mathbb{T}} \cdot K^{\mathbb{T}} = U$ and $K^{\mathbb{T}} \cdot F = F^{\mathbb{T}}$ hold.

Suppose that $K : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$ is any functor such that $U^{\mathbb{T}} \cdot K = U$ and $K \cdot F = F^{\mathbb{T}}$ hold. Consider $h : A \longrightarrow A'$. Put $KA = (X, a)$ and $KA = (X', a')$. From $U^{\mathbb{T}} \cdot K = U$ it follows that $X = UA$, $X' = UA'$ and $Kh = Uh$. We prove that $a = U\varepsilon_A$, the proof that $a' = U\varepsilon_{A'}$ is analogous.

Since $\varepsilon_A : FUA \longrightarrow A$ is in $\mathscr{A}$, we can consider $K\varepsilon_A : K(FUA) \longrightarrow KA$ in $\mathscr{X}^{\mathbb{T}}$. Since $K \cdot F = F^{\mathbb{T}}$ and by the above, $K\varepsilon_A$ is $U\varepsilon_A : (TUA, \mu_{UA}) \longrightarrow (UA, a)$.

Consider now the diagram

$$\begin{tikzcd}
TUA \arrow[r, "\eta_{TUA}"] \arrow[rr, bend left, "1_{TUA}"] \arrow[dr, equal] & TTUA \arrow[r, "TU\varepsilon_A"] \arrow[d, "\mu_{UA}"] & TUA \arrow[d, "a"] \\
& TUA \arrow[r, "U\varepsilon_A"'] & UA
\end{tikzcd}$$

The triangle commutes, since (TUA, μ_{UA}) is an algebra, the square commutes, since $U\varepsilon_A$ is a homomorphism. We have proved $a = U\varepsilon_A$. Therefore $K = K^{\mathbb{T}}$, as desired.

(2) Suppose $K : \mathscr{X}_{\mathbb{T}} \longrightarrow \mathscr{A}$ is a functor satisfying $U_{\mathbb{T}} = U \cdot K$ and $F = K \cdot F_{\mathbb{T}}$.

Clearly, $KX = FX$, for every object X of $\mathsf{Kl}(\mathbb{T})$. This follows from $K \cdot F_{\mathbb{T}} = F$.

To prove that, for $f : X \longrightarrow UFX'$ in $\mathsf{Kl}(\mathbb{T})$, the morphism $Kf : FX \longrightarrow FX'$ is the transpose of f under $F \dashv U$, consider

$$\frac{FX \xrightarrow{Kf} FX'}{X \xrightarrow{\eta_X} UFX \xrightarrow{UKf} UFX'}$$

Since $UKf = U_{\mathbb{T}}f$, we know that the transpose of Kf is the composite

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] & UFX \arrow[r, "UFf"] \arrow[rr, bend right, "U_{\mathbb{T}}f"'] & UFUFX' \arrow[r, "U\varepsilon_{FX'}"] & UFX'
\end{tikzcd}$$

But the diagram

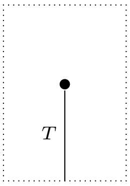
$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & UFX \arrow[d, "UFf"] \\
UFX' \arrow[r, "\eta_{UFX'}"] \arrow[rd, equal] & UFUFX' \arrow[d, "U\varepsilon_{FX'}"] \\
& UFX'
\end{tikzcd}$$

commutes, proving that the transpose of Kf is f. ■

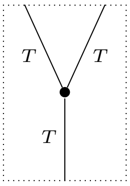
4.5 Exercises

4.5.1 Exercise (Monoids give rise to monads) Suppose $\mathbb{M} = (M, i, \circ)$ is a monoid. Prove that the assignment $T : X \mapsto M \times X$ extends to a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$. Prove that $\eta_X : X \longrightarrow TX$, sending x to (i, x), and $\mu_X : TTX \longrightarrow TX$, sending $(m_1, (m_2, x))$ to $(m_1 \circ m_2, x)$, form components of natural transformations $\eta : \mathrm{Id}_{\mathsf{Set}} \longrightarrow T$, $\mu : TT \longrightarrow T$. Finally, prove that $\mathbb{T} = (T, \eta, \mu)$ is a monad on Set and prove that $\mathsf{Set}^{\mathbb{T}}$ is isomorphic to $\mathbb{M}$-Acts.

4.5.2 Exercise (Monads are genuine monoids) Recall the string diagrams from Section 2.5. Using the diagram



for $\eta : \mathrm{Id}_{\mathscr{X}} \longrightarrow T$, and the diagram



for $\mu : TT \longrightarrow T$, write down the monad axioms. Compare your results with the tree representation of nullary and binary operations known from universal algebra.

4.5.3 Exercise (The unit monad) Prove that, for every category $\mathscr{X}$, the identity functor $\mathrm{Id} : \mathscr{X} \longrightarrow \mathscr{X}$ bears the canonical structure of a monad when we put η and μ to be the identity natural transformations. This monad is called a *unit monad* on $\mathscr{X}$ and we denote it by $\mathbb{I}$.

Prove that $\mathscr{X}^{\mathbb{I}}$ is isomorphic to $\mathscr{X}$.

4.5.4 Exercise (The trivial monad) Suppose 1 is a terminal object in a category $\mathscr{X}$ and denote, for every X, by $t_X : X \longrightarrow 1$ the respective unique morphism.

Prove that the assignment $X \mapsto 1$ can be extended to a functor $T : \mathscr{X} \longrightarrow \mathscr{X}$. Prove that there is a canonical structure of a monad $\mathbb{T} = (T, \eta, \mu)$, called a *trivial monad*.

Prove that if (X, a) is a $\mathbb{T}$-algebra, then $X \cong 1$ and a is identity.

4.5.5 Exercise (The double dualisation monad) Recall from Exercise 3.4.21 the necessary and sufficient condition such that $\mathscr{A}(-, D) : \mathscr{A}^{op} \longrightarrow \mathsf{Set}$ has a left adjoint.

Denote the left adjoint by F and describe explicitly the monad $\mathbb{D}$ of $F \dashv \mathscr{A}(-, D)$. The monad $\mathbb{D}$ is called a *double dualisation monad* and D is called a *dualisation object*.

4.5.6 Exercise (Kleisli algebras) Let $\mathscr{X}$ be a one-object category. Denote the unique object of $\mathscr{X}$ by $\star$ and put $X = \mathscr{X}(\star, \star)$. Observe that X becomes a monoid w.r.t. the composition in $\mathscr{X}$. What is a monad on $\mathscr{X}$? (The structure you come up with is called a *Kleisli algebra*.)

4.5.7 Exercise (Strength of a monad) Prove that every monad $\mathbb{T} = (T, \eta, \mu)$ on Set is *strong*, i.e., prove that there exists a natural transformation

$$\sigma_{X,Y} : TX \times Y \longrightarrow T(X \times Y)$$

called *strength* of $\mathbb{T}$, such that the diagrams

$$\begin{tikzcd}
TX \times 1 \arrow[rr, "\sigma_{X,1}"] & & T(X \times 1) \\
& X \times 1 \arrow[ul, "\eta_X \times 1"] \arrow[ur, "\eta_{X \times 1}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
TTX \times Y \arrow[r, "\sigma_{TX,Y}"] \arrow[d, "\mu_X \times Y"'] & T(TX \times Y) \arrow[r, "T\sigma_{X,Y}"] & TT(X \times Y) \arrow[d, "\mu_{X \times Y}"] \\
TX \times Y \arrow[rr, "\sigma_{X,Y}"'] & & T(X \times Y)
\end{tikzcd}$$

commute for every X and Y.

Hint: instead of defining $\sigma_{X,Y}$ you may want to define a function that assigns to each $y \in Y$ a function $s_y : TX \longrightarrow T(X \times Y)$ and then put $\sigma_{X,Y}(t, y) = s_y(t)$. To define s_y, consider the map $u_y : x \mapsto (x, y)$ and apply T to it to obtain s_y.

4.5.8 Exercise (When does $U^{\mathbb{T}} : \mathsf{Set}^{\mathbb{T}} \longrightarrow \mathsf{Set}$ preserve colimits?) Suppose that $\mathbb{T} = (T, \eta, \mu)$ is a monad on Set. Prove that $T1$ (where 1 is a one-element set) bears canonically the structure of a monoid. You may need Exercise 4.5.7 to define $\circ : T1 \times T1 \longrightarrow T1$.

Denote the resulting monoid by $\mathbb{M} = (T1, i, \circ)$. Prove that the following properties of the underlying functor $U^{\mathbb{T}} : \mathsf{Set}^{\mathbb{T}} \longrightarrow \mathsf{Set}$ are equivalent:

(1) $U^{\mathbb{T}}$ has a right adjoint.

(2) $U^{\mathbb{T}}$ preserves small colimits.

(3) $U^{\mathbb{T}}$ preserves small coproducts.

(4) $U^{\mathbb{T}}$ preserves small copowers.

(5) $\mathbb{T}$ is the monad coming from the monoid $\mathbb{M} = (T1, e, \circ)$ as in Exercise 4.5.1, i.e., $TX \cong T1 \times X$, $\eta_X(x) = (e, x)$ and $\mu_X(m_1, (m_2, x)) = (m_1 \circ m_2, x)$.

Hint: in proving that (4) implies (5), use that $T = U^{\mathbb{T}} F^{\mathbb{T}}$ preserves copowers, the fact that $X \cong X \bullet 1$, and the fact that $X \bullet T1 \cong T1 \times X$. For the proof that (5) implies (1), recall Example 2.1.4.

4.5.9 Exercise (Relations as a Kleisli category) Prove that the Kleisli category of the powerset monad $(P, \{.\}, \bigcup)$ of Example 4.1.5 is the category having sets as objects and binary relations as morphisms.

4.5.10 Exercise (Matrices as a Kleisli category) Let $\mathbb{R} = (R, +, \times, 0, 1)$ be a ring with a unit. A *vector* on a set X is a function $v : X \longrightarrow R$ with a finite support, i.e., all but finitely many $v(x)$'s are zero. Let TX be the set of all vectors on X.

(1) Prove that the assignment $X \mapsto TX$ can be extended to a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$.

(2) Call a map $m : X \longrightarrow TY$ an $X \times Y$*-matrix*. Think of $m(x)$ as of the x-th row of the matrix m.

(3) Given matrices $m : X \longrightarrow TY$, $n : Y \longrightarrow TZ$, define the matrix $n \cdot m : X \longrightarrow TZ$ by the usual matrix multiplication formula, i.e., put

$$(n \cdot m)(x)(z) = \sum_y m(x)(y) \times n(y)(z)$$

Observe that the above sum makes sense due to our definition of a vector.

(4) Prove that composition of matrices is associative and that there is an identity morphism $i_X : X \longrightarrow TX$, for each X. This identity morphism is called the *identity* $X \times X$*-matrix*.

(5) Denote by $\mathsf{Mat}(\mathbb{R})$ the category having sets as objects and matrices as morphisms. Prove that the assignment $X \mapsto TX$ can be extended to a functor $U : \mathsf{Mat}(\mathbb{R}) \longrightarrow \mathsf{Set}$ that has a left adjoint. Denote the left adjoint by F.

(6) Prove that $\mathsf{Mat}(\mathbb{R})$ is the Kleisli category of the monad $\mathbb{T} = (T, \eta, \mu)$ of $F \dashv U$.

(7) What is the Eilenberg-Moore category of $\mathbb{T}$? Hint: think of $a : TX \longrightarrow X$ as sending $v : X \longrightarrow R$ to a formal linear combination $\sum_x v(x) * x$ in X.

4.5.11 Exercise (The Eilenberg-Moore category for a Galois connection) Recall the notation of Exercise 2.6.2. Prove that the Eilenberg-Moore category of the monad $\mathbb{T}$ associated to the adjunction

$$(-)^u \dashv (-)^\ell : \mathscr{B}^{op} \longrightarrow \mathscr{A}$$

consists of *Galois closed subsets*, i.e., such subsets X of A that satisfy the equality $(X^u)^\ell = X$.

Using Exercise 2.6.3, prove that extended real numbers are an Eilenberg-Moore category for a suitable monad.

4.5.12 Exercise (Resolution of a monad in more than two ways) Let $\mathscr{A}$ be the category of *left cancellative monoids* and monoid homomorphisms. A monoid $(X, i, \circ)$ is called left cancellative, if the following holds

$$x = y, \text{ whenever } a \circ x = a \circ y$$

for all a, x, y in X.

(1) Prove that every free monoid is left cancellative.

(2) Find a left cancellative monoid that is not free and that is not a group.

(3) Prove that the obvious underlying functor $U : \mathscr{A} \longrightarrow \mathsf{Set}$ has a left adjoint. Denote the left adjoint by F.

(4) Consider the monad $\mathbb{T} = (T, \eta, \mu)$ of $F \dashv U$. Prove that $\mathsf{Set}^{\mathbb{T}}$ is isomorphic to the variety of *all* monoids and monoid homomorphisms.

(5) Prove that $\mathsf{Kl}(\mathbb{T})$ is isomorphic to the category of free monoids and their homomorphisms.

(6) Conclude that the monad $\mathbb{T}$ can be obtained from at least three different adjunctions.

4.5.13 Exercise (Liftings to $\mathscr{X}^{\mathbb{T}}$ and generalised Eilenberg-Moore algebras) Suppose $\mathbb{T}$ is a monad on $\mathscr{X}$. Prove that, for a general functor $X : \mathscr{K} \longrightarrow \mathscr{X}$, the following conditions are equivalent:

(1) There is a functor $X^\sharp : \mathscr{K} \longrightarrow \mathscr{X}^{\mathbb{T}}$ such that the triangle

$$\begin{tikzcd}
\mathscr{K} \arrow[r, "X^\sharp"] \arrow[dr, "X"'] & \mathscr{X}^{\mathbb{T}} \arrow[d, "U^{\mathbb{T}}"] \\
 & \mathscr{X}
\end{tikzcd}$$

commutes.

(2) The functor X has an *action* of $\mathbb{T}$, i.e., there exists a natural transformation $\alpha : T \cdot X \longrightarrow X$ such that the diagrams

$$\begin{tikzcd}
X \arrow[r, "\eta X"] \arrow[dr, equal] & TX \arrow[d, "\alpha"] \\
 & X
\end{tikzcd}
\qquad
\begin{tikzcd}
TTX \arrow[r, "T\alpha"] \arrow[d, "\mu X"'] & TX \arrow[d, "\alpha"] \\
TX \arrow[r, "a"'] & X
\end{tikzcd}$$

commute.

Prove that the assignment $(X, \alpha) \mapsto X^\sharp$ is a bijection. Think of two special cases of the above:

(i) If $\mathscr{K}$ is a one-morphism category, then to give a functor $X : \mathscr{K} \longrightarrow \mathscr{X}$ is to give an object X of $\mathscr{X}$. The action $\alpha : T \cdot X \longrightarrow X$ as above is exactly an Eilenberg-Moore algebra on the object X.

(ii) If $\mathbb{T}$ is the monad of $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$, then $U^\sharp : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$ is the Eilenberg-Moore comparison functor $K^{\mathbb{T}}$. And the action corresponding to $K^{\mathbb{T}}$ as above is $U\varepsilon : UFU \longrightarrow U$.

Given actions (X, α), (X', α'), say that a natural transformation $\tau : X \longrightarrow X'$ is a *morphism of actions*, provided the square

$$\begin{tikzcd}
TX \arrow[r, "T\tau"] \arrow[d, "\alpha"'] & TX' \arrow[d, "\alpha'"] \\
X \arrow[r, "\tau"'] & X'
\end{tikzcd}$$

commutes. Define an assignment $\tau \mapsto \tau^\sharp$, where $\tau^\sharp$ should be a natural transformation from $X^\sharp$ to $X'^\sharp$.

4.5.14 Exercise (Monoids of endomaps) You know most of the facts that follow. But go through this exercise nevertheless. It will help immensely in getting intuitions for Exercises 4.5.15 and 4.5.16. Denote, for sets X and Y, by $[X, Y]$ the set of all functions $f : X \longrightarrow Y$.

(1) Prove that $[X, X]$ is a monoid w.r.t. composition of functions.

(2) Prove that, for every Z and every $f : X \longrightarrow Y$, there are canonically defined functions

$$[Z, f] : [Z, X] \longrightarrow [Z, Y], \quad [f, Z] : [Y, Z] \longrightarrow [X, Z]$$

(3) Define, for every function $f : X \longrightarrow Y$, the set $[\![f, f]\!]$ as the vertex of a pullback

$$\begin{tikzcd}
{[\![f, f]\!]} \arrow[r, "p_1"] \arrow[d, "p_0"'] & {[Y, Y]} \arrow[d, "{[f, Y]}"] \\
{[X, X]} \arrow[r, "{[X, f]}"'] & {[X, Y]}
\end{tikzcd}$$

and prove that $[\![f, f]\!]$ is a monoid in the canonical way.

Prove that both p_0 and p_1 are morphisms of monoids.

(4) Prove that

  (a) To give a monoid homomorphism $\mathbb{M} \longrightarrow [X, X]$ is to give an action of $\mathbb{M}$ on the set X.

  (b) To give a monoid homomorphism $\mathbb{M} \longrightarrow [\![f, f]\!]$ is to say that f is equivariant (between the actions on X and Y, determined by composing the given monoid homomorphism with p_0 and p_1, respectively).

4.5.15 Exercise (Spitze Klammern in Set) In this exercise we generalise Exercise 4.5.14. The term *Spitze Klammern* refers to the German description of the symbols we introduce in this exercise.

(1) Suppose X and Y are sets. Define $\langle\!\langle X, Y \rangle\!\rangle S$ to be the set $\mathsf{Set}(S, X) \pitchfork Y$, for any set S. Prove that the assignment $S \mapsto \langle\!\langle X, Y \rangle\!\rangle S$ can be extended to a functor $\langle\!\langle X, Y \rangle\!\rangle : \mathsf{Set} \longrightarrow \mathsf{Set}$.

(2) Prove that, for any mapping $f : X \longrightarrow Y$ and any set Z, one can define natural transformations

$$\langle\!\langle Z, f \rangle\!\rangle : \langle\!\langle Z, X \rangle\!\rangle \longrightarrow \langle\!\langle Z, Y \rangle\!\rangle, \quad \langle\!\langle f, Z \rangle\!\rangle : \langle\!\langle Y, Z \rangle\!\rangle \longrightarrow \langle\!\langle X, Z \rangle\!\rangle$$

Hint: in defining, e.g., the S-th component $\langle\!\langle Z, f \rangle\!\rangle_S : \langle\!\langle Z, X \rangle\!\rangle S \longrightarrow \langle\!\langle Z, Y \rangle\!\rangle S$, try not to think of elements too much and use universal properties instead.

(3) Define, for a map $f : X \longrightarrow Y$ and any set S, the set $\{\!\{f,f\}\!\}S$ as the vertex of a pullback

$$\begin{tikzcd}
\{\!\{f,f\}\!\}S \arrow[r, "p^1_S"] \arrow[d, "p^0_S"'] & \langle\!\langle Y,Y\rangle\!\rangle S \arrow[d, "\langle\!\langle f,Y\rangle\!\rangle_S"] \\
\langle\!\langle X,X\rangle\!\rangle S \arrow[r, "\langle\!\langle X,f\rangle\!\rangle_S"'] & \langle\!\langle X,Y\rangle\!\rangle S
\end{tikzcd}$$

in Set.

(4) Prove that the assignment $S \mapsto \{\!\{f,f\}\!\}S$ extends to a functor $\{\!\{f,f\}\!\} : \mathsf{Set} \longrightarrow \mathsf{Set}$ and $S \mapsto p^0_S$, $S \mapsto p^1_S$ are natural in S. Hint: you will draw a cube with pullbacks on some faces and you will use a universal property.

(5) Prove that every $\langle\!\langle X,X\rangle\!\rangle$ and every $\{\!\{f,f\}\!\}$ bears canonically the structure of a monad.

Hint: use various universal properties that are involved in definitions of $\langle\!\langle X,X\rangle\!\rangle$ and of $\{\!\{f,f\}\!\}$.

(6) Try to guess what a monad morphism should be (if you fail, peek into Chapter 6) and prove that both $p_0 : \{\!\{f,f\}\!\} \longrightarrow \langle\!\langle X,X\rangle\!\rangle$ and $p_0 : \{\!\{f,f\}\!\} \longrightarrow \langle\!\langle Y,Y\rangle\!\rangle$ are monad morphisms.

(7) Prove that

  (a) To give a monad morphism $\mathbb{T} \longrightarrow \langle\!\langle X,X\rangle\!\rangle$ is to give a $\mathbb{T}$-algebra on the set X.

  (b) To give a monad morphism $\mathbb{T} \longrightarrow \{\!\{f,f\}\!\}$ is to say that f is a morphism of $\mathbb{T}$-algebras (between the $\mathbb{T}$-algebras on X and Y, determined by composing the given monad morphism with p_0 and p_1, respectively).

4.5.16 Exercise (Spitze Klammern in a general category) Generalise Exercise 4.5.15 to any category $\mathscr{X}$ having small limits.

Chapter 5

The analysis of the Eilenberg-Moore comparison functor

> *Perfection is only attained through true understanding, infinite patience and precise attention to detail.*
>
> Rapee Sagarik, Thai expert on orchids

We will give now a fine analysis of the properties of the Eilenberg-Moore comparison functor $K^{\mathbb{T}} : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$, induced by an adjunction $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$. We will be interested in the following questions:

(1) When is $K^{\mathbb{T}}$ fully faithful?

(2) When does $K^{\mathbb{T}}$ have a left adjoint?

We will harvest this analysis in Chapter 6. As we will see, the answers to the above questions will be closely connected to the existence of coequalisers of certain pairs in $\mathscr{A}$ and the behaviour of U with respect to these coequalisers.

In the whole chapter, we fix an adjunction $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$, we denote by $\mathbb{T}$ the monad on $\mathscr{X}$ that corresponds to $F \dashv U$, and we will relax the notation and write K, instead of $K^{\mathbb{T}}$, for the Eilenberg-Moore comparison functor.

5.1 Faithfulness and fullness

If $K : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$ is fully faithful, then the category $\mathscr{A}$ is a "piece" of the category $\mathscr{X}^{\mathbb{T}}$, since we will have a bijection $K_{A,A'} : \mathscr{A}(A, A') \longrightarrow \mathscr{X}^{\mathbb{T}}(KA, KA')$. Therefore morphisms in $\mathscr{A}$ could be understood as morphisms of $\mathbb{T}$-algebras. Such a property deserves a special name.

5.1.1 Definition An adjunction $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ is said to be *of descent type*, provided that the induced Eilenberg-Moore comparison functor is fully faithful.

A functor $U : \mathscr{A} \longrightarrow \mathscr{X}$ is said to be *of descent type* if has a left adjoint F and the adjunction $F \dashv U$ is of descent type.

Understanding when K is faithful is very easy:

5.1.2 Proposition *The following are equivalent:*

(1) *$K : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$ is faithful.*

(2) *$U : \mathscr{A} \longrightarrow \mathscr{X}$ is faithful.*

(3) *For every A, the counit $\varepsilon_A : FUA \longrightarrow A$ is an epimorphism in $\mathscr{A}$.*

Proof. That (1) is equivalent to (2) is trivial: recall that $U^{\mathbb{T}} \cdot K = U$. Conditions (2) and (3) are equivalent by Proposition 2.3.1. ■

Coming to K being fully faithful, we first prove an easy but useful result.

5.1.3 Lemma *For $h : FUA \longrightarrow A'$, the following are equivalent:*

(1) *The diagram*

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA \arrow[r, "h"] & A' \end{tikzcd}$$

commutes

(2) *The transpose $h^\flat : UA \longrightarrow UA'$ of h is a $\mathbb{T}$-algebra morphism from KA to KA', i.e., the diagram*

$$\begin{tikzcd} UFUA \arrow[r, "UFh^\flat"] \arrow[d, "U\varepsilon_A"'] & UFUA' \arrow[d, "U\varepsilon_{A'}"] \\ UA \arrow[r, "h^\flat"'] & UA' \end{tikzcd}$$

commutes.

Proof. Observe that $\varepsilon_{A'} \cdot UFh^\flat = Uh$ holds *always* — see the definition of transposes.

Consider now

$$\frac{FUFUA \xrightarrow{FU\varepsilon_A} FUA \xrightarrow{h} A'}{UFUA \xrightarrow{U\varepsilon_A} UA \xrightarrow{h^\flat} UA'} \qquad \frac{FUFUA \xrightarrow{\varepsilon_{FUA}} FUA \xrightarrow{h} A'}{UFUA \xrightarrow{1_{UFUA}} UFUA \xrightarrow{Uh} UA'}$$

see Remark 2.2.2. Hence (1) holds iff $Uh = h^\flat \cdot U\varepsilon_A$.

Now it is easy to conclude that (1) and (2) are equivalent. ■

Observe that, due to naturality, the square

$$\begin{tikzcd} FUFUA \arrow[r, "FU\varepsilon_A"] \arrow[d, "\varepsilon_{FUA}"'] & FUA \arrow[d, "\varepsilon_A"] \\ FUA \arrow[r, "\varepsilon_A"'] & A \end{tikzcd}$$

commutes for every A. Hence, using Lemma 5.1.3, the transpose $\varepsilon_A^\flat = 1_{UA}$ is a morphism from KA to KA. This is obvious and we would not have needed to apply any lemma to conclude it. However, if ε_A is a *coequaliser* of $FU\varepsilon_A$, ε_{FUA}, we derive a result concerning the case when K is fully faithful.

5.1.4 Proposition *The following are equivalent:*

(1) *$U : \mathscr{A} \longrightarrow \mathscr{X}$ is of descent type.*

(2) *For every A, the diagram*

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA \arrow[r, "\varepsilon_A"] & A \end{tikzcd} \tag{5.1}$$

is a coequaliser in $\mathscr{A}$.

Proof. By Lemma 5.1.3, $\mathbb{T}$-algebra morphisms $f : KA \longrightarrow KA'$ are in bijective correspondence with morphisms $f^\sharp : FUA \longrightarrow A'$ coequalising $FU\varepsilon_A$, ε_{FUA}. The latter morphisms are in bijective correspondence with morphisms $k : A \longrightarrow A'$ such that $k \cdot \varepsilon_A = f^\sharp$ iff ε_A is a coequaliser. See the diagram

$$\begin{tikzcd}
FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA \arrow[r, "\varepsilon_A"] \arrow[rd, "f^\sharp"'] & A \arrow[d, "k"] \\
 & & A'
\end{tikzcd}$$

But the equality $k \cdot \varepsilon_A = f^\sharp$ states that $Uk = f$. ■

As a first application we prove a result concerning colimits for adjunctions of descent type.

5.1.5 Proposition *Let $\mathscr{X}$ have small coproducts. Suppose that $U : \mathscr{A} \longrightarrow \mathscr{X}$ is of descent type. Then the following are equivalent:*

(1) *$\mathscr{A}$ has all small colimits.*

(2) *$\mathscr{A}$ has coequalisers of reflexive pairs.*

Proof. It clearly suffices to prove that (2) implies (1) and, by Theorem 3.2.2, it suffices to prove that (2) implies the existence of coproducts in $\mathscr{A}$.

To that end, consider a small family A_i, $i \in I$, in $\mathscr{A}$. Since $F \dashv U$ is of descent type, by Proposition 5.1.4 we know that

$$\begin{tikzcd}
FUFUA_i \arrow[r, shift left, "FU\varepsilon_{A_i}"] \arrow[r, shift right, "\varepsilon_{FUA_i}"'] & FUA_i \arrow[r, "\varepsilon_{A_i}"] & A_i
\end{tikzcd}$$

is a coequaliser, for every $i \in I$.

Since $\mathscr{X}$ is assumed to have small coproducts, the coproducts $\coprod_{i\in I} UA_i$ and $\coprod_{i\in I} UFUA_i$ exist in $\mathscr{X}$, and F preserves these coproducts, since it is a left adjoint.

We can therefore consider the following parallel pair

$$\begin{tikzcd}
\coprod_{i\in I} FUFUA_i \arrow[r, shift left, "\coprod_{i\in I} FU\varepsilon_{A_i}"] \arrow[r, shift right, "\coprod_{i\in I} \varepsilon_{FUA_i}"'] & \coprod_{i\in I} FUA_i
\end{tikzcd} \tag{5.2}$$

and we observe that it is clearly a reflexive pair: the common splitting is

$$\coprod_{i\in I} F\eta_{UA_i} : \coprod_{i\in I} FUA_i \longrightarrow \coprod_{i\in I} FUFUA_i$$

Use triangle identities for that.

The rest of the proof imitates the proof of Theorem 3.2.2. Namely, we prove that to give a cocone for (5.2) is to give a cocone for A_i, $i \in I$.

Consider the diagram

$$\begin{tikzcd}
\coprod_{i\in I} FUFUA_i \arrow[r, shift left, "\coprod_{i\in I} FU\varepsilon_{A_i}"] \arrow[r, shift right, "\coprod_{i\in I} \varepsilon_{FUA_i}"'] & \coprod_{i\in I} FUA_i & \\
FUFUA_i \arrow[u, "\mathrm{inj}'_i"] \arrow[r, shift left, "FU\varepsilon_{A_i}"] \arrow[r, shift right, "\varepsilon_{FUA_i}"'] & FUA_i \arrow[u, "\mathrm{inj}_i"'] \arrow[r, "\varepsilon_{A_i}"'] & A_i
\end{tikzcd}$$

where inj_i and inj'_i denote the respective coproduct injections and where the bottom row is a coequaliser.

Suppose $h : \coprod_{i\in I} FUA_i \longrightarrow A$ coequalises the top row as in

$$\begin{tikzcd}
\coprod_{i\in I} FUFUA_i \arrow[r, shift left, "\coprod_{i\in I} FU\varepsilon_{A_i}"] \arrow[r, shift right, "\coprod_{i\in I} \varepsilon_{FUA_i}"'] & \coprod_{i\in I} FUA_i \arrow[r, "h"] & A \\
FUFUA_i \arrow[u, "\mathrm{inj}'_i"] \arrow[r, shift left, "FU\varepsilon_{A_i}"] \arrow[r, shift right, "\varepsilon_{FUA_i}"'] & FUA_i \arrow[u, "\mathrm{inj}_i"'] \arrow[r, "\varepsilon_{A_i}"'] \arrow[ru, "h_i"] & A_i \arrow[u, "k_i"']
\end{tikzcd}$$

Then, for every $i \in I$, $h_i = h \cdot \mathsf{inj}_i$ coequalises $FU\varepsilon_{A_i}$ and ε_{FUA_i}. Therefore, every h_i induces a unique $k_i : A_i \longrightarrow A$. Since the passage $h \mapsto (k_i)$ is a bijection, the proof is finished. ∎

5.1.6 Example Consider the Eilenberg-Moore adjunction $F^{\mathbb{T}} \dashv U^{\mathbb{T}} : \mathscr{X}^{\mathbb{T}} \longrightarrow \mathscr{X}$. Then the corresponding Eilenberg-Moore comparison functor is identity, in particular, it is fully faithful. Therefore, the adjunction $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ is *always* of descent type.

Hence, by Proposition 5.1.4, the diagram

$$\begin{tikzcd}
F^{\mathbb{T}}U^{\mathbb{T}}F^{\mathbb{T}}U^{\mathbb{T}}(X,a) \arrow[r, shift left, "F^{\mathbb{T}}U^{\mathbb{T}}\varepsilon^{\mathbb{T}}_{(X,a)}"] \arrow[r, shift right, "\varepsilon^{\mathbb{T}}_{F^{\mathbb{T}}U^{\mathbb{T}}(X,a)}"'] & F^{\mathbb{T}}U^{\mathbb{T}}(X,a) \arrow[r, "\varepsilon^{\mathbb{T}}_{(X,a)}"] & (X,a)
\end{tikzcd} \tag{5.3}$$

is a coequaliser in $\mathscr{X}^{\mathbb{T}}$, for every $\mathbb{T}$-algebra (X,a).

Since $\varepsilon^{\mathbb{T}}_{(X',a')} = a'$ for any $\mathbb{T}$-algebra (X',a') (see the proof of Proposition 4.2.4), we can rewrite (5.3) to the diagram

$$\begin{tikzcd}
(TTX, \mu_{TX}) \arrow[r, shift left, "Ta"] \arrow[r, shift right, "\mu_X"'] & (TX, \mu_X) \arrow[r, "a"] & (X,a)
\end{tikzcd} \tag{5.4}$$

For the reasons so far unclear we will call the coequaliser (5.4) the *canonical presentation* of the algebra (X,a).

As an application, we can determine now when *colimits* exist in $\mathscr{X}^{\mathbb{T}}$. Recall from Proposition 4.2.5 that the computation of limits in $\mathscr{X}^{\mathbb{T}}$ is very easy: one computes a limit in $\mathscr{X}$ and the functor $U^{\mathbb{T}}$ takes care of the rest — $U^{\mathbb{T}}$ *creates* the limit, see Remark 4.2.7.

5.1.7 Proposition (Colimits in the Eilenberg-Moore category) *Suppose $\mathscr{X}$ has coproducts. Then the following are equivalent:*

(1) *$\mathscr{X}^{\mathbb{T}}$ has colimits.*

(2) *$X^{\mathbb{T}}$ has coequalisers of reflexive pairs.*

Proof. This is immediate from Proposition 5.1.5 and the fact that $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ is of descent type (see Example 5.1.6). ∎

5.1.8 Remark Let us realise that the proof of Proposition 5.1.5 tells us *how* to compute coproducts in $\mathscr{X}^{\mathbb{T}}$ as certain coequalisers. This is typical in varieties: coproducts of algebras are computed by "glueing" things together — this is what coequalisers do.

The diagram (5.1) will play the lead rôle in our future considerations and we will now analyse the diagram in somewhat greater detail.

5.1.9 Lemma *Consider the diagram*

$$\begin{tikzcd}
UFUFUA \arrow[r, shift left, "UFU\varepsilon_A"] \arrow[r, shift right, "U\varepsilon_{FUA}"'] & UFUA \arrow[r, "U\varepsilon_A"] & UA
\end{tikzcd} \tag{5.5}$$

in $\mathscr{X}$, resulting when U is applied to (5.1). Then (5.5) is a coequaliser in $\mathscr{X}$, and it remains a coequaliser after applying any *functor to it.*

Proof. We will give *equational* reasons why (5.5) is a coequaliser in $\mathscr{X}$. Since these equations will be preserved by any functor, the second assertion will immediately follow.

Observe that the diagram (5.5) may be augmented by two arrows to

$$\begin{tikzcd}
UFUFUA \arrow[r, shift left=2, "UFU\varepsilon_A"] \arrow[r, "U\varepsilon_{FUA}"'] & UFUA \arrow[r, shift left, "U\varepsilon_A"] \arrow[l, bend left, "\eta_{UFUA}"] & UA \arrow[l, shift left, "\eta_{UA}"]
\end{tikzcd}$$

and that the equalities

$$U\varepsilon_A \cdot \eta_{UA} = 1_{UA}, \quad UFU\varepsilon_A \cdot \eta_{UFUA} = \eta_{UA} \cdot U\varepsilon_A, \quad U\varepsilon_{FUA} \cdot \eta_{UFUA} = 1_{UFUA} \tag{5.6}$$

hold. In fact, the first and the last equalities are triangle equalities, the equality in the middle follows from naturality of η.

Consider now any $f : UFUA \longrightarrow X$ coequalising $UFU\varepsilon_A$ and $U\varepsilon_{FUA}$. We want to find a unique $g : UA \longrightarrow X$ such that the diagram

$$\begin{tikzcd}
UFUFUA \arrow[r, shift left, "UFU\varepsilon_A"] \arrow[r, shift right, "U\varepsilon_{FUA}"'] & UFUA \arrow[r, "U\varepsilon_A"] \arrow[dr, "f"'] & UA \arrow[d, "g"] \\
 & & X
\end{tikzcd}$$

commutes. Since $U\varepsilon_A \cdot \eta_{UA} = 1_{UA}$, we know that $U\varepsilon_A$ is an epimorphism, hence it suffices to find *some* g making the above diagram commutative. We prove that $g = f \cdot \eta_{UA}$ will do, i.e., we prove that the triangle

$$\begin{tikzcd}
UFUA \arrow[r, "U\varepsilon_A"] \arrow[ddr, "f"'] & UA \arrow[d, "\eta_{UA}"] \\
 & UFUA \arrow[d, "f"] \\
 & X
\end{tikzcd}$$

commutes.

Consider the diagram

$$\begin{tikzcd}
UFUA \arrow[r, "U\varepsilon_A"] \arrow[d, "\eta_{UFUA}"] \arrow[dd, bend right=60, "1_{UFUA}"'] & UA \arrow[d, "\eta_{UA}"'] \arrow[dd, bend left=60, "g"] \\
UFUFUA \arrow[r, "UFU\varepsilon_A"] \arrow[d, "U\varepsilon_{FUA}"] & UFUA \arrow[d, "f"'] \\
UFUA \arrow[r, "f"'] & X
\end{tikzcd}$$

where the top square is the equality in the middle of (5.6), the bottom square commutes since f coequalises $UFU\varepsilon_A$ and $U\varepsilon_{FUA}$, and the left vertical leg is identity due to the equation on the right of (5.6).

Hence $g \cdot U\varepsilon_A = f$, as desired — the proof is finished. ■

5.1.10 Remark Let us go once more through the proof above, pointing out the rôle of individual equalities in (5.6). From left to right:

(1) The equality $U\varepsilon_A \cdot \eta_{UA} = 1_{UA}$ ensures that $U\varepsilon_A$ is an epimorphism (and it will remain such after the application of *any* functor, see Exercise 1.4.6).

Hence we need not bother with uniqueness when verifying that $U\varepsilon_A$ is a coequaliser, *any* mediating morphism will do. Observe *how* the mediating mapping g is defined: one simply precomposes f with η_{UA}, i.e., we precompose with the split monomorphism, corresponding to $U\varepsilon_A$.

(2) The equality $UFU\varepsilon_A \cdot \eta_{UFUA} = \eta_{UA} \cdot U\varepsilon_A$ allows us to "trade" η_{UA} for $UFU\varepsilon_A$, i.e., we are "trading" the splitting for one of the morphisms that f coequalises.

The composite $f \cdot UFU\varepsilon_A$ can be now "traded" for the composite $f \cdot U\varepsilon_{FUA}$.

(3) The equality $U\varepsilon_{FUA} \cdot \eta_{UFUA} = 1_{UFUA}$ then ensures that both "trades" above cost us nothing: we can conclude $U\varepsilon_A \cdot g = f$.

Clearly, the considerations of Remark 5.1.10 can be done with an appropriate set of equalities concerning an arbitrary parallel pair. Since the ideas of Remark 5.1.10 will become a recurring theme, we introduce the following notions.

5.1.11 Definition A commutative diagram of the form

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & X_0 \arrow[r, "e"] & X \end{tikzcd}$$

is called

(1) An *absolute coequaliser*, if it is a coequaliser after applying any functor to it.

(2) A *split coequaliser*, it there exists a *splitting*, i.e., if there exist morphisms $s : X \longrightarrow X_0$, $t : X_0 \longrightarrow X_1$ such that the following equations

$$e \cdot s = 1_X, \quad d_1 \cdot t = s \cdot e, \quad d_0 \cdot t = 1_{X_0}$$

hold.

5.1.12 Example (Every split epi is a split coequaliser) Suppose $e : X \longrightarrow Y$ is split epi with the splitting $s : Y \longrightarrow X$. Then the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "s \cdot e"] \arrow[r, shift right, "1_X"'] & X \arrow[r, "e"] & Y \end{tikzcd}$$

is a split coequaliser with the splitting given by $s : Y \longrightarrow X$ and $t = 1_X$.

5.1.13 Proposition *Any split coequaliser is an absolute coequaliser. Any absolute coequaliser is a coequaliser.*

Proof. For the first asertion, go through the proof of Lemma 5.1.9 using Remark 5.1.10. For the second assertion, consider the image under the identity functor. ■

We therefore have the implications

split coequaliser ⇒ absolute coequaliser ⇒ coequaliser

none of which can be reversed:

5.1.14 Example

(1) An absolute coequaliser that is not split.

Consider the diagram

$$\begin{tikzcd} X_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "f"'] & X_0 \arrow[r, "1_{X_0}"] & X_0 \end{tikzcd}$$

that is clearly an absolute coequaliser. But it is not a split coequaliser, unless f is a split epi.

Hence

$$\begin{tikzcd} \emptyset \arrow[r, shift left, "\emptyset"] \arrow[r, shift right, "\emptyset"'] & \{x\} \arrow[r, "1_{\{x\}}"] & \{x\} \end{tikzcd}$$

is an example of an absolute coequaliser in Set that is not a split coequaliser.

(2) A coequaliser (of a reflexive pair) that is not absolute.

Consider the commutative diagram

$$\begin{tikzcd} \mathbb{N} + \mathbb{N} \arrow[r, shift left, "{[\mathsf{succ}, 1_{\mathbb{N}}]}"] \arrow[r, shift right, "{[1_{\mathbb{N}}, 1_{\mathbb{N}}]}"'] & \mathbb{N} \arrow[r, "e"] & 1 \end{tikzcd}$$

in Set, where $\mathbb{N}$ is the set of natural numbers, succ is the successor function, and e is the unique map to the one-element set 1.

The above diagram is a coequaliser and it is not preserved by the functor $\mathsf{Set}(\mathbb{N}, -) : \mathsf{Set} \longrightarrow \mathsf{Set}$. In fact, the elements $(0, 0, 0, \ldots)$ and $(0, 1, 2, \ldots)$ do not get merged by $\mathsf{Set}(\mathbb{N}, e)$.

A characterisation of absolute coequalisers gives the following result due to Robert Paré [17].

5.1.15 Proposition *Consider the diagram*

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & X_0 \arrow[r, "e"] & X \end{tikzcd} \tag{5.7}$$

Then the following are equivalent:

(1) *Diagram (5.7) is an absolute coequaliser.*

(2) *Either $d_0 = d_1$ and e is an isomorphism, or there exists an $n \geq 1$ and an augmentation*

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & X_0 \arrow[r, shift left, "e"] \arrow[l, bend left=50, "t_0"] \arrow[l, bend left=80, "t_{n-1}"'] & X \arrow[l, shift left, "s"] \end{tikzcd} \tag{5.8}$$

such that the equations

$$e \cdot s = 1_X \tag{5.9}$$

$$d_1 \cdot t_0 = s \cdot e,\ d_0 \cdot t_1 = s \cdot e,\ d_1 \cdot t_1 = d_0 \cdot t_2,\ d_1 \cdot t_2 = d_0 \cdot t_3,\ \ldots, d_1 \cdot t_{n-2} = d_0 \cdot t_{n-1}, \tag{5.10}$$

$$d_0 \cdot t_{n-1} = 1_{X_0}$$

hold.

Proof. Although the statement may seem horrifying, the proof is relatively easy.

(1) implies (2). To find s, consider the image

$$\begin{tikzcd} \mathscr{X}(X, X_1) \arrow[r, shift left, "\mathscr{X}(X{,}d_1)"] \arrow[r, shift right, "\mathscr{X}(X{,}d_0)"'] & \mathscr{X}(X, X_0) \arrow[r, "\mathscr{X}(X{,}e)"] & \mathscr{X}(X, X) \end{tikzcd}$$

of (5.7) under the functor $\mathscr{X}(X, -) : \mathscr{X} \longrightarrow \mathsf{Set}$. By assumption, it is a coequaliser in Set, hence, in particular, the mapping $\mathscr{X}(X, e)$ is surjective. Therefore we can find $s : X \longrightarrow X_0$ such that $e \cdot s = \mathscr{X}(X, e)(s) = 1_X$. We have established the equality (5.9).

To find t_0, ..., t_{n-1}, consider the image

$$\begin{tikzcd} \mathscr{X}(X_0, X_1) \arrow[r, shift left, "\mathscr{X}(X_0{,}d_1)"] \arrow[r, shift right, "\mathscr{X}(X_0{,}d_0)"'] & \mathscr{X}(X_0, X_0) \arrow[r, "\mathscr{X}(X_0{,}e)"] & \mathscr{X}(X_0, X) \end{tikzcd}$$

of (5.7) under the functor $\mathscr{X}(X_0, -) : \mathscr{X} \longrightarrow \mathsf{Set}$. It is a coequaliser by assumption and the elements 1_{X_0} and $s \cdot e$ in $\mathscr{X}(X_0, X_0)$ are merged by $\mathscr{X}(X_0, e)$, since the equalities

$$\mathscr{X}(X_0, e)(1_{X_0}) = e \cdot 1_{X_0} = e \quad \text{and} \quad \mathscr{X}(X_0, e)(s \cdot e) = e \cdot s \cdot e = e$$

hold. Due to the description of coequalisers in Set (see Example 3.1.8), we know that there is a sequence t_0, ..., t_{n-1} in $\mathscr{X}(X_0, X_1)$ witnessing that $s \cdot e$ and 1_{X_0} are merged by the map $\mathscr{X}(X_0, e) : \mathscr{X}(X_0, X_0) \longrightarrow \mathscr{X}(X_0, X)$. This gives precisely the equalities in (5.10).

(2) implies (1). By Example 5.1.14, the diagram

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & X_0 \arrow[r, "e"] & X \end{tikzcd}$$

is an absolute coequaliser, whenever $d_0 = d_1$ and e is an isomorphism.

It therefore suffices to show that (5.8) is a coequaliser. The reasoning is quite analogous to the analysis in Remark 5.1.10 above, except for that the "trading" gets longer. Namely: suppose f coequalises d_0 and d_1 and define $g = f \cdot s$. Equalities (5.10) then ensure that $f = g \cdot e$ holds. Since e is an epimorphism by (5.9), g is necessarily uniquely determined. ■

5.1.16 Remark The diagram (5.8), together with the equalities (5.9) and (5.10) collapses to the notion of a split coequaliser, if $n = 1$. Hence the split coequalisers are exactly those absolute coequalisers where $s \cdot e$ and 1_{X_0} get merged in one step.

Coming back to the problem when K is fully faithful, we prove now that requiring every ε_A to be a *particular* coequaliser as in Proposition 5.1.4 is not necessary. It turns out that it suffices that every ε_A is a coequaliser of *some* parallel pair, i.e., that every ε_A is a *regular epimorphism.*

5.1.17 Proposition *The following are equivalent:*

(1) *For every A, the diagram*

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA \arrow[r, "\varepsilon_A"] & A \end{tikzcd}$$

is a coequaliser in $\mathscr{A}$.

(2) *For every A, the morphism $\varepsilon_A : FUA \longrightarrow A$ is a coequaliser of* some *parallel pair.*

Proof. It suffices to prove that (2) implies (1). Suppose that

$$\begin{tikzcd} A' \arrow[r, shift left, "h_0"] \arrow[r, shift right, "h_1"'] & FUA \arrow[r, "\varepsilon_A"] & A \end{tikzcd}$$

is a coequaliser. We prove that, for any $k : FUA \longrightarrow B$, the morphism k coequalises h_0, h_1 iff k coequalises $FU\varepsilon_A$, ε_{FUA}.

(1) Suppose $k \cdot h_0 = k \cdot h_1$ and consider the unique $h : A \longrightarrow B$ with $h \cdot \varepsilon_A = k$ (the universal property of coequalisers):

$$\begin{tikzcd} A' \arrow[r, shift left, "h_0"] \arrow[r, shift right, "h_1"'] & FUA \arrow[r, "\varepsilon_A"] \arrow[dr, "k"'] & A \arrow[d, "h"] \\ & & B \end{tikzcd}$$

Then

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA \arrow[r, "\varepsilon_A"] \arrow[rr, bend right, "k"'] & A \arrow[r, "h"] & B \end{tikzcd}$$

and k coequalises $FU\varepsilon_A$, ε_{FUA}.

(2) Suppose $k \cdot FU\varepsilon_A = k \cdot \varepsilon_{FUA}$. Then there exists a *unique* $h : FUA \longrightarrow FUB$ such that the diagram

$$\begin{tikzcd} FUFUFUA \arrow[r, shift left, "FUFU\varepsilon_A"] \arrow[r, shift right, "FU\varepsilon_{FUA}"'] & FUFUA \arrow[r, "FU\varepsilon_A"] \arrow[dr, "FUk"'] & FUA \arrow[d, "h"] \\ & & FUB \end{tikzcd}$$

commutes (use that $U\varepsilon_A$ is an *absolute* coequaliser of $UFU\varepsilon_A$, $U\varepsilon_{FUA}$).

Therefore

$$\begin{tikzcd} FUA' \arrow[r, shift left, "FUh_0"] \arrow[r, shift right, "FUh_1"'] & FUFUA \arrow[r, "FU\varepsilon_A"] \arrow[rr, bend right, "FUk"'] & FUA \arrow[r, "h"] & FUB \end{tikzcd}$$

commutes, since the diagram

$$\begin{tikzcd} A' \arrow[r, shift left, "h_0"] \arrow[r, shift right, "h_1"'] & FUA \arrow[r, "\varepsilon_A"] & A \end{tikzcd}$$

commutes (it is assumed to be a coequaliser).

The diagram

$$\begin{tikzcd} FUA' \arrow[r, shift left, "FUh_0"] \arrow[r, shift right, "FUh_1"'] \arrow[d, "\varepsilon_{A'}"'] & FUFUA \arrow[r, "FUk"] \arrow[d, "\varepsilon_{FUA}"] & FUB \arrow[d, "\varepsilon_B"] \\ A' \arrow[r, shift left, "h_0"] \arrow[r, shift right, "h_1"'] & FUA \arrow[r, "k"] & B \end{tikzcd}$$

commutes by naturality of ε.

Use that $\varepsilon_{A'}$ is epi (it being a coequaliser), hence $k \cdot h_0 = k \cdot h_1$.

■

5.1.18 Remark Let us summarise all the facts that we learned about the commutative diagram

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA \arrow[r, "\varepsilon_A"] & A \end{tikzcd} \tag{5.11}$$

in $\mathscr{A}$.

(1) The parallel pair

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA \end{tikzcd} \tag{5.12}$$

is clearly *reflexive*, the common splitting is given by $F\eta_{UA} : FUA \longrightarrow FUFUA$. To wit: the equalities $FU\varepsilon_A \cdot F\eta_{UA} = 1_{FUA}$ and $\varepsilon_{FUA} \cdot F\eta_{UA} = 1_{FUA}$ hold by the triangle equalities for $F \dashv U$.

(2) The image of the diagram under U is a *split coequaliser*, hence an *absolute coequaliser* in $\mathscr{X}$.

Conditions (1) and (2), put together, will be phrased as follows:

Diagram (5.12) is a *reflexive U-split pair* (*reflexive U-absolute pair*, respectively).

In general: a *reflexive U-split pair* is a reflexive pair in $\mathscr{A}$, whose image under U can be completed to a split coequaliser in $\mathscr{X}$. Analogously, a *reflexive U-absolute pair* is a reflexive pair in $\mathscr{A}$, whose image under U can be completed to an absolute coequaliser in $\mathscr{X}$.

5.2 The left adjoint to the comparison functor

Proceeding in our analysis, we address now the question when K has a left adjoint. It turns out that the answer is related to the existence of coequalisers of certain pairs in $\mathscr{A}$.

We will start with a result that slightly generalises Lemma 5.1.3.

5.2.1 Lemma *Suppose (X, a) is a $\mathbb{T}$-alegbra. For $h : FX \longrightarrow A'$, the following are equivalent:*

(1) *The diagram*

$$\begin{tikzcd} FUFX \arrow[r, shift left, "Fa"] \arrow[r, shift right, "\varepsilon_{FX}"'] & FX \arrow[r, "h"] & A' \end{tikzcd}$$

commutes.

(2) *The transpose $h^\flat : X \longrightarrow UA'$ of h is a $\mathbb{T}$-algebra morphism from (X, a) to KA', i.e., the diagram*

$$\begin{tikzcd} UFX \arrow[r, "UFh^\flat"] \arrow[d, "a"'] & UFUA' \arrow[d, "U\varepsilon_{A'}"] \\ X \arrow[r, "h^\flat"'] & UA' \end{tikzcd}$$

commutes.

Proof. Observe that $\varepsilon_{A'} \cdot UFh^\flat = Uh$ holds *always* — see the definition of transposes.

Consider now

$$\begin{array}{c} FUFX \xrightarrow{Fa} FX \xrightarrow{h} A' \\ \hline UFX \xrightarrow{a} UA \xrightarrow{h^\flat} UA' \end{array} \qquad \begin{array}{c} FUFX \xrightarrow{\varepsilon_{FX}} FX \xrightarrow{h} A' \\ \hline UFX \xrightarrow{1_{UFX}} UFX \xrightarrow{Uh} UA' \end{array}$$

see Remark 2.2.2. Hence (1) holds iff $Uh = h^\flat \cdot a$ holds.

Now it is easy to conclude that (1) and (2) are equivalent. ■

5.2.2 Remark Observe that Lemma 5.1.3 follows from Lemma 5.2.1 by considering the $\mathbb{T}$-algebra $KA = (UA, U\varepsilon_A)$ in place of (X, a).

The characterisation of the existence of an adjunction $L \dashv K$ now follows immediately.

5.2.3 Proposition *The following are equivalent:*

(1) $K : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$ *has a left adjoint.*

(2) *For every algebra* (X, a)*, a coequaliser of the pair*

$$\begin{tikzcd} FUFX \arrow[r, shift left, "Fa"] \arrow[r, shift right, "\varepsilon_{FX}"'] & FX \end{tikzcd}$$

exists in $\mathscr{A}$.

Proof. (1) implies (2). Suppose $L \dashv K$ holds. Since $U^{\mathbb{T}} \cdot K = U$, we may assume that $L \cdot F^{\mathbb{T}} = F$ (by the essential uniqueness of left adjoints). Take a $\mathbb{T}$-algebra (X, a) and consider its canonical presentation

$$\begin{tikzcd} (TTX, \mu_{TX}) \arrow[r, shift left, "Ta"] \arrow[r, shift right, "\mu_X"'] & (TX, \mu_X) \arrow[r, "a"] & (X, a) \end{tikzcd}$$

of (X, a) and recall it is a coequaliser in $\mathscr{X}^{\mathbb{T}}$, see Example 5.1.6. The functor L, being a left adjoint, sends this coequaliser to a coequaliser

$$\begin{tikzcd} L(TTX, \mu_{TX}) \arrow[r, shift left, "LTa"] \arrow[r, shift right, "L\mu_X"'] & L(TX, \mu_X) \arrow[r, "La"] & L(X, a) \end{tikzcd}$$

in the category $\mathscr{A}$.

We will prove that the parallel pairs

$$\begin{tikzcd} L(TTX, \mu_{TX}) \arrow[r, shift left, "LTa"] \arrow[r, shift right, "L\mu_X"'] & L(TX, \mu_X) & FUFX \arrow[r, shift left, "Fa"] \arrow[r, shift right, "\varepsilon_{FX}"'] & FX \end{tikzcd}$$

are the same.

Since $L \cdot F^{\mathbb{T}} = F$, the equation $L(TX, \mu_X) = LF^{\mathbb{T}}X = FX$ holds. The equation $L(TTX, \mu_{TX}) = FUFX$ follows in a similar way. Since $LTa = LU^{\mathbb{T}}F^{\mathbb{T}}a = LF^{\mathbb{T}}a$, ($U^{\mathbb{T}}$ is identity on morphisms), the equality $LF^{\mathbb{T}}a = Fa$ follows. Since μ_X is the transpose of $1_{TX} : TX \longrightarrow U^{\mathbb{T}}(TX, \mu_X)$ under $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$, $L\mu_X$ is the transpose of $1_{UFX} : UFX \longrightarrow UFX$ under $F \dashv U$. The latter transpose is precisely ε_{FX}.

(2) implies (1). Suppose (2) holds. Fix a $\mathbb{T}$-algebra (X, a) and denote by

$$\begin{tikzcd} FUFX \arrow[r, shift left, "Fa"] \arrow[r, shift right, "\varepsilon_{FX}"'] & FX \arrow[r, "c_{(X,a)}"] & L_0(X, a) \end{tikzcd}$$

the coequaliser that is assumed to exist. We will prove that $L_0(X, a)$ is a free object on (X, a) w.r.t. K.

To conclude the proof, we need to define the "insertion of generators", i.e., we need to define $\alpha_{(X,a)} : (X,a) \longrightarrow KL_0(X,a)$ and extablish its universal property. Let $\alpha_{(X,a)}$ be the unique morphism in the diagram

$$\begin{tikzcd}
(TTX,\mu_{TX}) \arrow[r, shift left, "Ta"] \arrow[r, shift right, "\mu_X"'] & (TX,\mu_X) \arrow[r, "a"] \arrow[dr, "Kc_{(X,a)}"'] & (X,a) \arrow[d, "\alpha_{(X,a)}"] \\
& & KL_0(X,a)
\end{tikzcd}$$

defined by the universal property of the coequaliser in the top row.

Let A be any object of $\mathscr{A}$ and let $f : (X,a) \longrightarrow KA$ be a morphism in $\mathscr{X}^{\mathbb{T}}$. By Lemma 5.2.1 we know that the morphism $f^\sharp : FX \longrightarrow A$ coequalises Fa and ε_{FX} and it therefore defines a unique morphism $f^* : L_0(X,a) \longrightarrow A$ such that $f^* \cdot c_{(X,a)} = f^\sharp$:

$$\begin{tikzcd}
FUFX \arrow[r, shift left, "Fa"] \arrow[r, shift right, "\varepsilon_{FX}"'] & FX \arrow[r, "c_{(X,a)}"] \arrow[dr, "f^\sharp"'] & L_0(X,a) \arrow[d, "f^*"] \\
& & A
\end{tikzcd}$$

To prove that $Kf^* \cdot \alpha_{(X,a)} = f$, consider the following diagram

$$\begin{tikzcd}
(TTX,\mu_{TX}) \arrow[r, shift left, "Ta"] \arrow[r, shift right, "\mu_X"'] & (TX,\mu_X) \arrow[r, "a"] \arrow[dr, "Kc_{(X,a)}"'] \arrow[ddr, bend right, "Kf^\sharp"'] & (X,a) \arrow[d, "\alpha_{(X,a)}"] \\
& & KL_0(X,a) \arrow[d, "Kf^*"] \\
& & KA
\end{tikzcd}$$

in $\mathscr{X}^{\mathbb{T}}$, where the upper triangle commutes by the definition of $\alpha_{(X,c)}$ and the lower triangle commutes by the definition of f^*.

The prove that $Kf^* \cdot \alpha_{(X,a)} = f$ will be finished (using the universal property of coequalisers), when we show that the triangle

$$\begin{tikzcd}
(TX,\mu_{TX}) \arrow[r, "a"] \arrow[dr, "Kf^\sharp"'] & (X,a) \arrow[d, "f"] \\
& KA
\end{tikzcd}$$

commutes in $\mathscr{X}^{\mathbb{T}}$, or, since $U^{\mathbb{T}}$ is faithful, when we show that the triangle

$$\begin{tikzcd}
U^{\mathbb{T}}(TX,\mu_{TX}) \arrow[r, "U^{\mathbb{T}}a"] \arrow[dr, "U^{\mathbb{T}}Kf^\sharp"'] & U^{\mathbb{T}}(X,a) \arrow[d, "U^{\mathbb{T}}f"] \\
& U^{\mathbb{T}}KA
\end{tikzcd}$$

commutes in $\mathscr{X}$. The last triangle is, due to $U^{\mathbb{T}} \cdot K = U$, the triangle

$$\begin{tikzcd}
TX \arrow[r, "a"] \arrow[dr, "f^\sharp"'] & X \arrow[d, "f"] \\
& UA
\end{tikzcd}$$

and it commutes, using the definition of $f^\sharp$ and the fact that $f : (X,a) \longrightarrow (UA, U\varepsilon_A)$ is a morphism of $\mathbb{T}$-algebras:

$$\begin{tikzcd}
UFX \arrow[r, "a"] \arrow[d, "UFf"'] \arrow[dr, "f^\sharp"'] & X \arrow[d, "f"] \\
UFUA \arrow[r, "U\varepsilon_A"'] & UA
\end{tikzcd}$$

The proof is finished. ■

5.2.4 Corollary *Suppose $\mathscr{A}$ has coequalisers of reflexive U-split pairs. Then $K : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$ has a left adjoint.*

Proof. This is easy: for every $\mathbb{T}$-algebra (X, a), the pair

$$\begin{tikzcd} FUFX \arrow[r, shift left, "Fa"] \arrow[r, shift right, "\varepsilon_{FX}"'] & FX \end{tikzcd}$$

required for the existence of a left adjoint to K in Proposition 5.2.3 is reflexive. The common splitting is $F\eta_X : FX \longrightarrow FUFX$.

Moreover, the image of the above pair under U can be completed to a split coequaliser

$$\begin{tikzcd} UFUFX \arrow[r, shift left, "UFa"] \arrow[r, shift right, "U\varepsilon_{FX}"'] & UFX \arrow[r, "a"] & X \end{tikzcd}$$

the splitting being $\eta_X : X \longrightarrow UFX$ and $\eta_{UFX} : UFX \longrightarrow UFUFX$. ■

Having established the necessary and sufficient conditions for the existence of a left adjoint L of K, we want to have explicit formulas for the unit and the counit of $L \dashv K$. The formulas follow immediately from the proof of Proposition 5.2.3. Let us fix the notation

$$\begin{tikzcd} FUFX \arrow[r, shift left, "Fa"] \arrow[r, shift right, "\varepsilon_{FX}"'] & FX \arrow[r, "c_{(X,a)}"] & L(X,a) \end{tikzcd} \tag{5.13}$$

for the coequaliser defining $L(X, a)$.

5.2.5 Proposition (The unit of $L \dashv K$) *Suppose (X, a) is a $\mathbb{T}$-algebra. The unit $\alpha_{(X,a)}$ of $L \dashv K$ is the unique morphism in the diagram*

$$\begin{tikzcd} (TTX, \mu_{TX}) \arrow[r, shift left, "Ta"] \arrow[r, shift right, "\mu_X"'] & (TX, \mu_X) \arrow[r, "a"] \arrow[dr, "Kc_{(X,a)}"'] & (X, a) \arrow[d, "\alpha_{(X,a)}"] \\ & & KL(X,a) \end{tikzcd}$$

defined by the universal property of the coequaliser in the top row.

Proof. This is easy: the assertion is exactly how the transpose of $1_{L(X,a)} : L(X, a) \longrightarrow L(X, a)$ has been defined in Proposition 5.2.3. ■

5.2.6 Proposition (The counit of $L \dashv K$) *Suppose A is an object of $\mathscr{A}$. Then the counit β_A of $L \dashv K$ is the unique morphism in the diagram*

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA \arrow[r, "c_{KA}"] \arrow[dr, "\varepsilon_A"'] & LKA \arrow[d, "\beta_A"] \\ & & A \end{tikzcd}$$

defined by the universal property of the coequaliser in the top row.

Proof. This is easy: the assertion is exactly how the transpose of $1_{KA} : KA \longrightarrow KA$ has been defined in Proposition 5.2.3. ■

5.3 Exercises

5.3.1 Exercise (A sufficient condition for the existence of reflexive coequalisers in $\mathscr{X}^{\mathbb{T}}$) Suppose that $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ is an adjunction. Denote by $\mathbb{T} = (T, \eta, \mu)$ the respective monad on $\mathscr{X}$. Prove ([13], Corollary 3):

> Suppose $\mathscr{X}$ has and $T : \mathscr{X} \longrightarrow \mathscr{X}$ preserves coequalisers of reflexive pairs. Then $\mathscr{X}^{\mathbb{T}}$ has coequalisers of reflexive pairs.

5.3.2 Exercise (Zig-zags of John Isbell) A different approach to analysing when K is fully faithful is in terms of *zig-zags*, see [9].

> We say that the functor $U : \mathscr{A} \longrightarrow \mathscr{X}$ *satisfies the short zig-zag condition* provided that for any *short zig-zag*
>
> $$\begin{tikzcd} & A_1 \arrow[dl, "h_1"'] & \\ A_2 & & A_3 \arrow[ll, "h_2"'] \arrow[dl, "h_3"] \\ & A_4 & \end{tikzcd}$$
>
> in $\mathscr{A}$, whenever its image
>
> $$\begin{tikzcd} & UA_1 \arrow[dl, "Uh_1"'] \arrow[dr, dotted, "f"] & \\ UA_2 \arrow[dr, dotted, "g"'] & & UA_3 \arrow[ll, "Uh_2"'] \arrow[dl, "Uh_3"] \\ & UA_4 & \end{tikzcd}$$
>
> under U has a *fill-in*, denoted by the dotted arrows, then the morphism $g \cdot Uh_1 = Uh_3 \cdot f : UA_1 \longrightarrow UA_4$ has the form Uh for some $h : A_1 \longrightarrow A_4$.

Prove the following:

(1) The functor $U^{\mathbb{T}} : \mathscr{X}^{\mathbb{T}} \longrightarrow \mathscr{X}$ satisfies the short zig-zag condition.

(2) Suppose $U : \mathscr{A} \longrightarrow \mathscr{X}$ satisfies the short zig-zag condition and $K : \mathscr{B} \longrightarrow \mathscr{A}$ is a fully functor. Then $UK : \mathscr{B} \longrightarrow \mathscr{X}$ satisfies the short zig-zag condition.

Conclude that if U is of descent type, then it satisfies the short zig-zag condition.

(3) Prove that if U is faithful, then U satisfies the short zig-zag condition.

Hint: to prove K is full, consider $h : KA' \longrightarrow KA$ and the short zig-zag

$$\begin{tikzcd} & A' \arrow[dl, "1_{A'}"'] & \\ A' & & FUA' \arrow[ll, "\varepsilon_{A'}"'] \arrow[dl, "\varepsilon_A \cdot Fh"] \\ & A & \end{tikzcd}$$

Conclude that, for a faithful $U : \mathscr{A} \longrightarrow \mathscr{X}$, U is of descent type iff U satisfies the short zig-zag condition.

Generalise the above to zig-zags of arbitrary length.

Chapter 6

Beck's Theorem

Write down the evident diagram, apply the obvious argument, and obtain the usual result.

Phreilambud

In this chapter we will summarise what we know about the Eilenberg-Moore and Kleisli comparison functors for an adjunction $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ and answer the following questions:

(1) When is $K^{\mathbb{T}}$ an equivalence?

(2) When is $K^{\mathbb{T}}$ an isomorphism?

(3) When is $K_{\mathbb{T}}$ an equivalence?

(4) When is $K_{\mathbb{T}}$ an isomorphism?

The results for $K^{\mathbb{T}}$ — the so-called *Beck's Theorems* — will be stated in terms of the behaviour of U w.r.t. coequalisers of pairs studied on Chapter 5, see Sections 6.1 and 6.2 below. The results for functor $K_{\mathbb{T}}$ are fairly easy and they are summarised in Section 6.3 below.

6.1 Recognising algebras up to equivalence

Beck's monadicity theorems are results characterising situations when the Eilenberg-Moore comparison functor is an equivalence of categories. Since a functor is an equivalence iff it is an adjoint equivalence, we require the comparison to have a left adjoint and the unit and the counit should be natural isomorphisms. Since the existence of a left adjoint to the comparison functor is stated in terms of certain coequalisers, we expect the monadicity theorem to be stated in terms of these coequalisers. This is indeed the case.

6.1.1 Definition Say that $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ is a *monadic adjunction*, if the comparison functor $K : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$ is an equivalence of categories.

A functor $U : \mathscr{A} \longrightarrow \mathscr{X}$ is called *monadic*, provided it has a left adjoint F and the adjunction $F \dashv U$ is monadic.

6.1.2 Remark Some authors define monadic functors in the way that the comparison functor in an *isomorphism*. Since we take the stance that two categories are "abstractly the same", whenever they are equivalent, we stated monadicity in terms of K being an equivalence of categories. We will address the problem when K is an honest isomorphism in Section 6.2 below.

A perhaps surprising is the following example of a monadic functor.

6.1.3 Example (Fully faithful right adjoints are monadic) Suppose $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ is such that U is fully faithful. We claim that $F \dashv U$ is a monadic adjunction.

Denote by $\mathbb{T} = (T, \eta, \mu)$ the resulting monad on $\mathscr{X}$. By Proposition 2.3.1, U is fully faithful iff every ε_A is an isomorphism. Therefore $\mu_X = U\varepsilon_{FX}$ is an isomorphism for every X, and we have $\eta T = T\eta = \mu^{-1}$ from the monad axioms.

We prove that $K : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$ is an equivalence of categories, using Proposition 2.4.3:

(1) K is fully faithful.

Since every ε_A is an isomorphism, it is a regular epimorphism (Example 5.1.12 expresses ε_A as even a split, hence absolute, coequaliser). Now use Proposition 5.1.17.

(2) K is e.s.o.

Suppose (X, a) is a $\mathbb{T}$-algebra. We first prove that $a : TX \longrightarrow X$ is an isomorphism (having necessarily $\eta_X : X \longrightarrow TX$ as an inverse). To that end, consider the naturality square

$$\begin{tikzcd}
TX \arrow[r, "a"] \arrow[d, "\eta_{TX}"'] & X \arrow[d, "\eta_X"] \\
TTX \arrow[r, "Ta"'] & TX
\end{tikzcd}$$

and use $\eta T = T\eta$ to replace it by the commutative square

$$\begin{tikzcd}
TX \arrow[r, "a"] \arrow[d, "T\eta_X"'] & X \arrow[d, "\eta_X"] \\
TTX \arrow[r, "Ta"'] & TX
\end{tikzcd}$$

whose first-down-then-right passage gives identity, since (X, a) is a $\mathbb{T}$-algebra. Thus $\eta_X \cdot a = 1_{TX}$ as desired.

Define $A = FX$. Then $KA = (UA, U\varepsilon_A) = (UFX, U\varepsilon_{FX}) = (TX, \mu_X)$ is a $\mathbb{T}$-algebra, isomorphic to (X, a) by virtue of the $\mathbb{T}$-algebra morphism a:

$$\begin{tikzcd}
TTX \arrow[r, "Ta"] \arrow[d, "\mu_X"'] & TX \arrow[d, "a"] \\
TX \arrow[r, "a"'] & X
\end{tikzcd}$$

6.1.4 Remark The above example gives us a plethora of monadic functors: every full subcategory $U : \mathscr{A} \longrightarrow \mathscr{X}$, where U has a left adjoint, is monadic. The examples include:

(1) The full inclusion of all compact Hausdorff spaces into all (completely regular, if you wish) topological spaces. The left adjoint is given by the Stone-Čech compactification of a (completely regular) topological space.

(2) The full inclusion of the category of all posets into the category of all preorders and monotone maps. The left adjoint is given by antisymmetrisation of a preorder.

(3) The full inclusion of the category of all Abelian groups into the category of all groups and group homomorphisms. The left adjoint is given by the quotient by a commutator subgroup.

(4) And many others...

Of course, our main example of a monadic functor should be $U^{\mathbb{T}} : \mathscr{X}^{\mathbb{T}} \longrightarrow \mathscr{X}$. We will introduce the following property and prove that $U^{\mathbb{T}}$ has it.

6.1.5 Definition We say that $U : \mathscr{A} \longrightarrow \mathscr{X}$ *reflects isomorphisms*, h is an isomorphism in $\mathscr{A}$, whenever Uh is an isomorphism in $\mathscr{X}$.

6.1.6 Example (Properties of $U^{\mathbb{T}}$) We state now quite trivial but very important properties of the functor $U^{\mathbb{T}} : \mathscr{X}^{\mathbb{T}} \longrightarrow \mathscr{X}$.

(1) $U^{\mathbb{T}}$ reflects isomorphisms. Suppose $f : (X, a) \longrightarrow (X', a')$ is such that $U^{\mathbb{T}} f = f : X \longrightarrow X'$ is an isomorphism in $\mathscr{X}$. Denote by $g : X' \longrightarrow X$ the inverse of f. It suffices to prove that the square

$$\begin{tikzcd}
TX' \arrow[r, "Tg"] \arrow[d, "a'"'] & TX \arrow[d, "a"] \\
X' \arrow[r, "g"'] & X
\end{tikzcd}$$

commutes. Since Tf is an isomorphism (having Tg as an inverse), it is an epimorphism. It therefore suffices to prove that Tf equalises both paths in the above square. This is trivial:

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[rr, bend left, "1_{TX}"] & TX' \arrow[r, "Tg"] \arrow[d, "a'"'] & TX \arrow[d, "a"] \\
X \arrow[r, "f"'] \arrow[rr, bend right, "1_X"'] & X' \arrow[r, "g"'] & X
\end{tikzcd}$$

(2) $\mathscr{X}^{\mathbb{T}}$ has and $U^{\mathbb{T}}$ preserves coequalisers of all $U^{\mathbb{T}}$-absolute pairs.

Suppose that

$$\begin{tikzcd}
(X, a) \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & (Y, b)
\end{tikzcd}$$

is a parallel pair in $\mathscr{X}^{\mathbb{T}}$ such that

$$\begin{tikzcd}
X \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & Y \arrow[r, "e"] & Z
\end{tikzcd}$$

is an absolute coequaliser in $\mathscr{X}$.

We proceed in several steps

(a) First we prove that there is a structure of a $\mathbb{T}$-algebra on Z, such that e becomes a morphism of $\mathbb{T}$-algebras. Consider the diagram

$$\begin{tikzcd}
TX \arrow[r, shift left, "Td_1"] \arrow[r, shift right, "Td_0"'] \arrow[d, "a"'] & TY \arrow[r, "Te"] \arrow[d, "b"'] & TZ \arrow[d, dotted, "c"] \\
X \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & Y \arrow[r, "e"] & Z
\end{tikzcd}$$

where the top row is a coequaliser and define $c : TZ \longrightarrow Z$ as the unique mediating arrow, using the universal property of coequalisers.

Clearly, e will become a morphism of $\mathbb{T}$-algebras as soon as we prove that $c : TZ \longrightarrow Z$ satisfies the axioms of Eilenberg-Moore algebras.

(i) To prove that $c \cdot \eta_Z = 1_Z$, we consider the diagram

$$\begin{tikzcd}
X \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, "\eta_X"'] & Y \arrow[r, "e"] \arrow[d, "\eta_Y"'] & Z \arrow[d, "\eta_Z"] \\
TX \arrow[r, shift left, "Td_1"] \arrow[r, shift right, "Td_0"'] \arrow[d, "a"'] & TY \arrow[r, "Te"] \arrow[d, "b"'] & TZ \arrow[d, "c"] \\
X \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & Y \arrow[r, "e"] & Z
\end{tikzcd}$$

and, by using the fact that both (X, a) and (Y, b) are algebras, we conclude that $c \cdot \eta_Z = 1_Z$ by the universal property of coequalisers.

(ii) To prove $c \cdot Tc = c \cdot \mu_Z$, we consider the diagram

$$\begin{tikzcd}
TTX \arrow[r, shift left, "TTd_1"] \arrow[r, shift right, "TTd_0"'] \arrow[d, shift right, "Ta"'] \arrow[d, shift left, "\mu_X"] & TTY \arrow[r, "TTe"] \arrow[d, shift right, "Tb"'] \arrow[d, shift left, "\mu_Y"] & TTZ \arrow[d, shift right, "Tc"'] \arrow[d, shift left, "\mu_Z"] \\
TX \arrow[r, shift left, "Td_1"] \arrow[r, shift right, "Td_0"'] \arrow[d, "a"'] & TY \arrow[r, "Te"] \arrow[d, "b"'] & TZ \arrow[d, "c"] \\
X \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & Y \arrow[r, "e"] & Z
\end{tikzcd}$$

where the top row is a coequaliser and use the universal property of coequalisers again.

(b) We prove that $e : (Y, b) \longrightarrow (Z, c)$ is a coequaliser in $\mathscr{X}^{\mathbb{T}}$. It is clear from the construction that as soon as we prove it, we will also prove that $U^{\mathbb{T}}$ preserves this coequaliser.

Suppose therefore that $e' : (Y, b) \longrightarrow (Z', c')$ coequalises d_0 and d_1 in $\mathscr{X}^{\mathbb{T}}$. Then, in particular, e' coequalises d_0 and d_1 in $\mathscr{X}$. Thus there exists a unique $z : Z \longrightarrow Z'$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & Y \arrow[r, "e"] \arrow[dr, "e'"'] & Z \arrow[d, "z'"] \\
 & & Z'
\end{tikzcd}$$

commutes.

It remains to be proved that z' is a morphism of algebras. By considering the diagram

$$\begin{tikzcd}
TX \arrow[r, shift left, "Td_1"] \arrow[r, shift right, "Td_0"'] \arrow[d, "a"'] & TY \arrow[r, "Te"] \arrow[d, "b"] & TZ \arrow[r, "Tz'"] \arrow[d, "c"] & TZ' \arrow[d, "c'"] \\
X \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & Y \arrow[r, "e"'] & Z \arrow[r, "z'"'] & Z'
\end{tikzcd}$$

we see that Te is epi (it being a coequaliser). Therefore the square

$$\begin{tikzcd}
TZ \arrow[r, "Tz'"] \arrow[d, "c"] & TZ' \arrow[d, "c'"] \\
Z \arrow[r, "z'"'] & Z'
\end{tikzcd}$$

commutes as desired.

(3) Quite analogously to the above, one can prove that $\mathscr{X}^{\mathbb{T}}$ has and $U^{\mathbb{T}}$ preserves colimits of all $U^{\mathbb{T}}$-*absolute diagrams* (not just coequalisers of $U^{\mathbb{T}}$-absolute pairs).

That is, we want to prove that for every diagram $D : \mathscr{D} \longrightarrow \mathscr{X}^{\mathbb{T}}$ such that $U^{\mathbb{T}}D : \mathscr{D} \longrightarrow \mathscr{X}$ has a colimit (Z, inj_d) that is preserved by any functor, a colimit $(\widehat{Z}, \widehat{\mathrm{inj}}_d)$ exists in $\mathscr{X}^{\mathbb{T}}$ and $U^{\mathbb{T}}$ preserves it.

Perform the same calculations as for coequalisers above, using this time a colimit cocone of $TU^{\mathbb{T}}D$ to define the algebra structure c on Z:

$$\begin{tikzcd}
TX_d \arrow[r, "T\mathrm{inj}_d"] \arrow[d, "a_d"'] & TZ \arrow[d, dotted, "c"] \\
X_d \arrow[r, "\mathrm{inj}_d"'] & Z
\end{tikzcd}$$

where we have denoted $Dd = (X_d, a_d)$.

6.1.7 Remark Observe that if $E : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$ is *any equivalence* of categories, then $U^{\mathbb{T}} \cdot E$ reflects isomorphisms. Moreover, $\mathscr{A}$ has colimits of all $(U^{\mathbb{T}} \cdot E)$-absolute diagrams and $U^{\mathbb{T}} \cdot E$ preserves them.

Hence the properties of $U^{\mathbb{T}}$ from Example 6.1.6 are *stable under composition with equivalence of categories*. In particular, every monadic functor must have the above two properties.

We are now ready to state and prove the main result of this section.

6.1.8 Theorem (Beck's Monadicity Theorem [5]) *For $U : \mathscr{A} \longrightarrow \mathscr{X}$, the following are equivalent:*

(1) *U is monadic.*

(2) *U has a left adjoint, reflects isomorphisms, and $\mathscr{A}$ has colimits of* all *U-absolute diagrams and U preserves these coequalisers.*

(3) *U has a left adjoint, reflects isomorphisms, and $\mathscr{A}$ has coequalisers of* all *U-absolute pairs and U preserves these coequalisers.*

(4) *U has a left adjoint, reflects isomorphisms, and $\mathscr{A}$ has coequalisers of reflexive U-absolute pairs and U preserves these coequalisers.*

(5) *U has a left adjoint, reflects isomorphisms, and $\mathscr{A}$ has coequalisers of reflexive U-split pairs and U preserves these coequalisers.*

Proof. For the proof that (1) implies (2), recall from Example 6.1.6 that $U^{\mathbb{T}}$ reflects isomorphisms, that $\mathscr{X}^{\mathbb{T}}$ has colimits of all $U^{\mathbb{T}}$-absolute diagrams and that $U^{\mathbb{T}}$ preserves these colimits. Since we assume that K is an equivalence of categories and since $U = U^{\mathbb{T}} \cdot K$ holds, we proved (2).

That (2) implies (3) implies (4) implies (5) is trivial.

(5) implies (1). Since $\mathscr{A}$ has coequalisers of reflexive U-split pairs, there is an adjunction $L \dashv K$, see Corollary 5.2.4. We will prove that both the unit α and the counit β of $L \dashv K$ are isomorphisms. This will finish the proof.

Recall the definition of α and β of $L \dashv K$ from Propositions 5.2.5 and 5.2.6: see the diagrams

$$\begin{tikzcd}
(TTX, \mu_{TX}) \arrow[r, shift left, "Ta"] \arrow[r, shift right, "\mu_X"'] & (TX, \mu_X) \arrow[r, "a"] \arrow[dr, "Kc_{(X,a)}"'] & (X,a) \arrow[d, "\alpha_{(X,a)}"] \\
 & & KL(X,a)
\end{tikzcd}
\qquad
\begin{tikzcd}
FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA \arrow[r, "c_{KA}"] \arrow[dr, "\varepsilon_A"'] & LKA \arrow[d, "\beta_A"] \\
 & & A
\end{tikzcd}
\tag{6.1}$$

where the top rows are coequalisers.

(i) $\alpha_{(X,a)}$ is an isomorphism.

By applying $U^{\mathbb{T}}$ to the diagram on the left of (6.1), and using $U^{\mathbb{T}} \cdot K = U$, we obtain a diagram

$$\begin{tikzcd}
TTX \arrow[r, shift left, "Ta"] \arrow[r, shift right, "\mu_X"'] & TX \arrow[r, "a"] \arrow[dr, "Uc_{(X,a)}"'] & X \arrow[d, "\alpha_{(X,a)}"] \\
 & & UL(X,a)
\end{tikzcd}$$

in $\mathscr{X}$.

The top row is a coequaliser, since $U^{\mathbb{T}}$ preserves coequalisers of $U^{\mathbb{T}}$-absolute pairs by Example 6.1.6.

Since $c_{(X,a)}$ is a coequaliser of a reflexive U-split pair Fa, ε_{FX} (see the proof of Proposition 5.2.3), and since U preserves such coequalisers, $Uc_{(X,a)}$ is also a coequaliser of Ta and μ_X.

Since coequalisers are essentially unique, $\alpha_{(X,a)} = U^{\mathbb{T}}\alpha_{(X,a)} : U^{\mathbb{T}}(X,a) \longrightarrow U^{\mathbb{T}}KL(X,a)$ is an isomorphism.

Since $U^{\mathbb{T}}$ reflects isomorphisms by Example 6.1.6, $\alpha_{(X,a)} : (X,a) \longrightarrow KL(X,a)$ is an isomorphism in $\mathscr{X}^{\mathbb{T}}$.

(ii) β_A is an isomorphism.

By applying U to the diagram on the right of (6.1), we obtain

$$\begin{tikzcd}
UFUFUA \arrow[r, shift left, "UFU\varepsilon_A"] \arrow[r, shift right, "U\varepsilon_{FUA}"'] & UFUA \arrow[r, "Uc_{KA}"] \arrow[rd, "U\varepsilon_A"'] & ULKA \arrow[d, "U\beta_A"] \\
 & & UA
\end{tikzcd}$$

Now Uc_{KA} is a coequaliser of $UFU\varepsilon_A$ and $U\varepsilon_{FUA}$, since U preserves coequalisers of reflexive U-split pairs. But $U\varepsilon_A$ is also a coequaliser of $UFU\varepsilon_A$ and $U\varepsilon_{FUA}$ by Lemma 5.1.9. Therefore $U\beta_A$ is an isomorphism, since coequalisers are essentially unique.

Since U reflects isomorphisms, $\beta_A : LKA \longrightarrow A$ is an isomorphism.

■

6.1.9 Remark It is easy to see that in proving (5) implies (1) in Theorem 6.1.8 one could assume that U *reflects coequalisers* of reflexive U-split pairs in lieu of assuming U reflects isomorphisms. This is all one needs when proving that the unit $\alpha_{(X,a)}$ of $L \dashv K$ is an isomorphism.

> In general, we say that $U : \mathscr{A} \longrightarrow \mathscr{X}$ *reflects a colimit* of $D : \mathscr{D} \longrightarrow \mathscr{A}$, provided that every cocone (A, inj_d) for D, such that $(UA, U\mathrm{inj}_d)$ is a colimit of $U \cdot D$, is already a colimit of D.

However, one can readily seen the following

> Suppose U preserves and reflects a colimit of D. Then U reflects isomorphisms.

Thus, conditions (3)–(5) of Theorem 6.1.8 could be rewritten as follows:

> U has a left adjoint, $\mathscr{A}$ has coequalisers of $(*)$-pairs and U preserves and reflects these coequalisers.

where $(*)$ stands for the respective class of pairs in the individual conditions of Theorem 6.1.8.

The definition of *reflection of limits* is dual to reflection of colimits. Observe that the functor $U^{\mathbb{T}}$ (hence every monadic functor) reflects limits. This is clear from Proposition 4.2.5.

6.1.10 Remark The reader may be slightly dissapointed that Beck's Theorem does not fully support the intuition we have from Universal Algebra: where are the quotients of congruences as we know them? Namely, the parallel pair

$$\begin{tikzcd}
TTX \arrow[r, shift left, "Ta"] \arrow[r, shift right, "\mu_X"'] & TX
\end{tikzcd}$$

only tells us which pairs should our equivalence relation contain.

This discrepancy is due to the big generality of Beck's Theorem — the theorem works over an *arbitrary* category $\mathscr{X}$. Congruences play a major rôle in *Duskin's variant of Beck's Theorem* where the category $\mathscr{X}$ is supposed to "look more like sets", see [8].

6.2 Recognising algebras up to isomorphism

We want to strengthten the results of Section 6.1 and characterise adjunctions $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ where K is an *isomorphism* of categories.

6.2.1 Definition Say that $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ is a *precisely monadic adjunction*, if the comparison functor $K : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$ is an isomorphism of categories.

A functor $U : \mathscr{A} \longrightarrow \mathscr{X}$ is called *precisely monadic*, provided it has a left adjoint F and the adjunction $F \dashv U$ is precisely monadic.

6.2.2 Remark A closer look at Example 6.1.6 reveals that the behaviour of $U^{\mathbb{T}}$ w.r.t. colimits of $U^{\mathbb{T}}$-absolute diagrams is similar to its behaviour w.r.t. limits that we have observed in Remark 4.2.7. Namely, $U^{\mathbb{T}}$ *creates* colimits of all $U^{\mathbb{T}}$-absolute diagrams.

Let us spell out creation of colimits in detail.

6.2.3 Definition $U : \mathscr{A} \longrightarrow \mathscr{X}$ is said to *create a colimit* of $D : \mathscr{D} \longrightarrow \mathscr{A}$, provided that for a colimit (C, inj_d) of $U \cdot D$, there is a unique cocone $(\widehat{C}, \widehat{\mathrm{inj}}_d)$ for D such that $U\widehat{C} = C$ and $U\widehat{\mathrm{inj}}_d = \mathrm{inj}_d$ and, moreover, $(\widehat{C}, \widehat{\mathrm{inj}}_d)$ is a colimit of D.

6.2.4 Theorem (Precise Monadicity Theorem) *For $U : \mathscr{A} \longrightarrow \mathscr{X}$, the following are equivalent:*

(1) *U is precisely monadic.*

(2) *U has a left adjoint and creates colimits of* all *U-absolute diagrams.*

(3) *U has a left adjoint and creates coequalisers of* all *U-absolute pairs.*

(4) *U has a left adjoint and creates coequalisers of reflexive U-absolute pairs.*

(5) *U has a left adjoint and creates coequalisers of reflexive U-split pairs.*

Proof. For proving that (1) implies (2), recall from Example 6.1.6 that $U^{\mathbb{T}}$ creates colimits of $U^{\mathbb{T}}$-absolute diagrams. If K is an isomorphism, then $U = U^{\mathbb{T}} \cdot K$ creates colimits of U-absolute diagrams.

Implications (2) implies (3) and (3) implies (4) and (4) implies (5) are trivial.

(5) implies (1). We will prove that K is fully faithful and bijective on objects.

(i) First we prove a useful auxilliary result.

For every $\mathbb{T}$-algebra (X, a), we know that the diagram

$$\begin{tikzcd} UFUFX \arrow[r, shift left, "UFa"] \arrow[r, shift right, "U\varepsilon_{FX}"'] & UFX \arrow[r, "a"] & X \end{tikzcd}$$

is a coequaliser in $\mathscr{X}$ (it is, in fact, a split coequaliser). Moreover, the above coequaliser is a coequaliser of the image under U of the reflexive pair

$$\begin{tikzcd} FUFX \arrow[r, shift left, "Fa"] \arrow[r, shift right, "\varepsilon_{FX}"'] & FX \end{tikzcd}$$

in $\mathscr{A}$. Since U creates coequalisers of such pairs, there is a *unique* $\widehat{a} : FX \longrightarrow X^*$ such that $U\widehat{a} = a$ and

$$\begin{tikzcd} FUFX \arrow[r, shift left, "Fa"] \arrow[r, shift right, "\varepsilon_{FX}"'] & FX \arrow[r, "\widehat{a}"] & X^* \end{tikzcd}$$

is a coequaliser in $\mathscr{A}$. We claim that

$$UX^* = X, \quad \widehat{a} = \varepsilon_{X^*}$$

The first equality follows from $U\widehat{a} = a$. The second follows from the fact that the diagrams

$$\begin{tikzcd} UFUX^* \arrow[r, "U\widehat{a}"] & UX^* \\ UX^* \arrow[u, "\eta_{UX^*}"] \arrow[ur, equal] & \end{tikzcd} \qquad \begin{tikzcd} UFX \arrow[r, "a"] & X \\ X \arrow[u, "\eta_X"] \arrow[ur, equal] & \end{tikzcd}$$

are the same: $UX^* = X$ and $U\widehat{a} = a$ hold. Since (X, a) is a $\mathbb{T}$-algebra, both diagrams above commute. Therefore $\widehat{a} = (1_X)^\sharp = \varepsilon_{X^*}$.

(ii) K is fully faithful.

Bu choosing $(X,a)=(UA,U\varepsilon_A)$ in (i) above, we obtain a coequaliser

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA \arrow[r, "\varepsilon_A"] & A \end{tikzcd}$$

proving that K is fully faithful by Proposition 5.1.4.

(iii) K is bijective on objects.

In (i) we proved that $(X,a)=(UX^*,U\varepsilon_{X^*})=KX^*$ for a *unique* X^* in $\mathscr{A}$.

■

6.3 The characterisation of the Kleisli situation

Let us observe that the recognition of $F\dashv U:\mathscr{A}\longrightarrow\mathscr{X}$ as essentially (or, precisely) the *Kleisli adjunction*, i.e., when the Kleisli comparison functor $K_{\mathbb{T}}:\mathsf{Kl}(\mathbb{T})\longrightarrow\mathscr{A}$ is an equivalence (or, an isomorphism) of categories, is very easy.

6.3.1 Proposition *The following are equivalent:*

(1) $K_{\mathbb{T}}:\mathsf{Kl}(\mathbb{T})\longrightarrow\mathscr{A}$ *is an equivalence of categories.*

(2) F *is e.s.o.*

Proof. Observe that $K_{\mathbb{T}}:\mathsf{Kl}(\mathbb{T})\longrightarrow\mathscr{A}$ is *always* a fully faithful functor. This follows from the diagram

$$\begin{tikzcd} \mathsf{Kl}(\mathbb{T})(X,X') \arrow[r, equal] \arrow[rr, bend right, "(K_{\mathbb{T}})_{X,X'}"'] & \mathscr{X}(X,UFX') \arrow[r, "b^{-1}_{X,FX'}"] & \mathscr{A}(FX,FX') \end{tikzcd}$$

Hence, by Proposition 2.4.3, $K_{\mathbb{T}}$ is an equivalence iff $K_{\mathbb{T}}$ is e.s.o. But the latter condition is equivalent to F being e.s.o. ■

6.3.2 Proposition *The following are equivalent:*

(1) $K_{\mathbb{T}}:\mathsf{Kl}(\mathbb{T})\longrightarrow\mathscr{A}$ *is an isomorphism of categories.*

(2) F *is bijective on objects.*

Proof. Since $\mathscr{K}_{\mathbb{T}}$ is fully faithful, it will be an isomorphism of categories iff it is bijective on objects. The latter means precisely that F is bijective on objects. ■

6.4 Exercises

6.4.1 Exercise (A composition of monadic functors need not be monadic) Let Ab denote the category of Abelian groups and their homomorphisms. Denote by $U:\mathsf{Ab}\longrightarrow\mathsf{Set}$ the usual underlying functor, and denote by $E:\mathsf{TorFree}\longrightarrow\mathsf{Ab}$ the inclusion of the full subcategory spanned by torsion-free groups. (A group is torsion-free if it has no elements of finite order.)

Prove:

(1) Both E and U are monadic functors. Hint: the left adjoint of E sends the group A to the factor A/C where C is the subgroup of elements of finite order in A. Since E is fully faithful, it is monadic by Example 6.1.3.

(2) The composite $UE : \mathsf{TorFree} \longrightarrow \mathsf{Set}$ is not monadic. Hint: denote by (T, η, μ) the monad given by U and consider the split coequaliser

$$\begin{tikzcd} TT2 \arrow[r, shift left, "Tx"] \arrow[r, shift right, "\mu_2"'] & T2 \arrow[r, "x"] & 2 \end{tikzcd}$$

in Set, where $(2, x)$ is the two-element Abelian group. The above coequaliser cannot be lifted to $\mathsf{TorFree}$ since the two-element group is not torsion-free.

6.4.2 Exercise (A cancellation result for monadic functors) Suppose a chain

$$\begin{tikzcd} \mathscr{A} \arrow[r, shift right=1.5, "U"'] \arrow[r, phantom, "\perp"] & \mathscr{X} \arrow[l, shift right=1.5, "F"'] \arrow[r, shift right=1.5, "U'"'] \arrow[r, phantom, "\perp"] & \mathscr{X}' \arrow[l, shift right=1.5, "F'"'] \end{tikzcd}$$

of adjunctions is given. Denote by $\mathbb{T} = (T, \eta, \mu)$ the monad of $F \dashv U$ and by $\mathbb{T}' = (T', \eta', \mu')$ the monad of $F' \dashv U'$. Prove the following ([6], Section 7):

(1) Suppose $U' \cdot U$ is monadic. Then U is of descent type and the comparison functor $K^{\mathbb{T}} : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$ has a left adjoint.

(2) Suppose $U' \cdot U$ is monadic and suppose U' reflects isomorphisms. Then U is monadic.

Conclude that if $U' \cdot U$ and U' are monadic, so is U.

6.4.3 Exercise (An easy composition result for monadic functors) Suppose a chain

$$\begin{tikzcd} \mathscr{A} \arrow[r, shift right=1.5, "U"'] \arrow[r, phantom, "\perp"] & \mathscr{X} \arrow[l, shift right=1.5, "F"'] \arrow[r, shift right=1.5, "U'"'] \arrow[r, phantom, "\perp"] & \mathscr{X}' \arrow[l, shift right=1.5, "F'"'] \end{tikzcd}$$

of *monadic* adjunctions is given. Prove the following ([21], Remark 4.2):

Suppose $T = UF : \mathscr{X} \longrightarrow \mathscr{X}$ preserves all coequalisers

$$\begin{tikzcd} F'U'F'U'X \arrow[r, shift left, "F'U'\varepsilon'_X"] \arrow[r, shift right, "\varepsilon'_{F'U'X}"'] & F'U'X \arrow[r, "\varepsilon'_X"] & X \end{tikzcd}$$

in $\mathscr{X}$. Then $U' \cdot U$ is monadic.

6.4.4 Exercise (A not so easy composition result for monadic functors) Suppose a chain

$$\begin{tikzcd} \mathscr{A} \arrow[r, shift right=1.5, "U"'] \arrow[r, phantom, "\perp"] & \mathscr{X} \arrow[l, shift right=1.5, "F"'] \arrow[r, shift right=1.5, "U'"'] \arrow[r, phantom, "\perp"] & \mathscr{X}' \arrow[l, shift right=1.5, "F'"'] \end{tikzcd}$$

of *monadic* adjunctions is given. Prove ([14], ZHD Lemma):

The composite $U' \cdot U$ is monadic, whenever $\mathscr{X}$ is a ZHD category.

A category $\mathscr{X}$ is *ZHD* (it stands for *zero homological dimension*), provided that all objects X of $\mathscr{X}$ are either

(1) *projective* w.r.t. regular epis, i.e., for every regular epimorphism $e : A \longrightarrow B$ and every $f : X \longrightarrow B$ there exists (not necessarily unique) $g : X \longrightarrow A$ making the triangle

$$\begin{tikzcd} A \arrow[rr, "e"] & & B \\ & X \arrow[ul, dotted, "g"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutative,

or

(2) *artificially terminal*, i.e., X is a terminal object and every $f : X \longrightarrow X'$ is an isomorphism.

Observe that the category Set is ZHD.

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