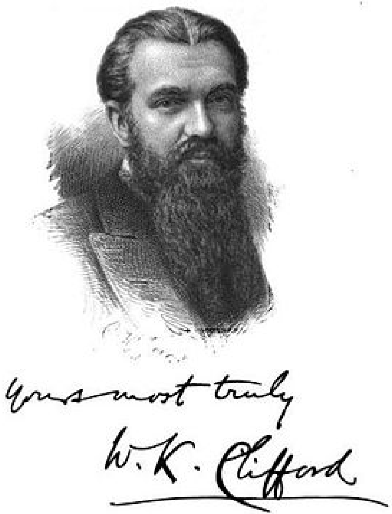
Let's have fun with Geometric algebra

Jirka Velebil
Department of Mathematics
Faculty of Electrical Engineering
Czech Technical University in Prague

We may always depend on it that algebra, which cannot be translated into good English and sound common sense, is bad algebra.

William Kingdon Clifford, *The common sense of the exact sciences*,
(Completed by Karl Pearson, 1885)

William Kingdon Clifford



William Kingdon Clifford (4 May 1845 – 3 March 1879)

- Born in Exeter, educated at King's College (London) and at Trinity College (Cambridge).
- In 1878, Clifford published his fundamental work
  - ☞ Applications of Grassmann's extensive algebra, *Amer. J. Math.* 1(4) (1878), 350–358.
- It is worth mentioning that Clifford suggested that gravity is the curvature of space (1870). Published in
  - ☞ On the space-theory of matter, *Proc. Cambridge Philos. Soc.* (1864–1876 — Printed 1876), 157–158.
- Clifford died of tuberculosis at age 34. His overall exhaustion surely had contributed; it is said that he devoted days to teaching and administration, and nights to research.

What is Geometric Algebra (GA)?

A slogan: GA = Clifford Algebra + "macros for doing geometry".

What is a Clifford algebra (over the reals)?

Suppose V is a vector space (over $\mathbb{R}$) and let $\mathbf{q}$ be a quadratic form on V. A Clifford algebra of $(V, \mathbf{q})$ is a unital associative algebra, free on V, subject to the equation $\mathbf{v}^2 = \mathbf{q}(\mathbf{v}) \cdot 1$ for every vector $\mathbf{v}$ from V. Notation: $\mathrm{Cl}(V, \mathbf{q})$.

Hmmmmm, sounds like a proper nightmare...

We will see: Clifford algebra is really what geometry is all about!

An overview of this talk

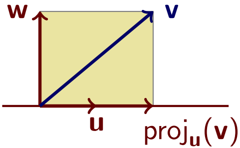
(I) We will do some truly anarchistic computations in basic geometry.

(II) We will make these anarchistic computations a part of the establishment.

(III) Using the establishment, we will show some quite charming ways of thinking about basic geometry.

Orthogonal rejections by division

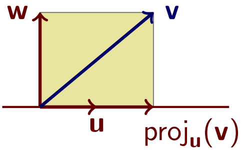
Given $\mathbf{u}$ and $\mathbf{v}$, find[a] $\mathbf{w}$, where $\langle \mathbf{w} \mid \mathbf{u} \rangle = 0$ and $\mathbf{v} = \mathbf{w} + \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$.



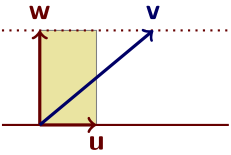
[a] The vector $\mathbf{w}$ is called an orthogonal rejection, $\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - \mathbf{w}$ is, of course, the orthogonal projection.

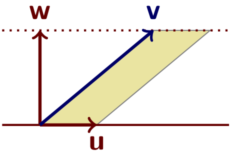
Orthogonal rejections by division

Given $\mathbf{u}$ and $\mathbf{v}$, find[a] $\mathbf{w}$, where $\langle \mathbf{w} \mid \mathbf{u} \rangle = 0$ and $\mathbf{v} = \mathbf{w} + \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$.



The areas



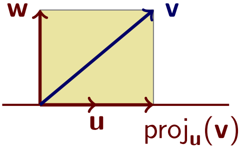


are equal.

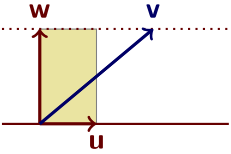
[a] The vector $\mathbf{w}$ is called an orthogonal rejection, $\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - \mathbf{w}$ is, of course, the orthogonal projection.

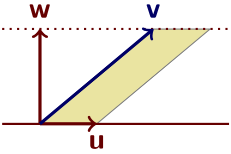
Orthogonal rejections by division

Given $\mathbf{u}$ and $\mathbf{v}$, find[a] $\mathbf{w}$, where $\langle \mathbf{w} \mid \mathbf{u} \rangle = 0$ and $\mathbf{v} = \mathbf{w} + \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$.



The areas



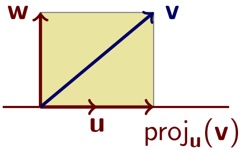


are equal. The equality $\mathrm{area}(\mathbf{u}, \mathbf{w}) = \mathbf{u} \cdot \mathbf{w}$ holds.

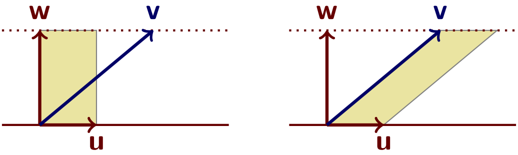
[a] The vector $\mathbf{w}$ is called an orthogonal rejection, $\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - \mathbf{w}$ is, of course, the orthogonal projection.

Orthogonal rejections by division

Given $\mathbf{u}$ and $\mathbf{v}$, find[a] $\mathbf{w}$, where $\langle \mathbf{w} \mid \mathbf{u} \rangle = 0$ and $\mathbf{v} = \mathbf{w} + \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$.



The areas



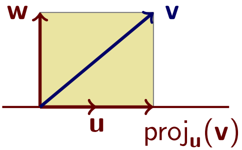
are equal. The equality $\mathrm{area}(\mathbf{u}, \mathbf{w}) = \mathbf{u} \cdot \mathbf{w}$ holds. Hence

$$\mathbf{w} = \frac{\mathrm{area}(\mathbf{u}, \mathbf{w})}{\mathbf{u}} = \frac{\mathrm{area}(\mathbf{u}, \mathbf{v})}{\mathbf{u}}$$

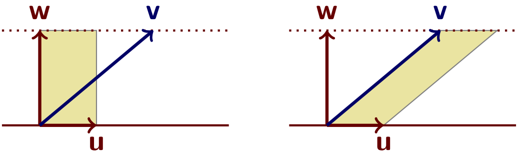
[a] The vector $\mathbf{w}$ is called an orthogonal rejection, $\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - \mathbf{w}$ is, of course, the orthogonal projection.

Orthogonal rejections by division

Given $\mathbf{u}$ and $\mathbf{v}$, find[a] $\mathbf{w}$, where $\langle \mathbf{w} \mid \mathbf{u} \rangle = 0$ and $\mathbf{v} = \mathbf{w} + \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$.



The areas



are equal. The equality $\mathrm{area}(\mathbf{u}, \mathbf{w}) = \mathbf{u} \cdot \mathbf{w}$ holds. Hence

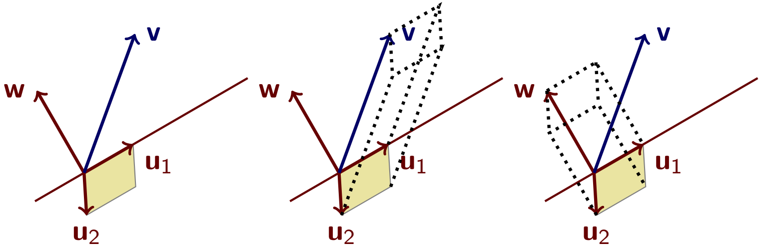
$$\mathbf{w} = \frac{\mathrm{area}(\mathbf{u}, \mathbf{w})}{\mathbf{u}} = \frac{\mathrm{area}(\mathbf{u}, \mathbf{v})}{\mathbf{u}}$$

Funny? This is how it will work after we introduce the formalism.

[a] The vector $\mathbf{w}$ is called an orthogonal rejection, $\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - \mathbf{w}$ is, of course, the orthogonal projection.

Hang on a minute! We spotted something rather important!

Entirely analogous pictures yield orthogonal rejections of vectors by planes!



Hence $\mathbf{w} = \dfrac{\text{volume}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v})}{\text{area}(\mathbf{u}_1, \mathbf{u}_2)}$ and $\text{proj}_{\text{span}(\mathbf{u}_1, \mathbf{u}_2)} = \mathbf{v} - \mathbf{w}$.

What happened is quite typical for GA

Not only it is a coordinate-free geometry, but proper thinking will allow us to see solutions of problems regardless of dimensions.

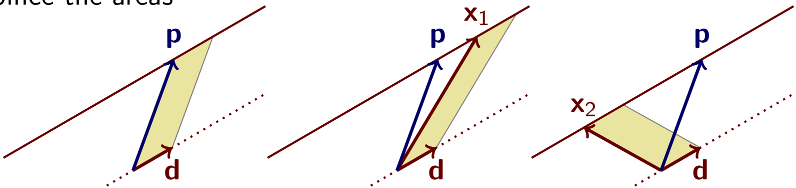
For example, I am sure you'd be able to write down the sequence giving (the anarchistic version of) the Gram-Schmidt orthogonalisation.

Yes, the Gram-Schmidt's process is a series of rejections!

Points on lines (a variation on the theme of rejections)

Describe all points $\mathbf{x}$ that lie on the line going through a point $\mathbf{p}$ and having a direction $\mathbf{d}$.

Since the areas



are all equal, we could say that

$$\text{area}(\mathbf{x}, \mathbf{d}) = \text{area}(\mathbf{p}, \mathbf{d})$$

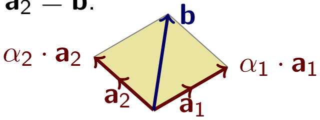
is the equation of the line![a] Notice that the above equation is very much like Kepler's second law.

[a] Observe that, by previous considerations, the offset of the line is $\dfrac{\text{area}(\mathbf{p}, \mathbf{d})}{\mathbf{d}}$.

It is indeed getting quite beautiful and truly bizarre. . .

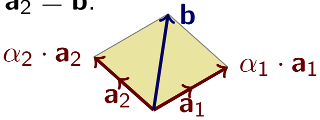
Solution of linear systems by division of areas

Suppose $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly independent vectors in the plane. Then, for any $\mathbf{b}$ in the plane, there are unique scalars α_1 and α_2 such that $\alpha_1 \cdot \mathbf{a}_1 + \alpha_2 \cdot \mathbf{a}_2 = \mathbf{b}$:

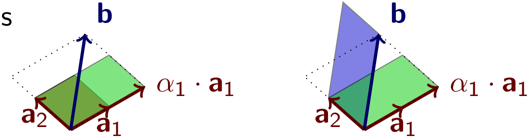


Solution of linear systems by division of areas

Suppose $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly independent vectors in the plane. Then, for any $\mathbf{b}$ in the plane, there are unique scalars α_1 and α_2 such that $\alpha_1 \cdot \mathbf{a}_1 + \alpha_2 \cdot \mathbf{a}_2 = \mathbf{b}$:



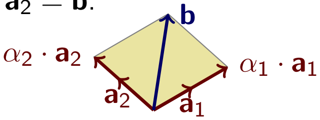
Consider the areas



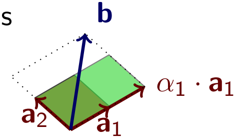
We have $\alpha_1 \cdot \text{area}(\mathbf{a}_1, \mathbf{a}_2) = \text{area}(\alpha_1 \cdot \mathbf{a}_1, \mathbf{a}_2) = \text{area}(\mathbf{b}, \mathbf{a}_2)$,

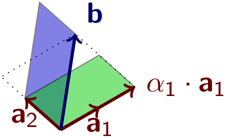
Solution of linear systems by division of areas

Suppose $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly independent vectors in the plane. Then, for any $\mathbf{b}$ in the plane, there are unique scalars α_1 and α_2 such that $\alpha_1 \cdot \mathbf{a}_1 + \alpha_2 \cdot \mathbf{a}_2 = \mathbf{b}$:



Consider the areas



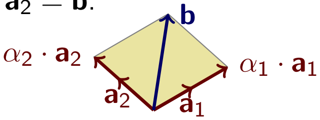


We have $\alpha_1 \cdot \text{area}(\mathbf{a}_1, \mathbf{a}_2) = \text{area}(\alpha_1 \cdot \mathbf{a}_1, \mathbf{a}_2) = \text{area}(\mathbf{b}, \mathbf{a}_2)$, hence

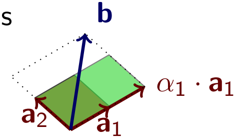
$\alpha_1 = \dfrac{\text{area}(\mathbf{b}, \mathbf{a}_2)}{\text{area}(\mathbf{a}_1, \mathbf{a}_2)}$ and (by similar reasoning) $\alpha_2 = \dfrac{\text{area}(\mathbf{a}_1, \mathbf{b})}{\text{area}(\mathbf{a}_1, \mathbf{a}_2)}$

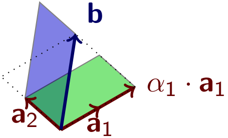
Solution of linear systems by division of areas

Suppose $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly independent vectors in the plane. Then, for any $\mathbf{b}$ in the plane, there are unique scalars α_1 and α_2 such that $\alpha_1 \cdot \mathbf{a}_1 + \alpha_2 \cdot \mathbf{a}_2 = \mathbf{b}$:



Consider the areas





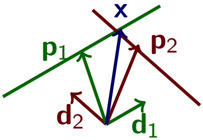
We have $\alpha_1 \cdot \text{area}(\mathbf{a}_1, \mathbf{a}_2) = \text{area}(\alpha_1 \cdot \mathbf{a}_1, \mathbf{a}_2) = \text{area}(\mathbf{b}, \mathbf{a}_2)$, hence

$$\alpha_1 = \frac{\text{area}(\mathbf{b}, \mathbf{a}_2)}{\text{area}(\mathbf{a}_1, \mathbf{a}_2)} \quad \text{and (by similar reasoning)} \quad \alpha_2 = \frac{\text{area}(\mathbf{a}_1, \mathbf{b})}{\text{area}(\mathbf{a}_1, \mathbf{a}_2)}$$

This is precisely what the celebrated Cramer's Rule teaches us!

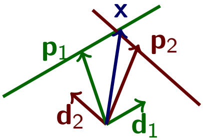
The intersection of two nonparallel lines

Suppose two nonparallel lines are given by directions $\mathbf{d}_1$, $\mathbf{d}_2$ and points $\mathbf{p}_1$, $\mathbf{p}_2$, respectively. Find their intersection $\mathbf{x}$.



The intersection of two nonparallel lines

Suppose two nonparallel lines are given by directions $\mathbf{d}_1$, $\mathbf{d}_2$ and points $\mathbf{p}_1$, $\mathbf{p}_2$, respectively. Find their intersection $\mathbf{x}$.



By the previous we have

$$\begin{aligned}\mathbf{x} &= \frac{\text{area}(\mathbf{x}, \mathbf{d}_2)}{\text{area}(\mathbf{d}_1, \mathbf{d}_2)} \cdot \mathbf{d}_1 + \frac{\text{area}(\mathbf{d}_1, \mathbf{x})}{\text{area}(\mathbf{d}_1, \mathbf{d}_2)} \cdot \mathbf{d}_2 \\ &= \frac{\text{area}(\mathbf{p}_2, \mathbf{d}_2)}{\text{area}(\mathbf{d}_1, \mathbf{d}_2)} \cdot \mathbf{d}_1 + \frac{\text{area}(\mathbf{d}_1, \mathbf{p}_1)}{\text{area}(\mathbf{d}_1, \mathbf{d}_2)} \cdot \mathbf{d}_2\end{aligned}$$

since (remember Kepler) the equations $\text{area}(\mathbf{d}_1, \mathbf{x}) = \text{area}(\mathbf{d}_1, \mathbf{p}_1)$ and $\text{area}(\mathbf{x}, \mathbf{d}_2) = \text{area}(\mathbf{p}_2, \mathbf{d}_2)$ hold.

We need to rectify our wishful thinking

We need to rectify our wishful thinking

1. So far we have seen that the following operations are very useful for basic geometry: multiplication of two vectors, division by a vector (even by an area).

We need to rectify our wishful thinking

1. So far we have seen that the following operations are very useful for basic geometry: multiplication of two vectors, division by a vector (even by an area).
2. The above operations surely could be combined to form hideous algebraic expressions like, say,

$$\mathbf{v} - (9 + \sqrt{2}) \cdot (\mathbf{a}_1 + \mathbf{a}_2) \cdot \frac{\mathbf{u}}{\text{area}(\mathbf{b}, \mathbf{c})} + 42$$

   Hence: scalars, vectors, areas, volumes, whatnots as above, . . . , should all be members of one almighty gang!

We need to rectify our wishful thinking

1. So far we have seen that the following operations are very useful for basic geometry: multiplication of two vectors, division by a vector (even by an area).
2. The above operations surely could be combined to form hideous algebraic expressions like, say,

$$\mathbf{v} - (9+\sqrt{2}) \cdot (\mathbf{a}_1 + \mathbf{a}_2) \cdot \frac{\mathbf{u}}{\text{area}(\mathbf{b}, \mathbf{c})} + 42$$

   Hence: scalars, vectors, areas, volumes, whatnots as above, . . . , should all be members of one almighty gang!

Yes, you have guessed the name of the gang.

We need to rectify our wishful thinking

1. So far we have seen that the following operations are very useful for basic geometry: multiplication of two vectors, division by a vector (even by an area).
2. The above operations surely could be combined to form hideous algebraic expressions like, say,

$$\mathbf{v} - (9 + \sqrt{2}) \cdot (\mathbf{a}_1 + \mathbf{a}_2) \cdot \frac{\mathbf{u}}{\text{area}(\mathbf{b}, \mathbf{c})} + 42$$

   Hence: scalars, vectors, areas, volumes, whatnots as above, . . . , should all be members of one almighty gang!

Yes, you have guessed the name of the gang.

Indeed, Ladies & Gentlemen, the time has come to face

THE CLIFFORD ALGEBRA.

A not so formal definition of $\mathrm{Cl}(V, \mathbf{q})$

We have a collection a, b, c, ... of objects,[a] plus two operations $+$ (addition), $\cdot$ (multiplication) on them, such that:

[a] Some of the objects are vectors in V (we will write them in boldface), some of the objects are real scalars (we will use Greek characters for them).

[b]

A not so formal definition of $\mathrm{Cl}(V, \mathbf{q})$

We have a collection a, b, c, ... of objects,[a] plus two operations $+$ (addition), $\cdot$ (multiplication) on them, such that:

1. $a + (b + c) = (a + b) + c$, and there is a zero (the zero scalar) $0 + a = a = a + 0$. The addition is commutative.

[a]Some of the objects are vectors in V (we will write them in boldface), some of the objects are real scalars (we will use Greek characters for them).
[b]

A not so formal definition of $\mathrm{Cl}(V, \mathbf{q})$

We have a collection a, b, c, ... of objects,[a] plus two operations $+$ (addition), $\cdot$ (multiplication) on them, such that:

1. $a + (b + c) = (a + b) + c$, and there is a zero (the zero scalar) $0 + a = a = a + 0$. The addition is commutative.
2. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, and there is a unit $1 \cdot a = a = a \cdot 1$.
    1. The multiplication is not commutative.
    2. $a \cdot (b + c) = a \cdot b + a \cdot c$, $(b + c) \cdot a = b \cdot a + c \cdot a$.
    3. And, for scalars, the multiplication coincides with the multiplication of scalars. And scalars commute with everything.

---

[a] Some of the objects are vectors in V (we will write them in boldface), some of the objects are real scalars (we will use Greek characters for them).

[b]

A not so formal definition of $\mathrm{Cl}(V, \mathbf{q})$

We have a collection a, b, c, ... of objects,[a] plus two operations $+$ (addition), $\cdot$ (multiplication) on them, such that:

1. $a + (b + c) = (a + b) + c$, and there is a zero (the zero scalar) $0 + a = a = a + 0$. The addition is commutative.
2. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, and there is a unit $1 \cdot a = a = a \cdot 1$.
    1. The multiplication is not commutative.
    2. $a \cdot (b + c) = a \cdot b + a \cdot c$, $(b + c) \cdot a = b \cdot a + c \cdot a$.
    3. And, for scalars, the multiplication coincides with the multiplication of scalars. And scalars commute with everything.
3. For any vector $\mathbf{v}$, the product $\mathbf{v} \cdot \mathbf{v}$ is the scalar $\mathbf{q}(\mathbf{v})$.[b]

[a] Some of the objects are vectors in V (we will write them in boldface), some of the objects are real scalars (we will use Greek characters for them).

[b] Recall that $\mathbf{q}$ is a quadratic form on V. It is there to introduce the metric on V, for $\mathbf{q}(\mathbf{v})$ "is" $\|\mathbf{v}\|^2$. Of course, $\|\mathbf{v}\|^2$ can be zero for non-zero vectors or it can be negative! Think of Minkowski spacetime, for example.

The inner and outer products are both present

Since the multiplication is not commutative, we can form:

1. The symmetric part $\frac{1}{2}(ab + ba)$ of ab.
   For vectors $\mathbf{a}$, $\mathbf{b}$, this gives their inner product!

$$\begin{aligned} \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) &= \frac{1}{2}\Big((\mathbf{a}+\mathbf{b})(\mathbf{a}+\mathbf{b}) - \mathbf{aa} - \mathbf{bb}\Big) \\ &= \frac{1}{2}\Big(\mathbf{q}(\mathbf{a}+\mathbf{b}) - \mathbf{q}(\mathbf{a}) - \mathbf{q}(\mathbf{b})\Big) \\ &= \langle \mathbf{a} \mid \mathbf{b} \rangle \end{aligned}$$

2. The antisymmetric part $\frac{1}{2}(ab - ba)$ of ab.
   For vectors $\mathbf{a}$, $\mathbf{b}$, this is neither a scalar, nor a vector.

   What is it? Let us call it the (Grassmann's) outer product of $\mathbf{a}$ and $\mathbf{b}$ and denote it by $\mathbf{a} \wedge \mathbf{b}$ (pronounced: $\mathbf{a}$ wedge $\mathbf{b}$).

Hence, for vectors $\mathbf{a}$, $\mathbf{b}$, we have obtained the fundamental equality $\mathbf{ab} = \langle \mathbf{a} \mid \mathbf{b} \rangle + \mathbf{a} \wedge \mathbf{b}$.

Hermann Günther Grassmann

Hermann Günther Grassmann (15 April 1809–26 September 1877)

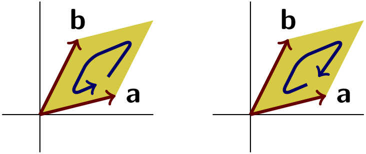
- Born in Sczeczin, studied theology and classical languages. He seems not to have taken any classes in mathematics.
- His essay *Theorie der Ebbe und Flut*, 200 pp., 1840 (that he had to write to become a head teacher at gymnasiums in Germany!) contains the first occurrence of linear algebra as we know it. Apparently, it also contains concepts of vector functions and vector differentiation.
- Grassmann's masterpiece

  ☞ *Die Lineale Ausdehnungslehre, ein neuer Zweig der Mathematik*, Leipzig, 1844

  was mostly neglected during Grassmann's life.
- Disappointed, Grassmann took over to linguistics and the study of Sanskrit. He translated *Rig-Veda* into German (1876).

The outer product in Euclidean $\mathbb{R}^2$

In the plane with the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$, the outer product $\mathbf{a} \wedge \mathbf{b}$ (called the 2-blade) is the oriented area of the parallelogram



$$\mathbf{a} \wedge \mathbf{b} \quad = \quad -(\mathbf{b} \wedge \mathbf{a})$$

The outer product: two important observations

Observe that

1. If $\mathbf{a} = \alpha_1 \cdot \mathbf{e}_1 + \alpha_2 \cdot \mathbf{e}_2$ and $\mathbf{b} = \beta_1 \cdot \mathbf{e}_1 + \beta_2 \cdot \mathbf{e}_2$, then the rules give

$$\mathbf{a} \wedge \mathbf{b} = (\alpha_1 \cdot \beta_2 - \alpha_2 \cdot \beta_1) \cdot \mathbf{e}_1 \wedge \mathbf{e}_2$$

Hence $\mathbf{a} \wedge \mathbf{b}$ differs from the unit 2-blade $\mathbf{i} = \mathbf{e}_1 \wedge \mathbf{e}_2$ by the scalar quantity $\alpha_1 \cdot \beta_2 - \alpha_2 \cdot \beta_1$, known as the determinant of the matrix $(\mathbf{a}, \mathbf{b})$ in some circles ☺.

The outer product: two important observations

Observe that

1. If $\mathbf{a} = \alpha_1 \cdot \mathbf{e}_1 + \alpha_2 \cdot \mathbf{e}_2$ and $\mathbf{b} = \beta_1 \cdot \mathbf{e}_1 + \beta_2 \cdot \mathbf{e}_2$, then the rules give

$$\mathbf{a} \wedge \mathbf{b} = (\alpha_1 \cdot \beta_2 - \alpha_2 \cdot \beta_1) \cdot \mathbf{e}_1 \wedge \mathbf{e}_2$$

   Hence $\mathbf{a} \wedge \mathbf{b}$ differs from the unit 2-blade $\mathbf{i} = \mathbf{e}_1 \wedge \mathbf{e}_2$ by the scalar quantity $\alpha_1 \cdot \beta_2 - \alpha_2 \cdot \beta_1$, known as the determinant of the matrix $(\mathbf{a}, \mathbf{b})$ in some circles ☺.

2. By orthogonality $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2 = -\mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{e}_2\mathbf{e}_1$ holds.

   Thus

$$\mathbf{i}^2 = (\mathbf{e}_1\mathbf{e}_2)^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -\mathbf{e}_1^2\mathbf{e}_2^2 = -(1 \cdot 1) = -1$$

   Is there an imaginary unit around? Yes (and you can draw it)!

The full strength of the Clifford product in Euclidean $\mathbb{R}^2$

If $\mathbf{a} = \alpha_1 \cdot \mathbf{e}_1 + \alpha_2 \cdot \mathbf{e}_2$ and $\mathbf{b} = \beta_1 \cdot \mathbf{e}_1 + \beta_2 \cdot \mathbf{e}_2$, then the rules give

$$\begin{aligned} \mathbf{ab} &= \langle \mathbf{a} \mid \mathbf{b} \rangle + \mathbf{a} \wedge \mathbf{b} \\ &= (\alpha_1 \cdot \beta_1 + \alpha_2 \cdot \beta_2) + (\alpha_1 \cdot \beta_2 - \alpha_2 \cdot \beta_1) \cdot \mathbf{i} \end{aligned}$$

where we know that $\mathbf{i}^2 = -1$ holds.

What do we obtain by computing **abc**? Easy: a vector again.

Hence, in $\mathrm{Cl}(\mathbb{R}^2, \mathbf{q})$ we have three "basic" types of things: scalars, vectors and 2-blades. All members of $\mathrm{Cl}(\mathbb{R}^2, \mathbf{q})$ can be obtained from the basic building blocks

$$1, \quad \mathbf{e}_1, \quad \mathbf{e}_2, \quad \mathbf{e}_1 \wedge \mathbf{e}_2$$

by linear combinations and multiplication, regardless of "dimensions"!

Some notable inverses in the Euclidean plane

Some notable inverses in the Euclidean plane

1. Every nonzero vector is invertible.

   If $\mathbf{v} \neq 0$, then $\mathbf{q}(\mathbf{v}) = \|\mathbf{v}\|^2 \neq 0$.

   Hence $\mathbf{v}^2 = \mathbf{q}(\mathbf{v})$ implies $\mathbf{v}^{-1} = \dfrac{1}{\mathbf{q}(\mathbf{v})} \cdot \mathbf{v} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|^2}$.

Some notable inverses in the Euclidean plane

1. Every nonzero vector is invertible.

   If $\mathbf{v} \neq 0$, then $\mathbf{q}(\mathbf{v}) = \|\mathbf{v}\|^2 \neq 0$.
   Hence $\mathbf{v}^2 = \mathbf{q}(\mathbf{v})$ implies $\mathbf{v}^{-1} = \dfrac{1}{\mathbf{q}(\mathbf{v})} \cdot \mathbf{v} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|^2}$.

2. Every product of nonzero vectors is invertible.

   By the Socks & Shoes Theorem: $(\mathbf{v}_1 \cdot \mathbf{v}_2)^{-1} = \mathbf{v}_2^{-1} \cdot \mathbf{v}_1^{-1}$.

Some notable inverses in the Euclidean plane

1. Every nonzero vector is invertible.

   If $\mathbf{v} \neq 0$, then $\mathbf{q}(\mathbf{v}) = \|\mathbf{v}\|^2 \neq 0$.
   Hence $\mathbf{v}^2 = \mathbf{q}(\mathbf{v})$ implies $\mathbf{v}^{-1} = \dfrac{1}{\mathbf{q}(\mathbf{v})} \cdot \mathbf{v} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|^2}$.

2. Every product of nonzero vectors is invertible.

   By the Socks & Shoes Theorem: $(\mathbf{v}_1 \cdot \mathbf{v}_2)^{-1} = \mathbf{v}_2^{-1} \cdot \mathbf{v}_1^{-1}$.

   Moreover (recall that $\mathbf{i}^2 = -1$), if $\mathbf{v}_1 \cdot \mathbf{v}_2 = \alpha + \mathbf{i} \cdot \beta$, then
   $(\mathbf{v}_1 \cdot \mathbf{v}_2)^{-1} = \dfrac{1}{\alpha^2 + \beta^2} \cdot (\alpha - \mathbf{i} \cdot \beta)$.

Some notable inverses in the Euclidean plane

1. Every nonzero vector is invertible.

   If $\mathbf{v} \neq 0$, then $\mathbf{q}(\mathbf{v}) = \|\mathbf{v}\|^2 \neq 0$.
   Hence $\mathbf{v}^2 = \mathbf{q}(\mathbf{v})$ implies $\mathbf{v}^{-1} = \dfrac{1}{\mathbf{q}(\mathbf{v})} \cdot \mathbf{v} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|^2}$.

2. Every product of nonzero vectors is invertible.

   By the Socks & Shoes Theorem: $(\mathbf{v}_1 \cdot \mathbf{v}_2)^{-1} = \mathbf{v}_2^{-1} \cdot \mathbf{v}_1^{-1}$.

   Moreover (recall that $\mathbf{i}^2 = -1$), if $\mathbf{v}_1 \cdot \mathbf{v}_2 = \alpha + \mathbf{i} \cdot \beta$, then
   $(\mathbf{v}_1 \cdot \mathbf{v}_2)^{-1} = \dfrac{1}{\alpha^2 + \beta^2} \cdot (\alpha - \mathbf{i} \cdot \beta)$.

3. Every nonzero 2-blade is invertible.

   If $\mathbf{a} \wedge \mathbf{b} = \alpha \cdot \mathbf{i}$ for $\alpha \neq 0$, then $(\mathbf{a} \wedge \mathbf{b})^{-1} = -\dfrac{1}{\alpha} \cdot \mathbf{i}$.

Now we can write down the careless computations properly

Now we can write down the careless computations properly

1. Orthogonal projection and orthogonal rejection:

$$\mathbf{v} = \mathbf{v} \cdot (\mathbf{u} \cdot \mathbf{u}^{-1}) = (\mathbf{v} \cdot \mathbf{u}) \cdot \mathbf{u}^{-1} = \Big(\langle \mathbf{v} \mid \mathbf{u} \rangle + (\mathbf{v} \wedge \mathbf{u}) \Big) \cdot \mathbf{u}^{-1}$$
$$= \langle \mathbf{v} \mid \mathbf{u} \rangle \cdot \mathbf{u}^{-1} + (\mathbf{v} \wedge \mathbf{u}) \cdot \mathbf{u}^{-1} = \mathrm{proj}_{\mathbf{u}}(\mathbf{v}) + (\mathbf{v} \wedge \mathbf{u}) \cdot \mathbf{u}^{-1}.$$

recall the informal version

Use: $\langle \mathbf{v} \mid \mathbf{u} \rangle \cdot \mathbf{u}^{-1} = \dfrac{\langle \mathbf{v} \mid \mathbf{u} \rangle}{\|\mathbf{u}\|^2} \cdot \mathbf{u} = \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$.

Now we can write down the careless computations properly

1. Orthogonal projection and orthogonal rejection:
$$\mathbf{v} = \mathbf{v}\cdot(\mathbf{u}\cdot\mathbf{u}^{-1}) = (\mathbf{v}\cdot\mathbf{u})\cdot\mathbf{u}^{-1} = \Big(\langle\mathbf{v}\mid\mathbf{u}\rangle + (\mathbf{v}\wedge\mathbf{u})\Big)\cdot\mathbf{u}^{-1}$$
$$= \langle\mathbf{v}\mid\mathbf{u}\rangle\cdot\mathbf{u}^{-1} + (\mathbf{v}\wedge\mathbf{u})\cdot\mathbf{u}^{-1} = \mathrm{proj}_{\mathbf{u}}(\mathbf{v}) + (\mathbf{v}\wedge\mathbf{u})\cdot\mathbf{u}^{-1}.$$
recall the informal version
Use: $\langle\mathbf{v}\mid\mathbf{u}\rangle\cdot\mathbf{u}^{-1} = \dfrac{\langle\mathbf{v}\mid\mathbf{u}\rangle}{\|\mathbf{u}\|^2}\cdot\mathbf{u} = \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$.

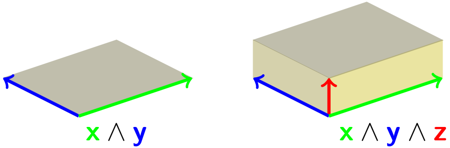
2. Equation of the line: $\mathbf{x}\wedge\mathbf{d} = \mathbf{p}\wedge\mathbf{d}$.
recall the informal version

3. Cramer's Rule (the 2-blade $\mathbf{a}_1\wedge\mathbf{a}_2$ is invertible):
$$\mathbf{b} = (\mathbf{b}\wedge\mathbf{a}_2)\cdot(\mathbf{a}_1\wedge\mathbf{a}_2)^{-1}\cdot\mathbf{a}_1 + (\mathbf{a}_1\wedge\mathbf{b})\cdot(\mathbf{a}_1\wedge\mathbf{a}_2)^{-1}\cdot\mathbf{a}_2.$$
recall the informal version

4. Intersection of two lines (the 2-blade $\mathbf{d}_1\wedge\mathbf{d}_2$ is invertible):
$$\mathbf{x} = (\mathbf{p}_2\wedge\mathbf{d}_2)\cdot(\mathbf{d}_1\wedge\mathbf{d}_2)^{-1}\cdot\mathbf{d}_1 + (\mathbf{d}_1\wedge\mathbf{p}_1)\cdot(\mathbf{d}_1\wedge\mathbf{d}_2)^{-1}\cdot\mathbf{d}_2.$$
recall the informal version

$Cl(\mathbb{R}^3, \mathbf{q})$ for the Euclidean $\mathbf{q}$

This is quite similar to $Cl(\mathbb{R}^2, \mathbf{q})$. In addition to 2-blades $\mathbf{x} \wedge \mathbf{y}$, we also have 3-blades, such as $\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}$ below

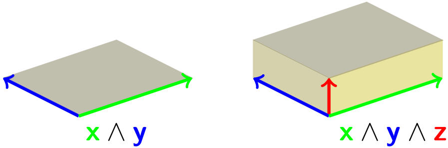


3-blades represent oriented volume.

There are three different unit 2-blades $\mathbf{i} = \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2$, $\mathbf{j} = \mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_2\mathbf{e}_3$ and $\mathbf{k} = \mathbf{e}_1 \wedge \mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_3$.

$\mathrm{Cl}(\mathbb{R}^3, \mathbf{q})$ for the Euclidean q

This is quite similar to $\mathrm{Cl}(\mathbb{R}^2, \mathbf{q})$. In addition to 2-blades $\mathbf{x} \wedge \mathbf{y}$, we also have 3-blades, such as $\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}$ below



3-blades represent oriented volume.

There are three different unit 2-blades $\mathbf{i} = \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2$, $\mathbf{j} = \mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_2\mathbf{e}_3$ and $\mathbf{k} = \mathbf{e}_1 \wedge \mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_3$.

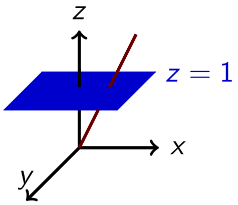
Yes, you have guessed it right; the Brougham Bridge identities[a] $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ hold.

[a] On Monday 16 October 1843, while walking with his wife to the Royal Irish Academy, sir William Rowan Hamilton discovered these identities and carved them on a stone of Brougham Bridge, Dublin. The same discovery was made by Benjamin Olinde Rodrigues three years earlier.

Some quite exotic but useful Clifford algebras

By clever choices of the pair $(V, \mathbf{q})$ one can end up with a lot more exotic Clifford algebras than the Euclidean ones. Some of the choices yield arenas with rich geometric contents. In particular:

1. The homogeneous model. For example, in the projective extension of the plane



2-blades become lines in the projective plane (including the lines at infinity).

Some quite exotic but useful Clifford algebras, cont'd

2. The conformal model.[a]

   This is a rather technical model (introduced in 1999) and it is definitely non-Euclidean.

   For example, to represent $\mathbb{R}^3$, one needs $\mathbb{R}^{3+1+1}$ with the quadratic form $\mathbf{q}$ having the signature $(4, 1)$. This essentially means that we work in a "Minkowski space with four space-like coordinates and one time-like coordinate".

   Advantages: besides the "usual" things, it allows one to work with circles and spheres as with blades. All linear conformal mappings can be made to be a part of the Clifford algebra.

[a] Warning: Commercial applications of conformal GA are protected by U.S. Patent 6,853,964, "System for encoding and manipulating models of objects". The patent is held by A. Rockwood, D. Hestenes and H. Li (8 February 2005).

The ZOO of terms

In a minute we will have to meet various elements of Clifford algebras. They will be named rather colourfully:

1. A vector $\mathbf{n}$ with $\mathbf{n}^2 = 1$ is often called a versor, since it can re-verse the sense of direction. Recall the reflection through a plane.

   In general, a versor is a product of unit vectors.

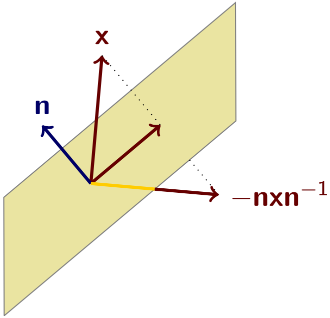
2. A product of two unit vectors is often called a rotor.

   Why? Since it can rotate things. (Obviously ☺.)

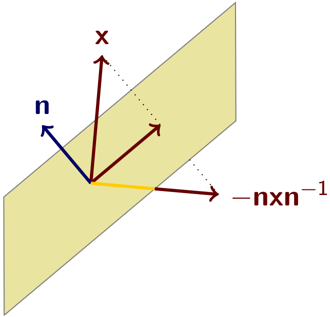
3. A special kind of a rotor is called a spinor.

   Does it spin things? If yes, what does that mean?
   Let's wait. . .

The reflection in $\mathbb{R}^3$ through a plane

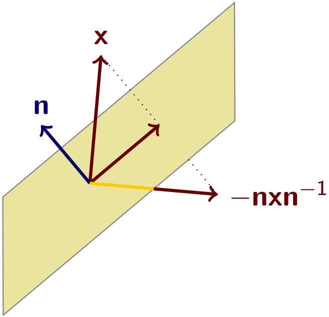


The reflection in $\mathbb{R}^3$ through a plane



$$\mathbf{x} = \mathbf{x}(\mathbf{n}\mathbf{n}^{-1}) = (\mathbf{x}\mathbf{n})\mathbf{n}^{-1} = \Big(\langle \mathbf{x} \mid \mathbf{n}\rangle + \mathbf{x}\wedge\mathbf{n}\Big)\mathbf{n}^{-1} = \underbrace{\langle \mathbf{x} \mid \mathbf{n}\rangle\mathbf{n}^{-1} + (\mathbf{x}\wedge\mathbf{n})\mathbf{n}^{-1}}_{\text{projection plus rejection}}$$

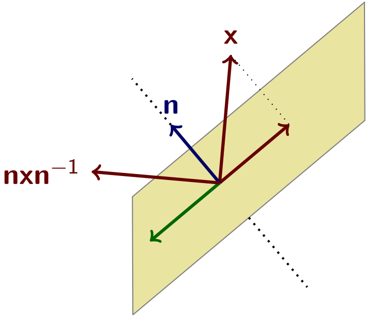
The reflection in $\mathbb{R}^3$ through a plane



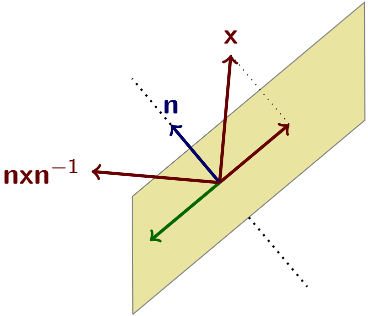
$$\mathbf{x} = \mathbf{x}(\mathbf{n}\mathbf{n}^{-1}) = (\mathbf{x}\mathbf{n})\mathbf{n}^{-1} = \Big(\langle \mathbf{x} \mid \mathbf{n}\rangle + \mathbf{x}\wedge\mathbf{n}\Big)\mathbf{n}^{-1} = \underbrace{\langle \mathbf{x} \mid \mathbf{n}\rangle\mathbf{n}^{-1} + (\mathbf{x}\wedge\mathbf{n})\mathbf{n}^{-1}}_{\text{projection plus rejection}}$$

$$\underbrace{(\mathbf{x}\wedge\mathbf{n})\mathbf{n}^{-1} - \langle \mathbf{x} \mid \mathbf{n}\rangle\mathbf{n}^{-1}}_{\text{rejection minus projection}} = \Big(\mathbf{x}\wedge\mathbf{n} - \langle \mathbf{x} \mid \mathbf{n}\rangle\Big)\mathbf{n}^{-1} = -\Big(\langle \mathbf{n} \mid \mathbf{x}\rangle + \mathbf{n}\wedge\mathbf{x}\Big)\mathbf{n}^{-1} = -\mathbf{n}\mathbf{x}\mathbf{n}^{-1}$$

The reflection in $\mathbb{R}^3$ through an axis



The reflection in $\mathbb{R}^3$ through an axis

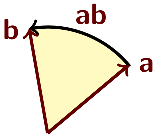


$$\underbrace{\langle \mathbf{x} \mid \mathbf{n}\rangle \mathbf{n}^{-1} - (\mathbf{x} \wedge \mathbf{n})\mathbf{n}^{-1}}_{\text{projection minus rejection}} = \Big(\langle \mathbf{x} \mid \mathbf{n}\rangle - (\mathbf{x}\wedge\mathbf{n})\Big)\mathbf{n}^{-1} = \Big(\langle \mathbf{n} \mid \mathbf{x}\rangle + \mathbf{n}\wedge\mathbf{x}\Big)\mathbf{n}^{-1}$$

$$= \mathbf{n}\mathbf{x}\mathbf{n}^{-1}$$

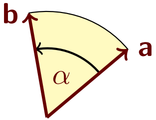
More general versors yield rotations in the plane

Consider two unit vectors **a**, **b**. Their product **ab** is a versor and it can be identified with an oriented arc:



The product $R = \mathbf{ab}$ is called a rotor, since $\mathbf{a}R = \mathbf{b}$ and $R\mathbf{b} = \mathbf{a}$.

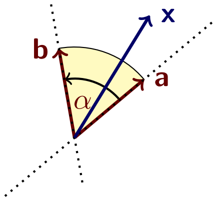
Moreover, $R = \mathbf{ab} = \cos\alpha + \mathbf{i}\sin\alpha = e^{\mathbf{i}\alpha}$, where α is the oriented angle between **a** and **b**:



Clearly: $e^{-\mathbf{i}\alpha} = (\mathbf{ab})^{-1} = \mathbf{b}^{-1}\mathbf{a}^{-1} = \mathbf{ba}$.

Rotations in the plane by reflections through axes

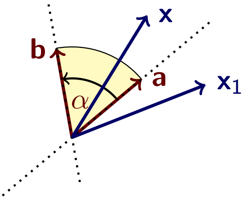
Two consecutive reflections through axes **a** and **b** (in that order!) yield a rotation by twice the oriented angle α between **a** and **b** (in that order!):[a]



[a] This nifty trick was discovered by sir William Rowan Hamilton. Not while on another walk with his wife, I hope.

Rotations in the plane by reflections through axes

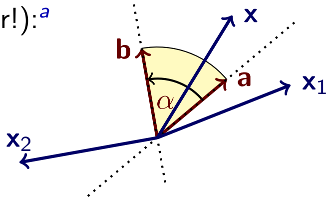
Two consecutive reflections through axes **a** and **b** (in that order!) yield a rotation by twice the oriented angle α between **a** and **b** (in that order!):[a]



[a] This nifty trick was discovered by sir William Rowan Hamilton. Not while on another walk with his wife, I hope.

Rotations in the plane by reflections through axes

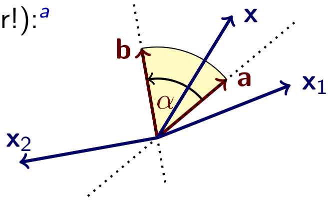
Two consecutive reflections through axes **a** and **b** (in that order!) yield a rotation by twice the oriented angle α between **a** and **b** (in that order!):[a]



[a] This nifty trick was discovered by sir William Rowan Hamilton. Not while on another walk with his wife, I hope.

Rotations in the plane by reflections through axes

Two consecutive reflections through axes **a** and **b** (in that order!) yield a rotation by twice the oriented angle α between **a** and **b** (in that order!):[a]

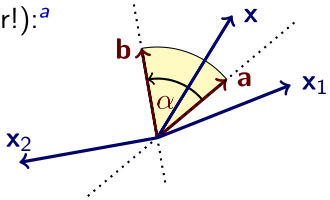


$$\text{Algebraically: } \mathbf{x} \mapsto \underbrace{\mathbf{b}(\overbrace{\mathbf{a}\mathbf{x}\mathbf{a}^{-1}}^{\mathbf{x}_1})\mathbf{b}^{-1}}_{\mathbf{x}_2} = (\mathbf{ba})\mathbf{x}(\mathbf{ba})^{-1} = e^{-\mathbf{i}\alpha}\mathbf{x}e^{\mathbf{i}\alpha}.$$

[a] This nifty trick was discovered by sir William Rowan Hamilton. Not while on another walk with his wife, I hope.

Rotations in the plane by reflections through axes

Two consecutive reflections through axes **a** and **b** (in that order!) yield a rotation by twice the oriented angle α between **a** and **b** (in that order!):[a]



Algebraically: $\mathbf{x} \mapsto \underbrace{\mathbf{b}(\overbrace{\mathbf{axa}^{-1}}^{\mathbf{x}_1})\mathbf{b}^{-1}}_{\mathbf{x}_2} = (\mathbf{ba})\mathbf{x}(\mathbf{ba})^{-1} = e^{-\mathbf{i}\alpha}\mathbf{x}e^{\mathbf{i}\alpha}$.

Thus, rotation by α is given by $\mathbf{x} \mapsto e^{-\mathbf{i}\frac{\alpha}{2}}\mathbf{x}e^{\mathbf{i}\frac{\alpha}{2}}$. The versor $e^{\mathbf{i}\frac{\alpha}{2}}$ was named a spinor by Wolfgang Ernst Pauli for reasons we will see.

[a] This nifty trick was discovered by sir William Rowan Hamilton. Not while on another walk with his wife, I hope.

The shining beauty of spinors: the sense of rotation

Clearly, we have the equality

$$e^{-\mathbf{i}\frac{\alpha}{2}}\mathbf{x}e^{\mathbf{i}\frac{\alpha}{2}} = (-e^{-\mathbf{i}\frac{\alpha}{2}})\mathbf{x}(-e^{\mathbf{i}\frac{\alpha}{2}})$$

The shining beauty of spinors: the sense of rotation

Clearly, we have the equality

$$e^{-\mathbf{i}\frac{\alpha}{2}}\mathbf{x}e^{\mathbf{i}\frac{\alpha}{2}} = (-e^{-\mathbf{i}\frac{\alpha}{2}})\mathbf{x}(-e^{\mathbf{i}\frac{\alpha}{2}})$$

Thus, the spinors $e^{\mathbf{i}\frac{\alpha}{2}}$ and $-e^{\mathbf{i}\frac{\alpha}{2}}$ represent the same rotation, but they are rotations with the opposite senses. More precisely:

1. $e^{\mathbf{i}\frac{\alpha}{2}}$ represents the counterclockwise rotation for α in $[0; 2\pi]$.
2. $e^{\mathbf{i}\frac{\alpha}{2}}$ represents the clockwise rotation for α in $[2\pi; 4\pi]$.

Algebraically:

$$-e^{\mathbf{i}\frac{\alpha}{2}} = (-1)\cdot e^{\mathbf{i}\frac{\alpha}{2}} = e^{\mathbf{i}\pi}\cdot e^{\mathbf{i}\frac{\alpha}{2}} = e^{\mathbf{i}\frac{2\pi+\alpha}{2}} = e^{\mathbf{i}\frac{\alpha+2\pi}{2}},$$

and

$$e^{\mathbf{i}\frac{\alpha+4\pi}{2}} = e^{\mathbf{i}\frac{\alpha}{2}}$$

Thus spinors are periodic with period 4π.

If spinors seem awkward to you. . .

First of all: spinors are awkward!

Do the trick with the plate to overcome the fear of spinors:

1. Stretch your hand with a plate on top of your open palm.
2. Rotate the plate by 2π in its horizontal plane. Your elbow should now point upwards in a quirky way.
3. Rotate by another 2π. The plate and your hand are now in the original position.

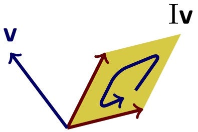
You just proved that your hand holding a plate is a spinorial object!

The trick sheds some light on the one-half spin of the electron. This is why it is sometimes called Dirac's trick.

Rotations in the Euclidean $(\mathbb{R}^3, \mathbf{q})$

Consider any versor $\mathbf{v}$ in $\mathbb{R}^3$ and any angle α. How do we find the spinor performing the rotation by α along the axis given by $\mathbf{v}$?

1. The 2-blade in which the angle α lives should be the "complement" of $\mathbf{v}$ (together with orientation, given by $\mathbf{v}$). The complement is easy to find: it is the 2-blade $\mathrm{I}\mathbf{v}$, where $\mathrm{I} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is the unit 3-blade in $\mathrm{Cl}(\mathbb{R}^3, \mathbf{q})$!



2. Hence $\mathbf{x} \mapsto e^{-\mathrm{I}\mathbf{v}\frac{\alpha}{2}}\mathbf{x}e^{\mathrm{I}\mathbf{v}\frac{\alpha}{2}}$ is the rotation by α along the axis $\mathbf{v}$.

This is most beautiful and most economical: from the spinor $e^{\mathrm{I}\mathbf{v}\frac{\alpha}{2}}$ you can read all the information about the rotation.

Composition of rotations in $\mathbb{R}^3$

The spinors $e^{\mathrm{I}\mathbf{v}\frac{\alpha}{2}}$ and $e^{\mathrm{I}\mathbf{w}\frac{\beta}{2}}$ represent rotations in $\mathbb{R}^3$ by α and β along axes given by versors $\mathbf{v}$ and $\mathbf{w}$, respectively.
Hence $e^{\mathrm{I}\mathbf{v}\frac{\alpha}{2}}e^{\mathrm{I}\mathbf{w}\frac{\beta}{2}}$ is the spinor of their composition: the composite transformation is

$$\mathbf{x} \mapsto e^{-\mathrm{I}\mathbf{w}\frac{\beta}{2}}e^{-\mathrm{I}\mathbf{v}\frac{\alpha}{2}}\mathbf{x}e^{\mathrm{I}\mathbf{v}\frac{\alpha}{2}}e^{\mathrm{I}\mathbf{w}\frac{\beta}{2}}$$

Since $e^{\mathrm{I}\mathbf{v}\frac{\alpha}{2}}e^{\mathrm{I}\mathbf{w}\frac{\beta}{2}}$ has the form $e^{\mathrm{I}\mathbf{u}\frac{\gamma}{2}}$ (think of the product of unit complex numbers!), it can be proved rather easily that the composition of rotations in $\mathbb{R}^3$ is a rotation.
Moreover, the resulting axis $\mathbf{u}$ and angle γ can be found from how complex exponentials are expanded by cosines and sines.[a]

[a] Warning: one has to be a bit careful, the equation $e^{\mathrm{I}\mathbf{v}\frac{\alpha}{2}}e^{\mathrm{I}\mathbf{w}\frac{\beta}{2}} = e^{\mathrm{I}(\mathbf{v}\frac{\alpha}{2}+\mathbf{w}\frac{\beta}{2})}$ does not hold in general.

The Cartan-Dieudonné Theorem

By this theorem we know that any orthogonal transformation (also called a rigid motion) of a finitely dimensional $(V, \mathbf{q})$ is a composite of a certain number of reflections (through hyperplanes).

Thus, any orthogonal transformation of $(V, \mathbf{q})$ can be given by

$$\mathbf{x} \mapsto (-1)^k A\mathbf{x}A^{-1}$$

where $A = \mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_k$ is a versor (hence $A^{-1} = \mathbf{a}_k \ldots \mathbf{a}_2\mathbf{a}_1$).

Notice that the versor A gives you all the necessary information about the rigid motion in the most satisfactory way!

Moreover (the explanation is left to a possible future talk)

The assignment $\mathbf{x} \mapsto (-1)^k A\mathbf{x}A^{-1}$ makes sense if we replace a vector $\mathbf{x}$ by any element x of $\mathrm{Cl}(V, \mathbf{q})$.

Hence we can reflect blades, rotate blades, etc by the same formulas as for vectors!

There is more: we can rotate rotors, reflect spinors, . . .

References

This talk was much inspired by very readable expositions of GA in

- ☞ Leo Dorst, *Geometric (Clifford) algebra: a practical tool for efficient geometric representation*, 1999.
- ☞ Alan Macdonald, *A survey of geometric algebra and geometric calculus*, 2014.

A very nice and quite thorough intro to GA is in the arXiv paper

- ☞ Eric Chisolm, *Geometric algebra*, 2012.

As a further proper reading in GA for computer scientists I would recommend the 600-something pages of the book

- ☞ Leo Dorst, Daniel Fontijne, Stephen Mann, *Geometric algebra for Computer science*, Elsevier, 2007.

And, as always, you can use Google™ to find your favourite sources.