České Vysoké Učení Technické v Praze

Fakulta elektrotechnická

Categorical Domain Theory

Disertační práce

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katedra matematiky
FEL ČVUT, Praha
květen 1998

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[1]Podporováno grantem č. 201/96/0119 Grantové Agentury České republiky.

Abstract

Basic concepts and results of the theory of domains are generalized from (special) posets to (special) categories, following the ideas of D. Lehmann, S. Abramsky and J. Adámek. For example, the rather technical but fruitful concept of approximable relation (as a presentation of a Scott continuous function) is shown to correspond to a more natural concept of flat distributor (as a presentation of a finitary functor) in Chapter 4. The fixed point calculus which enables recursive definitions of domains is extended to finitary functors by examining a limit-colimit coincidence of chains of finitary adjunctions. The basic result that every locally continuous functor has a least fixed point is generalized from domains as posets to domains as categories (Chapter 8). And universal domains are shown to have their categorical counterparts, see Chapter 7.

Contents

Chapter 1

Preface

1.1 Passage from Posets to Categories

This work is based on an observation that categories can be viewed as generalized posets. The question arises which results known from posets can be generalized to categories. The subject of this text is to find out how far one can go in extending Domain Theory (used in formal semantics of programming languages) to category theory.

The passage from posets to categories in domain theory can be of practical importance. In a domain $\langle X, \sqsubseteq \rangle$, the relation $x \sqsubseteq y$ denotes the fact that element x "approximates" element y. In a poset one cannot distinguish different "ways" how x relates to y. The concept of a category overcomes this problem: a witness of the approximation is a morphism $f : x \longrightarrow y$.

On the other hand, categorical generalizations can bring new insight into Domain Theory. For example, one can find new, efficient proofs of classical results, or classical definitions can be restated in a modern, condensed way.

Let us point out that the investigation of domains as categories instead of posets has been started by Daniel Lehmann [Leh76] and Samson Abramsky [Ab83] where the authors justify the use of categories for studying non-determinism.

The basic natural generalizations are as follows:

poset	$\mapsto$	category,
DCPO	$\mapsto$	category with filtered colimits,
algebraic DCPO	$\mapsto$	finitely accessible category,
pointed algebraic DCPO	$\mapsto$	generalized domain (=finitely accessible category with an initial object),
continuous map	$\mapsto$	finitary functor.

Daniel Lehmann chose a category having *colimits of countable chains* as a basic concept of a domain in [Leh76], Jiří Adámek, on the other hand, defined a *Scott complete category* in [Ad97] as a finitely accessible category which satisfies additional properties (see Definition 4.2.1). The latter approach to domains, i.e. that a domain is a category having *colimits of filtered diagrams* has the following advantage:

> Filtered diagrams in category theory provide a natural generalization of directed sets in the theory of posets. Directed-complete posets (DCPOs) in Domain Theory

form a starting point for definitions of theoretically important notions of *algebraic DCPOs* and *Scott domains*.

Another advantage of Adámek's definition is the fact that there is a link of two subjects: the study of Scott complete categories is a study of (rather special) accessible categories. Theory of accessible categories form a well-established part of category theory by now. It originated in the work of Charles Ehresmann [Eh68] as a study of categories which are models of *sketches*, i.e. categories which are "axiomatized". Logical aspects of accessible categories are stressed in the monograph [MP89]. Viewing accessible categories as domains leads to interesting questions on accessible categories which are inspired by Domain Theory, e.g. the study of adjoint pairs of accessible functors between accessible categories (see Chapter 4).

1.2 Main Results of the Dissertation

Structure of the text. The thesis is divided into eight chapters which are divided into sections. Definitions, theorems etc. are numbered within each section and they are referred to by their number. End of proofs, remarks and examples is denoted by □. At the end of the text there is index and list of references. Definitions and results which are not original are referred to. If there is no reference, they are, up to my best knowledge, original.

Here is a brief summary of the thesis:

Chapters 2 and 3. These chapters are excerpts from the standard literature on Domain Theory (Chapter 2) and Category Theory (Chapter 3).

Chapter 4. This chapter contains categorical generalizations of basic classical notions introduced in Chapter 2.

The notion of a *Scott complete category* has been defined by Jiří Adámek in [Ad97]. Adámek investigates there the basic properties of Scott complete categories which I sum up here.

Scott complete categories are a generalization of Scott domains. Scott domains themselves are special *pointed algebraic DCPOs*. Pointed algebraic DCPOs have their natural categorical counterpart in *finitely accessible categories* with *initial objects*. Following the paper [TrV97], I call such categories *generalized domains*.

The crux of Chapter 4 lies in *representation theorems*. There are two main results (published in [Ve97b]):

1. Theorem 4.3.12 states that the 2-quasicategory of generalized domains, left adjoints which have a finitary right adjoints and natural transformations is *biequivalent* to a legitimate 2-category of *normal functors*.

   This generalizes the well-known fact from Domain Theory: embedding-projection pairs between Scott domains correspond to *CUSL embeddings* between their posets of compact elements (see e.g. [SLG94], Chapter 4, Proposition 5.11).

   The concept of an embedding-projection pair between domains is traditionally used in Domain Theory. It however lacks the symmetry of a general adjunction. It turns

out that working with adjunctions in full generality in fact simplifies the reasoning. Moreover, one obtains more general results even in the case of domains as posets. Also, as a byproduct, we obtain in Corollary 4.3.6 an *adjoint functor theorem* for λ-accessible categories.

2. Theorem 4.4.8 states that the 2-quasicategory of generalized domains, finitary functors and natural transformations is *biequivalent* to a biquasicategory of *flat distributors*.

   This generalizes the fact that Scott continuous maps can alternatively be described by *approximable relations* (see e.g. [SLG94], Chapter 6, Remark 3.3). The notion of a flat distributor, introduced in 4.4.2, ties up two important properties: a *flat distributor* is a "relation" between categories (i.e. it is a *distributor*) and whenever one fixes the second argument, it becomes a *flat* functor.

Both representation theorems mentioned above can be "restricted" to Scott complete categories.

The last part of Chapter 4 deals with permanence results for generalized domains. It is proved that certain generalized domains are closed under rather general limit constructions. The proof is based on the observation that the 2-category of sketches is 2-topological over small categories. As a consequence one obtains another proof of the fact that Scott complete categories form a cartesian closed quasicategory. Another case of a limit construction is forming a category of Eilenberg-Moore (co)algebras for a finitary (co)monad. Since finitary monads and comonads are natural generalizations of finitary projections and finitary closures on a poset, one obtains generalizations of results on closedness of certain domains under finitary projections and finitary closures.

I have not pursued the concept of a *2-topological* 2-category any further, but it might be worthwhile.

Chapter 5. It contains the construction of a *free conservative cocompletion* of an arbitrary category — this concept has been introduced in [Ve97c]. Free conservative cocompletions generalize free cocompletions of categories in the aspect that certain prescribed class of colimits is preserved by the cocompletion. In Theorem 5.1.12 it is proved that such a cocompletion exists for any category, moreover, one can talk about a "canonical" choice of such a cocompletion. I took a direction in the spirit of how cocompletions of categories were studied e.g. in [Lam66] or [Ke82], namely the desired cocompletion is a certain category of $\underline{\mathrm{Set}}$-valued functors which preserve a prescribed class of limits. A more common approach to cocompletions of posets is to work with special types of *ideals* — see e.g. [Er86]. It turns out that one can proceed analogously for categories too. Ideals in posets must be replaced with *discrete op-fibrations*. I give an equivalent description of a free conservative cocompletion as a category of discrete op-fibrations in Section 5.2.

The final section of Chapter 5 is devoted to a free cocompletion w.r.t. small filtered colimits. Anders Kock showed in [Ko93] that such a cocompletion yields a special type of a monad called a *KZ doctrine*. Categories having filtered colimits then can naturally be identified with *algebras* for a KZ doctrine and finitary functors are precisely *homomorphisms* of such algebras.

Chapter 6. This chapter gives the basic facts about a categorical generalization of *contin-*

uous domains. The basic properties of the resulting concept of a *continuous category* have been established by Peter Johnstone and André Joyal in their paper [JJ82]. The definition of a continuous category uses the free cocompletion of a category w.r.t. small filtered colimits. The theory of KZ doctrines from [Ko93] allows us to view continuous categories naturally as certain *coalgebras.* These coalgebras have naturally defined *homomorphisms.* It turns out that, when one restricts to continuous posets, these homomorphisms are precisely monotone maps which preserve the way-below relation. Therefore one can introduce the concept of a *way-below preserving* functor.

Another approach to the way-below relation is the notion of a *wavy arrow* from [JJ82]. This concept leads to studying idempotent comonads of flat distributors over a category. I think that such comonads are a natural candidate for a general notion of an *abstract base* introduced by Samson Abramsky and Achim Jung in Definition 2.2.20 of [AbJ96]. This is a work in progress and therefore I have included no results in this spirit.

Chapter 7. This chapter is devoted to the concept of a *subobject* of a domain and to a categorical generalization of the *existence of a universal Scott domain.*

The first section of Chapter 7 contains two notions of a subobject: a *finitary retract* and an *(e,p)-subdomain.* I give a few basic results on the closedness of 2-quasicategories of domains under subobjects in the 2-quasicategory of all categories.

The existence of a universal domain has been proved by Dana Scott in [Sc76]. I give a categorical generalization of Scott's result based on a very general embedding theorem of Věra Trnková from her paper [Tr66a]. The proof given in Chapter 7 is independent of Scott's proof and the technique of my proof also allows to give another proof of Scott's result. These results come from [Ve97a].

The third variety of results of Chapter 7 deals with the question whether a universal generalized Scott domain exists within the realm of *finite* categories. It is shown that this question must be answered in negative: there is no such universal object. This result is a joint work of Věra Trnková and myself ([TrV97]).

From the paper [TrV97] also comes the notion of an $\aleph_0$-*Plotkin category* as a natural generalization of a *bifinite domain.* I give a definition of a general *Plotkin category* in Chapter 7 and I show that a universal Plotkin category exists. Again, the technique used here gives an independent proof that a universal bifinite domain exists.

Chapter 8. This chapter contains the proof of the basic ingredience for solving recursive domain equations — the so called *limit-colimit coincidence.* The result is based on the ideas of Paul Taylor from [Tay87]. It was Taylor's idea that in proving the limit-colimit coincidence for posets as domains, all one really uses are general properties of *adjunctions* and not properties of their special instance, namely embedding-projection pairs.

The proof of the limit-colimit coincidence for categories having filtered colimits uses Penrose diagrams as the main technical tool. I believe that the use of Penrose diagrams make the proof quite readable.

Index. All items of the index refer to the first page where the relevant notion has occured.

Glossary of Symbols. It contains references to the page where the symbols have been used for the first time.

1.3 Acknowledgements

I am much indebted to my supervisor, Jiří Adámek. He made me interested in accessible categories. He was always patient with me and he has been a brilliant teacher, not of category theory alone.

I would also like to thank to Věra Trnková for her interest in my work. She turned my attention to her results on universal categories. She also made possible for me to give several talks in the inspiring atmosphere at her *Seminar in General Mathematical Structures* at the Charles University in Prague.

I had many fruitful discussions on the subject with Teo Sturm and Robert El Bashir.

Jürgen Koslowski initiated me to Penrose diagrams. He also generously offered me his XY-pic macros for drawing them.

I would also like to thank to my family and to my friends for support and understanding.

The whole text has been typeset by Leslie Lamport's LaTeX. All the diagrams were typeset by the XY-pic package version 3.5 created by Kristoffer Rose and Ross Moore.

Chapter 2

Classical Notions of Domain Theory

In this chapter we collect the very basic notions of Domain Theory for further reference. The terminology used here comes from the texts [Gun92] and [SLG94].

2.1 Various Types of Domains

Domain Theory, as a theoretical tool for computer science, was founded by Dana Scott in the late 60's (we refer to Chapter 1 of [AbJ96] for a historical account of a development of Domain Theory). Mathematical structures introduced by Scott are special partially ordered sets called domains and they have been shown to be suitable for modelling the semantics of programming languages and their data types. There are various types of partially ordered sets which are frequently called a domain. There, however, seems to be no standard of a domain — different definitions suit different practical purposes. Roughly speaking, a domain is a partially ordered set, where the elements of the underlying set represent a *piece of knowledge*, the order represents the fact that *one piece of knowledge approximates another one*, and sometimes the partial order bears additional information about which pieces of knowledge are *finite* in an intuitive computational sense. In this section we present the most common notions of domains, namely,

- *pointed, directed complete partial orders* (pointed DCPOs) — Definition 2.1.1,
- *pointed, algebraic, directed complete partial orders* (pointed algebraic DCPOs) — Definition 2.1.3,
- *Scott domains* — Definition 2.1.5.

These structures (and also some other notions) will be generalized in further text.

Definition 2.1.1 A partially ordered set $\langle X, \sqsubseteq \rangle$ is called a *pointed directed complete partial order*, if it has a least element $\perp$ and has suprema of directed subsets of X.

Intuitively, we view $\perp$ as *no knowledge* and a directed subset D of X as a set of *approximations* of an ideal piece of knowledge. The requirement that $\bigsqcup D$ exists intuitively states that the ideal piece of knowledge can be reached by a (possibly infinite) process.

Example 2.1.2 Let `nat` denote the set $\{0, 1, \ldots\}$ of natural numbers. A function from `nat` to `nat` not neccessarily everywhere defined is called a *partial function*. Let X be the set of all partial functions from `nat` to `nat`, denoted by $X = \texttt{nat} \rightsquigarrow \texttt{nat}$. Define a relation $\sqsubseteq$ on X as follows:

$$f \sqsubseteq g \text{ iff whenever } f(n) \text{ is defined, then } g(n) \text{ is defined and } f(n) = g(n).$$

Then $\sqsubseteq$ is a partial order and the structure $\langle X, \sqsubseteq \rangle$ is a pointed DCPO:

1. The least element $\perp$ is the partial function defined *nowhere*.
2. Suppose that $M = \{f_i \mid i \in I\}$ is a directed subset of X. Then we can define f as follows: $f(n)$ is defined iff there is $i \in I$ such that $f_i(n)$ is defined and $f(n) = f_i(n)$. Then f is a partial function (since M was directed) and it is a supremum of M.

□

Definition 2.1.3 Let $\langle X, \sqsubseteq \rangle$ be a pointed DCPO. An element $a \in X$ is called *compact*, if whenever $a \sqsubseteq \bigsqcup D$ for a directed subset $D \subseteq X$, then $a \sqsubseteq d$ for some $d \in D$. $\langle X, \sqsubseteq \rangle$ is called a *pointed algebraic DCPO*, if there is a set X_{fin} of compact elements such that for any $x \in X$ the set $\{a \mid a \sqsubseteq x,\ a \in X_{fin}\}$ is directed and $x = \bigsqcup\{a \mid a \sqsubseteq x,\ a \in X_{fin}\}$.

Intuitively, a compact element is a *finite piece of knowledge*. The possibility to express each element as a supremum of compact elements below it means that each piece of knowledge can be approximated by its finite approximations.

Example 2.1.4 Suppose that $\langle X, \sqsubseteq \rangle$ is the pointed DCPO of partial functions from Example 2.1.2. One can easily find out that this is in fact a pointed algebraic DCPO. The set X_{fin} of compact elements is the set of all partial functions whose definition domain is a finite subset of `nat`. Clearly, the set $\{a \mid a \in X_{fin}, a \sqsubseteq f\}$ is directed and the equality $f = \bigsqcup\{a \mid a \in X_{fin}, a \sqsubseteq f\}$ holds for any $f \in X$. □

Definition 2.1.5 A pointed algebraic DCPO $\langle X, \sqsubseteq \rangle$ is called a *Scott domain*, if any non-empty subset of X bounded from above has a supremum.

Example 2.1.6 The pointed algebraic DCPO $\langle X, \sqsubseteq \rangle$ of partial functions from Example 2.1.2 is a Scott domain. If M is any subset of X with an upper bound u, then $f = \bigsqcup M$ exists and is defined as follows: $f(n)$ is defined iff there is a function $g \in M$ such that $g(n)$ is defined and $f(n) = g(n)$. □

Example 2.1.7 The last example can be easily generalized as follows: for any pair of sets A, B, the set $X = A \rightsquigarrow B$ of all partial functions from A to B together with a relation $\sqsubseteq$ defined as follows:

$$f \sqsubseteq g \text{ iff whenever } f(a) \text{ is defined, then } g(a) \text{ is defined and } f(a) = g(a).$$

form a Scott domain $\langle X, \sqsubseteq \rangle$. □

Example 2.1.8 To any set X there can be added an element $\perp \notin X$ to obtain a Scott domain, called *flat domain on* X, by defining a relation $\sqsubseteq$ on the set $X \cup \{\perp\}$ as follows:

$$a \sqsubseteq b \text{ iff } a = \perp \text{ or } a = b.$$

A flat domain on the set X is frequently denoted by $X_\perp$. □

For a pointed algebraic DCPO $\langle X, \sqsubseteq\rangle$, the partially ordered set $\langle X_{fin}, \sqsubseteq\rangle$ is of great importance: it suffices for the reconstruction the original partially ordered set $\langle X, \sqsubseteq\rangle$. The reconstruction is provided by the so called *ideal completion*, which is the free completion w.r.t. directed suprema.

Definition 2.1.9 An *ideal* of a poset $\langle X, \sqsubseteq\rangle$ is a set $I \subseteq X$ with the following properties:

1. If $x \in I$ and $y \sqsubseteq x$, then $y \in I$, i.e. I is *downward closed.*
2. I is a directed subset of X.

It is easy to verify that given a poset $\langle X, \sqsubseteq\rangle$ and an element $x \in X$, then the set $\downarrow x = \{y \in X \mid y \sqsubseteq x\}$ is an ideal, called a *principal ideal generated by* x.

Definition 2.1.10 The *ideal completion* $\langle X, \sqsubseteq\rangle^*$ of a poset $\langle X, \sqsubseteq\rangle$ is the set of all ideals of $\langle X, \sqsubseteq\rangle$ ordered by inclusion.

The above mentioned reconstruction is as follows:

Theorem 2.1.11 *The ideal completion $\langle X, \sqsubseteq\rangle^*$ of a poset $\langle X, \sqsubseteq\rangle$ with a least element is a pointed, algebraic DCPO. The set of all compact elements of $\langle X, \sqsubseteq\rangle^*$ is $\{\downarrow x \mid x \in X\}$. Moreover, the map $x \mapsto \downarrow x$ is a monotone embedding of $\langle X, \sqsubseteq\rangle$ to $\langle X, \sqsubseteq\rangle^*$.*

Proof. See e.g. [AbJ96], Proposition 2.2.22. □

Thus the posets $\langle A, \sqsubseteq\rangle$ arising as $\langle X_{fin} \sqsubseteq\rangle$ are precisely

- posets having a least element (for pointed algebraic DCPOs),
- complete upper semilattices (for Scott domains), see e.g. [SLG94], Chapter 3, Proposition 2.6. A *complete upper semilattice* (CUSL) is a partially ordered set $\langle A, \sqsubseteq\rangle$ which has a least element $\perp$ and is finitely boundedly cocomplete — that is, any finite subset of A bounded from above has the least upper bound.

Remark 2.1.12 It is obvious that any notion of a domain introduced so far has its "unpointed" version. One can obtain it by dropping the requirement that the least element exist. E.g. a *DCPO* is a poset having suprema of all directed subsets. Analogously one can define an *algebraic DCPO.* □

In the rest of this chapter we will use the word "domain" as a substitute for either a pointed DCPO, a pointed algebraic DCPO or a Scott domain.

2.2 Domains are Suitable for Recursive Definitions

In many areas of mathematics, recursive definitions give a neat description of objects. Sometimes the feature of recursiveness is built-in in the whole subject — e.g. when defining a syntax of programming languages such as Pascal, Ada etc. (cf. [Wa91] or [Wi75]).

We show in this section how the machinery of pointed DCPOs works for specifying semantics of recursively defined objects. Our example, however, will be of much simpler nature than that of a programming language semantics.

Example 2.2.1 Suppose we want to define recursively a partial function $\mathtt{fac} : \mathtt{nat} \rightsquigarrow \mathtt{nat}$ in the following way:

$$\mathtt{fac}(n) = \begin{cases} 1, & \text{provided } n = 0 \\ n * \mathtt{fac}(n-1), & \text{provided } n > 0 \end{cases}$$

Note that the function `fac` should be defined *everywhere* but nevertheless we want to work in the pointed DCPO of *partial* functions.

The defining conditions on the function `fac` can be described as a solution of the equation $f = Y(f)$, where

$$\begin{array}{rcl} Y : (\mathtt{nat} \rightsquigarrow \mathtt{nat}) & \longrightarrow & (\mathtt{nat} \rightsquigarrow \mathtt{nat}) \\ f & \mapsto & n \mapsto \begin{cases} 1, & \text{provided } n = 0 \\ n * f(n-1), & \text{provided } n > 0 \end{cases} \end{array}$$

The desired fixed point `fac` of Y will be obtained via approximations f_i as follows:

$$\begin{array}{rcl} f_0(n) & = & \bot \text{ (undefined for all } n \in \mathtt{nat}) \\ f_1(n) & = & \begin{cases} 1, & \text{for } n = 0 \\ \text{undefined}, & \text{for all } n > 0 \end{cases} \\ f_2(n) & = & \begin{cases} 1, & \text{for } n = 0 \\ 1, & \text{for } n = 1 \\ \text{undefined}, & \text{for all } n > 1 \end{cases} \\ f_3(n) & = & \begin{cases} 1, & \text{for } n = 0 \\ 1, & \text{for } n = 1 \\ 2, & \text{for } n = 2 \\ \text{undefined}, & \text{for all } n > 2 \end{cases} \\ & \vdots & \end{array}$$

Intuitively, the approximations f_i represent a partial finite knowledge of the solution. Moreover, our knowledge increases, since $f_0 \sqsubseteq f_1 \sqsubseteq \ldots$ holds. Having in mind that the structure $\langle \mathtt{nat} \rightsquigarrow \mathtt{nat}, \sqsubseteq \rangle$ is a pointed DCPO and that every countable chain is directed, the supremum $\bigsqcup_{i=0}^{\infty} f_i$ exists.

If we denote $\mathtt{fac} = \bigsqcup_{i=0}^{\infty} f_i$, then it is easy to show that $Y(\mathtt{fac}) = \mathtt{fac}$. □

The last example is an instance of a much more general principle of solving recursive equations. The principle (Theorem 2.2.3 below) also reveals that we have to work not only with suitable objects (pointed DCPOs), but also with suitable morphisms — the so called continuous mappings.

Definition 2.2.2 A monotone map $f : \langle X_1, \sqsubseteq_1 \rangle \longrightarrow \langle X_2, \sqsubseteq_2 \rangle$ between pointed DCPOs is called *continuous*, if it preserves directed suprema, i.e. if for any directed set $D \subseteq X_1$ it holds that

$$f(\bigsqcup D) = \bigsqcup \{ f(d) \mid d \in D \}$$

Note that a continuous mapping is *not* required to preserve least elements.

It is easy to show that each identity mapping is continuous, and that continuous mappings compose — therefore we obtain a *category* <u>DCPO</u> of pointed DCPOs and continuous mappings.

Theorem 2.2.3 (Kleene's Fixed Point Theorem) *Suppose that $\langle X, \sqsubseteq \rangle$ is a pointed DCPO and let $f : \langle X, \sqsubseteq \rangle \longrightarrow \langle X, \sqsubseteq \rangle$ be a continuous mapping. Then f has a unique least fixed point, i.e. there exists a unique $\mu f \in X$ such that the following hold:*

1. *$f(\mu f) = \mu f$ (μf is a fixed point of f).*
2. *If $f(x) = x$ for some $x \in X$, then $\mu f \sqsubseteq x$ (μf is a least fixed point of f).*

Proof. See e.g. [SLG94], Chapter 2, Theorem 3.6. □

Various categorical generalizations of the previous theorem are known — see e.g. [SP82] and [AT90]. These general theorems have been used to prove that a big number of recursive equations do have a solution. Also, working in a more general context provide recursively defined domains, i.e. solutions of equations of the form $D = F(D)$, where F is a *functor* defined on the category of domains and continuous mappings. The functor F cannot be arbitrary, it must satisfy conditions analogous to continuity. See [SLG94], Section 6.4 and [Ad97] for details.

2.3 Subdomains

If we view domains as universes in which we interpret syntactic types of programming languages, then a notion of subtyping requires a proper notion of a subobject. There are two (essentially equivalent) approaches which appeared to be suitable (see [SLG94], Section 4.5):

1. An embedding-projection pair — this notion mimics the standard way of how subobjects are treated in category theory: subobjects are special morphisms in a category.
2. A subdomain — this notion mimics a standard way of how subobjects are treated in universal algebra: subobjects are substructures.

We present both definitions for Scott domains in this section. They can easily be modified for other types of domains.

Embedding-projection Pairs

An *embedding-projection pair* $(e, p) : \langle X, \sqsubseteq_1 \rangle \longrightarrow \langle Y, \sqsubseteq_2 \rangle$ from a Scott domain $\langle X, \sqsubseteq_1 \rangle$ to a Scott domain $\langle Y, \sqsubseteq_2 \rangle$ is a pair of continuous functions $e : \langle X, \sqsubseteq_1 \rangle \longrightarrow \langle Y, \sqsubseteq_2 \rangle$ (embedding part) and $p : \langle Y, \sqsubseteq_2 \rangle \longrightarrow \langle X, \sqsubseteq_1 \rangle$ (projection part) such that $p \cdot e = 1_D$ and $e \cdot p \sqsubseteq 1_E$. The embedding part of any embedding-projection pair has a domain-codomain restriction which is a monotone map between the corresponding CUSLs $f : \langle X_{fin}, \sqsubseteq_1 \rangle \longrightarrow \langle Y_{fin}, \sqsubseteq_2 \rangle$ such that

1. f is a full embedding,
2. f preserves the least element,
3. if $A \subseteq X_{fin}$ is a finite non-empty set and $f[A]$ is bounded from above, then A is bounded from above and $f[\bigsqcup A] = \bigsqcup f[A]$.

A monotone map between CUSLs having the above properties 1.–3. is called a *CUSL embedding* ([SLG94], Chapter 4, Definition 1.4). Any CUSL embedding between CUSLs can be extended into an embedding part of some embedding-projection pair between Scott domains (see [SLG94] Chapter 4, Proposition 5.11). Embedding-projection pairs determine the notion of a subdomain, thus $\langle X, \sqsubseteq_1 \rangle$ is an *embedding-projection subdomain* of $\langle Y, \sqsubseteq_2 \rangle$, if there exists an embedding-projection pair $(e, p) : \langle X, \sqsubseteq_1 \rangle \longrightarrow \langle Y, \sqsubseteq_2 \rangle$.

Subdomains

Given a Scott domain $\langle Y, \sqsubseteq \rangle$ and $X \subseteq Y$, then $\langle X, \sqsubseteq_{/X} \rangle$ is called a *subdomain* of $\langle Y, \sqsubseteq \rangle$, provided that the following conditions hold:

1. $\perp \in X$,
2. if $D \subseteq X$ is directed, then $\bigsqcup D \in X$ (the supremum is taken in $\langle Y, \sqsubseteq \rangle$),
3. if a is a compact element of $\langle X, \sqsubseteq_{/X} \rangle$, then it is compact in $\langle Y, \sqsubseteq \rangle$,
4. if a, b are compact elements in $\langle X, \sqsubseteq_{/X} \rangle$ and have an upper bound in $\langle Y, \sqsubseteq \rangle$, then their supremum lies in X,
5. for any $x \in X$ it holds that $x = \bigsqcup \{a \mid a \in Y_{fin} \cap X, a \sqsubseteq x\}$ (the supremum is taken in $\langle Y, \sqsubseteq \rangle$).

It is easy to see that subdomains of $\langle Y, \sqsubseteq \rangle$ are precisely the substructures of $\langle Y, \sqsubseteq \rangle$ which are Scott domains.

Subdomains are related to embedding-projection subdomains as follows:

Theorem 2.3.1 *$\langle X, \sqsubseteq_1 \rangle$ is an embedding-projection subdomain of $\langle Y, \sqsubseteq_2 \rangle$ iff $\langle X, \sqsubseteq_1 \rangle$ is isomorphic to a subdomain of $\langle Y, \sqsubseteq_2 \rangle$.*

Proof. See e.g. [SLG94], Chapter 4, Theorem 5.14. □

Chapter 3

Category Theory Tool Kit

This chapter serves as a review of facts from category theory which we need to formulate the results of later chapters. Most of these facts are fairly standard and may be found e.g. in the classical book [ML71] or in the more recent Handbook [Bo94]. For details on accessible categories we refer to the books [MP89] and [AR94].

3.1 Size Classification

Whenever we work with categories we must distinguish between "sizes" of collections. Usually, distinction between sets and classes suffices. We will, however, often want to work with the "category of all categories" etc., which would not be legitimate within such a set theory. This problem is avoided by using *Grothendieck universes*.

A Grothendieck universe is a model of set theory. More precisely, a *Grothendieck universe* is a non-empty set $\mathbf{U}$ with the following properties (see [SGA4], Exposé I, Section 0):

1. If $x \in \mathbf{U}$ and $y \in x$ then $x \in \mathbf{U}$.
2. If $x, y \in \mathbf{U}$ then $\{x, y\} \in \mathbf{U}$.
3. If $x \in \mathbf{U}$ then the set of all subsets of x is an element of $\mathbf{U}$.
4. If $I \in \mathbf{U}$ and if $x_i \in \mathbf{U}$ for all $i \in I$ then $\bigcup\{x_i \mid i \in I\} \in \mathbf{U}$.

The existence of a Grothendieck universe is ensured, provided that an inaccessible cardinal exists. Recall from e.g. [Je78] that a cardinal κ is *inaccessible*, provided that $\kappa > \aleph_0$, κ is regular (i.e. $cf(\kappa) = \kappa$) and κ is a strong limit cardinal (i.e. $2^\lambda < \kappa$ for any cardinal $\lambda < \kappa$).

If κ is an inaccessible cardinal then the κ-th level $\mathbf{V}_\kappa$ of the cumulative hierarchy is a model of ZFC (cf. [Je78], Lemma 10.2), thus $\mathbf{V}_\kappa$ is a Grothendieck universe.

We assume the existence of three fixed Grothendieck universes: $\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{U}_3$ such that $\mathbf{U}_1 \in \mathbf{U}_2 \in \mathbf{U}_3$. Elements of $\mathbf{U}_1$ are called (small) *sets*. Elements of $\mathbf{U}_2$ are called *classes*. Elements of $\mathbf{U}_3$ are called *conglomerates*.

A category is called *small*, if its morphisms form a set. A category is called *large* (we will often drop the adjective "large" and talk just about a category), if its morphisms form a class and there is just a set of morphisms between any pair of its objects. A *quasicategory*

is defined just like a category, except that its morphisms can form a conglomerate. A quasicategory which is a category is called *legitimate*. For details on the problems of size see [AHS90], Chapter 0, Section 2.

Sometimes we need a finer control of the size of a category:

Definition 3.1.1 Let κ be an infinite cardinal. Let us say that a category $\underline{\mathrm{K}}$ is a *κ-category*, it it has at most κ objects and each hom-set of $\underline{\mathrm{K}}$ has less than κ elements.

A category $\underline{\mathrm{K}}$ is called *κ-small* if it is a κ-category having less than κ objects. In case $\kappa = \aleph_0$ we say a *finite category* instead of an $\aleph_0$-small one.

3.2 Adjunctions

We say that categories $\underline{\mathrm{A}}$ and $\underline{\mathrm{B}}$ are *isomorphic* ($\underline{\mathrm{A}} \cong \underline{\mathrm{B}}$) if there exists a pair of functors $G : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ and $F : \underline{\mathrm{B}} \longrightarrow \underline{\mathrm{A}}$ such that $FG = 1_{\underline{\mathrm{A}}}$ and $GF = 1_{\underline{\mathrm{B}}}$.

We say that categories $\underline{\mathrm{A}}$ and $\underline{\mathrm{B}}$ are *equivalent* ($\underline{\mathrm{A}} \simeq \underline{\mathrm{B}}$) if there exists a pair of functors $G : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ and $F : \underline{\mathrm{B}} \longrightarrow \underline{\mathrm{A}}$ and a pair of natural isomorphisms $\eta : 1_{\underline{\mathrm{B}}} \Longrightarrow GF$ and $\varepsilon : FG \Longrightarrow 1_{\underline{\mathrm{A}}}$. It is clear that isomorphic categories are also equivalent, but not vice versa.

The fact that a functor $F : \underline{\mathrm{B}} \longrightarrow \underline{\mathrm{A}}$ is a *left adjoint* to $G : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ is denoted by $F \dashv G$. Adjoint functors determine two natural transformations:

$$\eta : 1_{\underline{\mathrm{B}}} \Longrightarrow GF \ (\textit{unit}) \qquad \varepsilon : FG \Longrightarrow 1_{\underline{\mathrm{A}}} \ (\textit{counit})$$

such that the *triangle identities* $G\varepsilon \cdot \eta G = 1_G$ and $\varepsilon F \cdot F\eta = 1_F$ hold.

The quadruple $(F, G, \eta, \varepsilon)$ is then called an *adjoint situation*.

Sometimes, to assert that $G : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ has a left adjoint, only one of the two triangle identities needs to be verified — see Lemma 3.2.3. The result requires the category $\underline{\mathrm{A}}$ to have the following property:

Definition 3.2.1 We say that *idempotents split* in the category $\underline{\mathrm{A}}$ (or that $\underline{\mathrm{A}}$ is *Cauchy complete*), if for each $\underline{\mathrm{A}}$-morphism $f : a \longrightarrow a$ such that $f \cdot f = f$ (an *idempotent in* $\underline{\mathrm{A}}$) there exists a pair of morphisms $r : a \longrightarrow b$ and $s : b \longrightarrow a$ such that $s \cdot r = f$ and $r \cdot s = 1_b$. The pair r, s is called a *splitting of* f.

Cauchy completeness is indeed a mild (co)completeness requirement:

Lemma 3.2.2 *Let $f : a \longrightarrow a$ be an idempotent in $\underline{\mathrm{A}}$. The following are equivalent:*

1. *f splits.*
2. *An equalizer of f and 1_a exists.*
3. *A coequalizer of f and 1_a exists.*

Proof. See [Bo94], Volume 1, Proposition 6.5.4. □

By the above lemma, all idempotents split in the category $\underline{\mathrm{A}}$, whenever one of the following is true:

- $\underline{\mathrm{A}}$ has equalizers.
- $\underline{\mathrm{A}}$ has coequalizers.
- $\underline{\mathrm{A}}$ has filtered colimits (see Definition 3.5.1), since a coequalizer of an idempotent and an identity morphism can be obtained as a filtered colimit.

If all idempotents split in $\underline{\mathrm{A}}$, then so they do in any functor (quasi)category $[\underline{\mathrm{B}}, \underline{\mathrm{A}}]$, since (co)limits are formed pointwise.

We now formulate a very useful observation due to Robert Paré (see [ML71], Chapter IV.1, Exercise 4):

Lemma 3.2.3 *Suppose that $F : \underline{\mathrm{B}} \longrightarrow \underline{\mathrm{A}}$ and $G : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ are functors, $\eta : 1_{\underline{\mathrm{B}}} \Longrightarrow GF$ and $\varepsilon : FG \Longrightarrow 1_{\underline{\mathrm{A}}}$ are natural transformations such that $G\varepsilon \cdot \eta G = 1_G$. Then $\varepsilon F \cdot F\eta$ is an idempotent in the functor (quasi)category $[\underline{\mathrm{B}}, \underline{\mathrm{A}}]$. This idempotent splits iff G has a left adjoint.*

Remark 3.2.4 Note that the above lemma does *not* assert that F is a left adjoint of G. □

Theorem 3.2.5 *Suppose that $(F, G, \eta, \varepsilon) : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ and $(F', G', \eta', \varepsilon') : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ are adjoint situations. For any natural transformation $\mu : F \Longrightarrow F'$ there exists a unique natural transformation $\nu : G' \Longrightarrow G$ such that the following diagrams commute:*

$$\begin{tikzcd} G' \arrow[r, "\nu"] \arrow[d, "\eta G'"'] & G \\ GFG' \arrow[r, "G\mu G'"'] & GF'G' \arrow[u, "G\varepsilon"'] \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\mu"] \arrow[d, "F\eta'"'] & F' \\ FG'F' \arrow[r, "F\nu F'"'] & FGF' \arrow[u, "\varepsilon F'"'] \end{tikzcd}$$

$$\begin{tikzcd} FG' \arrow[r, "F\eta"] \arrow[d, "\mu G'"'] & FG \arrow[d, "\varepsilon"] \\ F'G' \arrow[r, "\varepsilon'"'] & 1_{\underline{\mathrm{A}}} \end{tikzcd} \qquad \begin{tikzcd} 1_{\underline{\mathrm{B}}} \arrow[r, "\eta"] \arrow[d, "\eta'"'] & GF \arrow[d, "G\nu"] \\ G'F' \arrow[r, "\mu F'"'] & GF' \end{tikzcd}$$

Moreover, all four diagrams commute, provided that one of them does.

Proof. See e.g. [ML71], Section IV.8, Theorem 2. □

Definition 3.2.6 The natural transformation ν corresponding to μ by virtue of Theorem 3.2.5 is called a *right mate* of μ. Conversely, μ is called a *left mate* of ν. The pair (μ, ν) is called a morphism from $(F \dashv G, \eta, \varepsilon)$ to $(F' \dashv G', \eta', \varepsilon')$.

The precise correspondence between left and right mates (the so-called *mate calculus*) will be explained later (see Theorem 3.3.14).

3.3 2-Categories and Bicategories

In what follows we will often work with categories having certain categories as objects and functors as morphisms. These structures naturally bear a "two-dimensional" structure. The axiomatic treatment of these notions is provided by the definitions of a 2-category and bicategory. For a detailed survey we refer to [Gr74] and [Bo94].

We denote by $\underline{\mathbf{1}}$ the *category* having just one morphism (this morphism is neccessarily the identity on the unique object of $\underline{\mathbf{1}}$).

Definition 3.3.1 A *2-category* K is given by the following data:

1. A class $Ob(\mathsf{K})$ of K-objects.
2. For each pair a, b of K-objects a small category $\mathsf{K}(a,b)$.
3. For each triple a, b, c of K-objects a *composition functor* $C_{a,b,c} : \mathsf{K}(a,b) \times \mathsf{K}(b,c) \longrightarrow \mathsf{K}(a,c)$.
4. For each K-object a a functor $U_a : \underline{\mathbf{1}} \longrightarrow \mathsf{K}(a,a)$.

such that the following axioms hold:

(i) Associativity axiom: the diagram

$$\begin{tikzcd}
\mathsf{K}(a,b) \times \mathsf{K}(b,c) \times \mathsf{K}(c,d) \arrow[r, "1\times C_{b,c,d}"] \arrow[d, "C_{a,b,c}\times 1"'] & \mathsf{K}(a,b) \times \mathsf{K}(b,d) \arrow[d, "C_{a,b,d}"] \\
\mathsf{K}(a,c) \times \mathsf{K}(c,d) \arrow[r, "C_{a,c,d}"'] & \mathsf{K}(a,d)
\end{tikzcd} \tag{3.1}$$

commutes for any quadruple a, b, c, d of K-objects.

(ii) Identity axiom: the diagrams

$$\begin{tikzcd}
\underline{\mathbf{1}} \times \mathsf{K}(a,b) \arrow[r, "\cong"] \arrow[d, "U_a\times 1"'] & \mathsf{K}(a,b) \arrow[d, "1"] \\
\mathsf{K}(a,a) \times \mathsf{K}(a,b) \arrow[r, "C_{a,a,b}"'] & \mathsf{K}(a,b)
\end{tikzcd} \tag{3.2}$$

$$\begin{tikzcd}
\mathsf{K}(a,b) \times \underline{\mathbf{1}} \arrow[r, "\cong"] \arrow[d, "1\times U_b"'] & \mathsf{K}(a,b) \arrow[d, "1"] \\
\mathsf{K}(a,b) \times \mathsf{K}(b,b) \arrow[r, "C_{a,b,b}"'] & \mathsf{K}(a,b)
\end{tikzcd} \tag{3.3}$$

commute for any pair a, b of K-objects.

Objects of K are called *0-cells*, objects of each $\mathsf{K}(a,b)$ are called *1-cells*, denoted by $f : a \longrightarrow b$, and morphisms of $\mathsf{K}(a,b)$ are called *2-cells*, denoted by $\alpha : f \Longrightarrow g$. The *identity 1-cell* on a is denoted by 1_a, the *identity 2-cell* on f is denoted by i_f. Composition in both K and all categories $\mathsf{K}(a,b)$ will be both denoted by a dot. We will call composition for 2-cells *vertical composition*. Instead of $C_{a,b,c}(\alpha,\beta)$ for 2-cells $\alpha : f \Longrightarrow g$ and $\beta : h \Longrightarrow k$ we will write $\beta \star \alpha : h \cdot f \longrightarrow k \cdot g$ and we will call $\star$ *horizontal composition*.

Remark 3.3.2 We say that a 2-category is *small*, if all 0-cells form a set. We also define the notion of a 2-quasicategory similarly to the notion of a quasicategory: a *2-quasicategory* K is given by a conglomerate $Ob(\mathsf{K})$ of objects, quasicategories $\mathsf{K}(a,b)$ for each pair a, b of objects, functors $C_{a,b,c}$ and U_a such that (i) and (ii) above hold. □

Example 3.3.3 Each category $\underline{\mathrm{K}}$ can be considered as a (so called *discrete*) 2-category: take $\underline{\mathrm{K}}$-objects for 0-cells, $\underline{\mathrm{K}}$-morphisms for 1-cells and let 2-cells be only the identity 2-cells. Conversely, each 2-category K has its *underlying category* K_o: forget the 2-cells. The two processes form in fact an adjunction — see [Bo94], Volume 2, Proposition 6.4.7. □

Example 3.3.4 The following example of a 2-category is useful: $\mathcal{I}$ has only one 0-cell $*$, only one identity 1-cell 1_* and only one identity 2-cell i_{1_*}. $\mathcal{I}$ is called a *unit 2-category*. □

Example 3.3.5 Let us give examples of 2-(quasi)categories we use later:

1. The "paradigmatic" 2-category is Cat with small categories as 0-cells, functors as 1-cells and natural transformations as 2-cells. One can easily verify all axioms of a 2-category. For functors (i.e. 1-cells) $F : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$, $G : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$, $H : \underline{\mathrm{C}} \longrightarrow \underline{\mathrm{D}}$, $K : \underline{\mathrm{C}} \longrightarrow \underline{\mathrm{D}}$ and natural transformations (i.e. 2-cells) $\sigma : F \Longrightarrow G$ and $\tau : H \Longrightarrow K$, the horizontal composition $\tau \star \sigma : H \cdot F \Longrightarrow K \cdot G$ is defined as $\tau \star \sigma = (\tau G) \cdot (K\sigma) = (H\sigma) \cdot (\tau F)$.

2. CAT is the 2-quasicategory having all categories as 0-cells, all functors as 1-cells and all natural transformations as 2-cells.

3. CAT^l is the 2-quasicategory having all categories as 0-cells, all left adjoints (i.e. all functors having right adjoints) as 1-cells and all natural transformations as 2-cells. We call CAT^l the *2-quasicategory of left adjoints*.

4. CAT^r is the 2-quasicategory having all categories as 0-cells, all right adjoints (i.e. all functors having left adjoints) as 1-cells and all natural transformations as 2-cells. We call CAT^r the *2-quasicategory of right adjoints*.

□

Notation 3.3.6 In 2-category theory, a weakening of a commutativity of a diagram appears — namely, *the commutativity of a diagram up to a 2-cell.* In fact, diagrams commutative up to a 2-cell are rather typical.

In any 2-category, the diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\tau}{\Longrightarrow}"] & b \arrow[d, "g"] \\ c \arrow[r, "k"'] & d \end{tikzcd} \tag{3.4}$$

represents the 2-cell $\tau : k \cdot h \Longrightarrow g \cdot f$. In the diagram above, τ is called a *comparison 2-cell.*

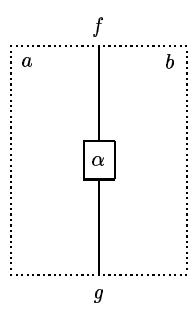
Analogously, the diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\tau}{\Longleftarrow}"] & b \arrow[d, "g"] \\ c \arrow[r, "k"'] & d \end{tikzcd} \tag{3.5}$$

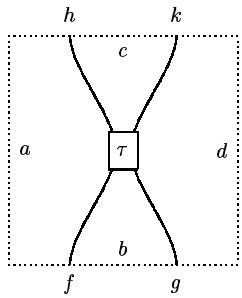
represents the 2-cell $\tau : g \cdot f \Longrightarrow k \cdot h$.

We will often use a different notation for diagrams commuting up to a 2-cell — the so called *Penrose diagrams.* Penrose diagrams represent what really matters, namely the 2-cells.

A single Penrose diagram is a rectangle with a dotted boundary. The rectangle is divided by curves labelled by 1-cells into sections labelled by 0-cells. A curve representing a 1-cell may either terminate at a boundary of the rectangle or at a box labelled by a 2-cell. For example, the following Penrose diagram represents a 2-cell $\alpha : f \Longrightarrow g$, where $f, g : a \longrightarrow b$ are 1-cells:



Instead of diagram (3.4) we can draw the following:



(3.6)

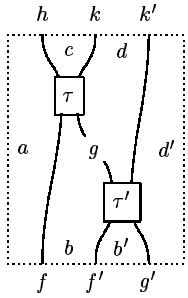
The important feature of Penrose diagrams is that the dotted rectangles are of a "rubber length". This allows us to "paste" them together whenever labellings of neighbouring areas match. First, we describe what pasting is in terms of squares commuting up to a 2-cell.

Squares commuting up to a 2-cell can be *pasted* (composed) horizontally and vertically:

- the diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\tau}{\Longrightarrow}"] & b \arrow[r, "f'"] \arrow[d, "g"] \arrow[dr, phantom, "\overset{\tau'}{\Longrightarrow}"] & b' \arrow[d, "g'"] \\ c \arrow[r, "k"'] & d \arrow[r, "k'"'] & d' \end{tikzcd} \tag{3.7}$$

represents the 2-cell $(\tau' \star i_f) \cdot (i_{k'} \star \tau) : k' \cdot k \cdot h \Longrightarrow g' \cdot f' \cdot f$. In the notation of Penrose diagrams, we draw instead the following:



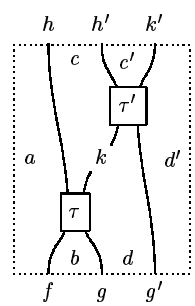
(3.8)

- the diagram

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\tau}{\Longrightarrow}"] & b \arrow[d, "g"] \\
c \arrow[r, "k"'] \arrow[d, "h'"'] \arrow[dr, phantom, "\overset{\tau'}{\Longrightarrow}"] & d \arrow[d, "g'"] \\
c' \arrow[r, "k'"'] & d'
\end{tikzcd} \tag{3.9}$$

represents the 2-cell $(i_{g'} \star \tau) \cdot (\tau' \star i_h) : k' \cdot h' \cdot h \Longrightarrow g' \cdot g \cdot f$.

The corresponding Penrose diagram of (3.9) is:



(3.10)

□

Remark 3.3.7 Each notation has its own advantages. In what follows we use *both* diagrams commutative up to a 2-cell *and* Penrose diagrams. Usually, when using Penrose diagrams, we omit the explicit mention of 0-cells. □

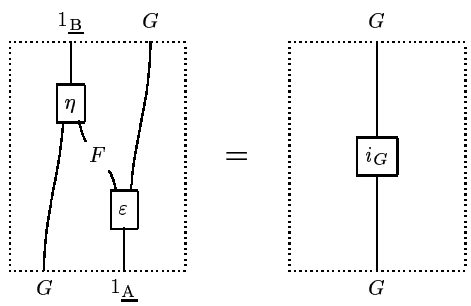
Example 3.3.8 Perhaps the most common instance of squares commutative up to a 2-cell is an adjunction. A pair of adjoint functors can be seen as a "pair of functors inverse to each other up to a 2-cell": see the following familiar diagrams

$$\begin{tikzcd}
\underline{\mathrm{B}} \arrow[r, "G"] \arrow[d, "1_{\underline{\mathrm{B}}}"'] \arrow[dr, phantom, "\overset{\eta}{\Longrightarrow}"] & \underline{\mathrm{A}} \arrow[d, "F"] \\
\underline{\mathrm{B}} \arrow[r, "1_{\underline{\mathrm{B}}}"'] & \underline{\mathrm{B}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathrm{A}} \arrow[r, "1_{\underline{\mathrm{A}}}"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\varepsilon}{\Longrightarrow}"] & \underline{\mathrm{A}} \arrow[d, "1_{\underline{\mathrm{A}}}"] \\
\underline{\mathrm{B}} \arrow[r, "G"'] & \underline{\mathrm{A}}
\end{tikzcd}$$

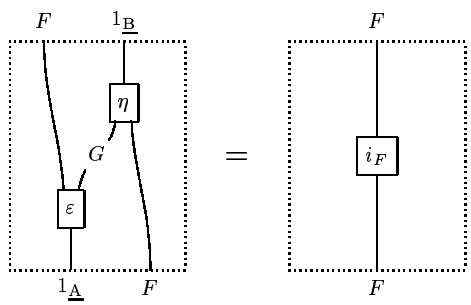
for $F : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$, $G : \underline{\mathrm{B}} \longrightarrow \underline{\mathrm{A}}$ with $G \dashv F$ and the unit transformation $\eta : 1_{\underline{\mathrm{B}}} \Longrightarrow FG$, the counit transformation $\varepsilon : GF \Longrightarrow 1_{\underline{\mathrm{A}}}$, which should be read "$\eta$ is a transformation

from $1_{\underline{B}}$ to FG" (the diagram on the left) and "ε is a transformation from GF to $1_{\underline{A}}$" (the diagram on the right). Of course, the above two diagrams *do not* reflect the main thing about η and ε — namely their universality and couniversality!

To desribe the triangle identities we use Penrose diagrams:



(3.11)



(3.12)

The triangle identities thus express the fact that η and ε are, in a sense, mutually inverse. □

Definition 3.3.9 A *bicategory* K is given by the following data:

1. A class $Ob(\mathsf{K})$ of K-*objects* (also called *0-cells*).
2. For each pair a, b of K-objects a small category $\mathsf{K}(a,b)$. $\mathsf{K}(a,b)$-objects are called *1-cells*, $\mathsf{K}(a,b)$-morphisms are called *2-cells*.
3. For each triple a, b, c of K-objects a *composition functor* $C_{a,b,c} : \mathsf{K}(a,b) \times \mathsf{K}(b,c) \longrightarrow \mathsf{K}(a,c)$.
4. For each K-object a a functor $U_a : \underline{\mathbf{1}} \longrightarrow \mathsf{K}(a,a)$.

such that the following axioms hold:

(i) Associativity isomorphism: for any 4-tuple a, b, c, d of K-objects there is a natural isomorphism $\alpha_{a,b,c,d}$

$$\begin{tikzcd}
\mathsf{K}(a,b) \times \mathsf{K}(b,c) \times \mathsf{K}(c,d) \arrow[r, "1 \times C_{b,c,d}"] \arrow[d, "C_{a,b,c} \times 1"'] \arrow[dr, phantom, "\overset{\alpha_{a,b,c,d}}{\Longrightarrow}"] & \mathsf{K}(a,b) \times \mathsf{K}(b,d) \arrow[d, "C_{a,b,d}"] \\
\mathsf{K}(a,c) \times \mathsf{K}(c,d) \arrow[r, "C_{a,c,d}"'] & \mathsf{K}(a,d)
\end{tikzcd} \tag{3.13}$$

i.e., for any triple of 1-cells $f : a \longrightarrow b$, $g : b \longrightarrow c$, $h : c \longrightarrow d$ there is an isomorphism 2-cell $\alpha_{f,g,h} : h \cdot (g \cdot f) \Longrightarrow (h \cdot g) \cdot f$, where we write $\alpha_{f,g,h}$, instead of the precise but clumsy $(\alpha_{a,b,c,d})_{f,g,h}$.

(ii) Identity isomorphisms: for any pair of K-objects a, b there are natural isomorphisms

$$\begin{tikzcd}
\underline{\mathbf{1}} \times \mathsf{K}(a,b) \arrow[r, "\cong"] \arrow[d, "U_a \times 1"'] \arrow[dr, phantom, "\overset{\rho_a}{\Longrightarrow}"] & \mathsf{K}(a,b) \arrow[d, "1"] \\
\mathsf{K}(a,a) \times \mathsf{K}(a,b) \arrow[r, "C_{a,a,b}"'] & \mathsf{K}(a,b) \\
\mathsf{K}(a,b) \times \underline{\mathbf{1}} \arrow[r, "\cong"] \arrow[d, "1 \times U_b"'] \arrow[dr, phantom, "\overset{\lambda_b}{\Longrightarrow}"] & \mathsf{K}(a,b) \arrow[d, "1"] \\
\mathsf{K}(a,b) \times \mathsf{K}(b,b) \arrow[r, "C_{a,b,b}"'] & \mathsf{K}(a,b)
\end{tikzcd} \tag{3.14}$$

i.e. for each 1-cell $f : a \longrightarrow b$ there are isomorphism 2-cells $\rho_f : f \cdot 1_a \Longrightarrow f$ and $\lambda_f : 1_b \cdot f \Longrightarrow f$, where we abbreviate e.g. ρ_f for the precise but clumsy $(\rho_a)_f$.

(iii) Associativity coherence: for any 4-tuple of 1-cells $f : a \longrightarrow b$, $g : b \longrightarrow c$, $h : c \longrightarrow d$, $k : d \longrightarrow e$ the following diagram

$$\begin{tikzcd}
k\cdot(h\cdot(g\cdot f)) \arrow[r, "i_k\star\alpha_{f,g,h}"] \arrow[dd, "\alpha_{gf,h,k}"'] & k\cdot((h\cdot g)\cdot f) \arrow[d, "\alpha_{f,hg,k}"] \\
 & (k\cdot(h\cdot g))\cdot f \arrow[d, "\alpha_{g,h,k}\star i_f"] \\
(k\cdot h)\cdot(g\cdot f) \arrow[r, "\alpha_{f,g,kh}"'] & ((k\cdot h)\cdot g)\cdot f
\end{tikzcd} \tag{3.15}$$

commutes.

(iv) Identity coherences: for any pair of 1-cells $f : a \longrightarrow b$, $g : b \longrightarrow c$ the following two diagrams

$$\begin{tikzcd}
g\cdot(f\cdot 1_a) \arrow[rr, "\alpha_{1_a,f,g}"] \arrow[dr, "i_g\star\rho_f"'] & & (g\cdot f)\cdot 1_b \arrow[dl, "\rho_{fg}"] \\
 & g\cdot f & \\
1_c\cdot(g\cdot f) \arrow[rr, "\alpha_{f,g,1_c}"] \arrow[dr, "\lambda_{gf}"'] & & (1_c\cdot g)\cdot f \arrow[dl, "\lambda_g\star i_f"] \\
 & g\cdot f &
\end{tikzcd} \tag{3.16}$$

commute.

When we want to emphasise the relevant isomorphisms we write the bicategory as the quadruple $(\mathsf{K}, \alpha, \rho, \lambda)$.

Remark 3.3.10 It is clear that the concept of a bicategory generalizes the notion of a 2-category (take the isomorphisms α, ρ and λ to be identities).

We can also speak about *biquasicategories* in the expected sense. □

Definition 3.3.11 Given bicategories $(\mathsf{A}, \alpha^{\mathsf{A}}, \rho^{\mathsf{A}}, \lambda^{\mathsf{A}})$ and $(\mathsf{B}, \alpha^{\mathsf{B}}, \rho^{\mathsf{B}}, \lambda^{\mathsf{B}})$ we say that $\Phi : \mathsf{A} \longrightarrow \mathsf{B}$ is a *lax functor* from A to B, if the following conditions are satisfied:

1. For each 0-cell a in A a 0-cell Φa in B is assigned.

2. For any pair of 0-cells a, b in A there is a *functor* $\Phi_{a,b} : \mathsf{A}(a, b) \longrightarrow \mathsf{B}(\Phi a, \Phi b)$.

3. For any triple of 0-cells a, b, c in A there is a natural transformation $\varphi_{a,b,c}$ (we denote by C^{A}, C^{B} the respective composition functors):

$$\begin{tikzcd}
\mathsf{A}(a,b) \times \mathsf{A}(b,c) \arrow[r, "C^{\mathsf{A}}{}_{a,b,c}"] \arrow[d, "\Phi_{a,b} \times \Phi_{b,c}"'] \arrow[dr, phantom, "\overset{\varphi_{a,b,c}}{\Longrightarrow}"] & \mathsf{A}(a,c) \arrow[d, "\Phi_{a,c}"] \\
\mathsf{B}(\Phi a, \Phi b) \times \mathsf{B}(\Phi b, \Phi c) \arrow[r, "C^{\mathsf{B}}{}_{\Phi a, \Phi b, \Phi c}"'] & \mathsf{B}(\Phi a, \Phi c)
\end{tikzcd} \tag{3.17}$$

i.e. for each pair of 1-cells $f : a \longrightarrow b$, $g : b \longrightarrow c$ in A we have a 2-cell in B $\varphi_{f,g} : \Phi g \cdot \Phi f \Longrightarrow \Phi(g \cdot f)$, where we write $\varphi_{f,g}$ instead of the precise but clumsy $(\varphi_{a,b,c})_{f,g}$ and analogously Φf instead of $\Phi_{a,b} f$.

4. For any 0-cell a in A there is a 2-cell $\psi_a : \Phi_{a,a}(1_a) \Longrightarrow 1_{\Phi a}$ in B.

5. Associativity coherence: for any triple of 1-cells $f : a \longrightarrow b$, $g : b \longrightarrow c$, $h : c \longrightarrow d$ in A the following diagram

$$\begin{tikzcd}
\Phi h \cdot (\Phi g \cdot \Phi f) \arrow[r, "i_{\Phi h} \star \varphi_{f,g}"] \arrow[d, "\alpha^{\mathsf{B}}{}_{\Phi f, \Phi g, \Phi h}"'] & \Phi h \cdot \Phi(g \cdot f) \arrow[d, "\varphi_{gf,h}"] \\
(\Phi h \cdot \Phi g) \cdot \Phi f \arrow[d, "\varphi_{g,h} \star i_{\Phi f}"'] & \Phi(h \cdot (g \cdot f)) \arrow[d, "\Phi(\alpha^{\mathsf{A}}{}_{f,g,h})"] \\
\Phi(h \cdot g) \cdot \Phi f \arrow[r, "\varphi_{f,hg}"'] & \Phi((h \cdot g) \cdot f)
\end{tikzcd} \tag{3.18}$$

commutes, where we write Φ instead of e.g. $\Phi_{a,b}$.

6. Identity coherences: for any 1-cell $f : a \longrightarrow b$ in A the following two diagrams

$$\begin{tikzcd}
\Phi f \cdot \Phi 1_a \arrow[r, "i_{\Phi f} \star \psi_a"] \arrow[d, "\varphi_{1_a, f}"'] & \Phi f \cdot 1_{\Phi a} \arrow[d, "\rho^{\mathsf{B}}{}_{\Phi f}"] \\
\Phi(f \cdot 1_a) \arrow[r, "\rho^{\mathsf{A}}{}_f"'] & \Phi f
\end{tikzcd}
\qquad
\begin{tikzcd}
\Phi 1_b \cdot \Phi f \arrow[r, "\psi_b \star i_{\Phi f}"] \arrow[d, "\varphi_{f, 1_b}"'] & 1_{\Phi b} \cdot \Phi f \arrow[d, "\lambda^{\mathsf{B}}{}_{\Phi f}"] \\
\Phi(1_b \cdot f) \arrow[r, "\lambda^{\mathsf{A}}{}_f"'] & \Phi f
\end{tikzcd} \tag{3.19}$$

commute.

When we want to emphasise the relevant comparison 2-cells we refer to a lax functor as to the triple (Φ, φ, ψ).

In case φ and ψ are isomorphism 2-cells, Φ is called a *pseudofunctor*. In case A and B are 2-categories and φ and ψ are identity 2-cells, Φ is called a *2-functor*.

The proper concept of an "equivalence" between 2-categories (or, more generally, between bicategories) is the following (see [Str80]):

Definition 3.3.12 If A and B are 2-categories (or bicategories), a pseudofunctor $\Phi : \mathsf{A} \longrightarrow \mathsf{B}$ is called a *biequivalence*, if

(a) for any 0-cell b in B there is a 0-cell a in A, a pair of 1-cells $f : \Phi a \longrightarrow b$, $g : b \longrightarrow \Phi a$ and a pair of isomorphism 2-cells $\alpha : fg \Longrightarrow 1_b$, $\beta : 1_{\Phi a} \Longrightarrow gf$, and

(b) for any pair of 0-cells a, a' in A, the functor $\Phi_{a,a'} : \mathsf{A}(a, a') \longrightarrow \mathsf{B}(\Phi a, \Phi a')$ is an equivalence of categories.

Recall the definition of CAT^l and CAT^r from Example 3.3.5. It is intuitively clear that CAT^l and CAT^r are "dually equivalent". To state this precisely (and this result is probably a folklore), we need the notion of a dual for a 2-category:

Notation 3.3.13 If K is a 2-category, then by K^{op} we denote the 2-category obtained from K by reversing *both* all 1-cells *and* all 2-cells. □

Theorem 3.3.14 CAT^r *and* $(\mathsf{CAT}^l)^{op}$ *are biequivalent.*

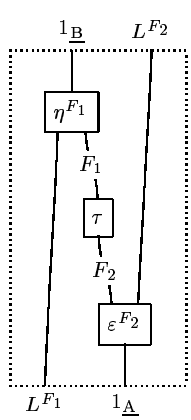
Proof. We are going to define a *contravariant* pseudofunctor $(\Phi, \varphi, \psi) : \mathsf{CAT}^r \longrightarrow \mathsf{CAT}^l$ and show that it is a biequivalence.

Suppose that for each right adjoint $F : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ a fixed left adjoint $L^F : \underline{\mathrm{B}} \longrightarrow \underline{\mathrm{A}}$ is chosen in such a way that $L^{1_{\underline{\mathrm{A}}}} = 1_{\underline{\mathrm{A}}}$. Moreover, let η^F and ε^F be unit and counit of $L^F \dashv F$ (with $\eta^{1_{\underline{\mathrm{A}}}} = \varepsilon^{1_{\underline{\mathrm{A}}}} = i_{1_{\underline{\mathrm{A}}}}$).

Let $\Phi(\underline{\mathrm{A}}) = \underline{\mathrm{A}}$ for each category $\underline{\mathrm{A}}$. Let

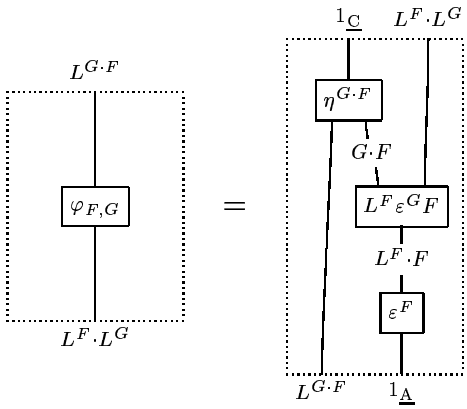
$$\Phi_{\underline{\mathrm{A}},\underline{\mathrm{B}}} : \tau : F_1 \Longrightarrow F_2 \mapsto \tau^* : L^{F_2} \Longrightarrow L^{F_1}$$

where $\tau^* : L^{F_2} \Longrightarrow L^{F_1}$ is the left mate of τ. Explicitly, τ^* is defined as the following 2-cell:



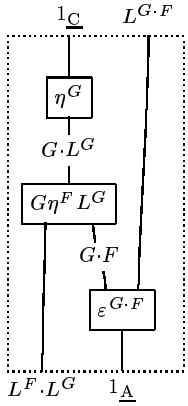
(3.20)

For $F : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$, $G : \underline{\mathrm{B}} \longrightarrow \underline{\mathrm{C}}$, the isomorphism $\varphi_{F,G} : \Phi(F) \cdot \Phi(G) \Longrightarrow \Phi(G \cdot F)$ is defined as follows:



(3.21)

Using the fact that $L^F \cdot L^G \dashv G \cdot F$ with $G\eta^F L^G \cdot \eta^G$ as a unit and $\varepsilon^F \cdot L^F \varepsilon^G F$ as a counit, $L^F \dashv F$ and $L^G \dashv G$, it is easy to show that $\varphi_{F,G}$ is an isomorphism with the inverse:



(3.22)

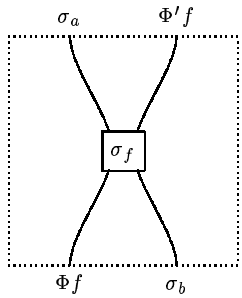
Define $\psi_{\underline{\mathrm{A}}} : \Phi(1_{\underline{\mathrm{A}}}) \Longrightarrow 1_{\Phi(\underline{\mathrm{A}})}$ to be the identity natural transformation.

It is a routine to verify that (Φ, φ, ψ) is a (contravariant) pseudofunctor which is a biequivalence. We omit these straightforward computations. □

Remark 3.3.15 In the previous proof we have not made any specific use of the fact that we work in the whole 2-quasicategory CAT^r. In fact, suppose that K is any sub-2-quasicategory of CAT which is full on 2-cells, i.e. that $\mathsf{K}(a, b)$ has all natural transformations as morphisms. If we define K^r and K^l in the obvious way, then K^r and $(\mathsf{K}^l)^{op}$ are biequivalent. We will use this fact later in Chapter 8. □

Definition 3.3.16 Suppose (Φ, φ, ψ) and (Φ', φ', ψ') are lax functors from a 2-category A to a 2-category B. A *lax natural transformation* $\sigma : (\Phi, \varphi, \psi) \Longrightarrow (\Phi', \varphi', \psi')$ consists of a 1-cell $\sigma_a : \Phi a \longrightarrow \Phi' a$ for each 0-cell a of A, and a comparison 2-cell σ_f for each 1-cell

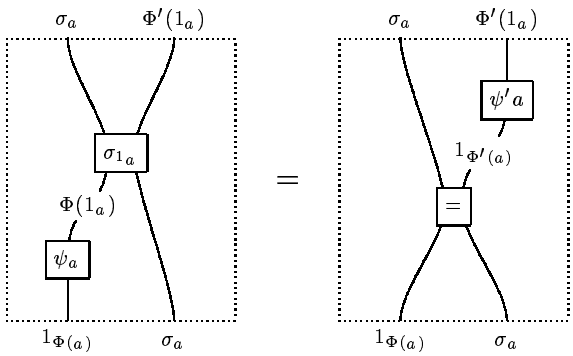
$f : a \longrightarrow b$:



(3.23)

subject to the following conditions:

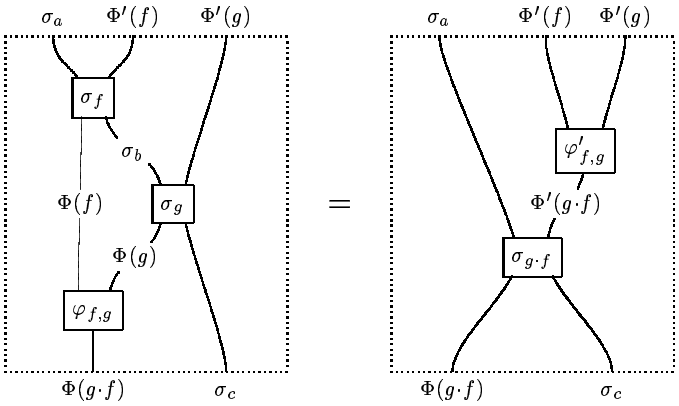
1. The following equality



(3.24)

holds for any 0-cell a.

2. For any pair of 1-cells $f : a \longrightarrow b$, $g : b \longrightarrow c$, the following equality



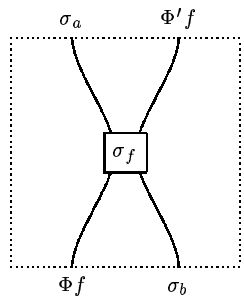
(3.25)

holds.

In case all the comparison 2-cells σ's are isomorphisms, σ is called a *pseudonatural transformation*.

Let us spell out the special case of the previous definition for 2-functors:

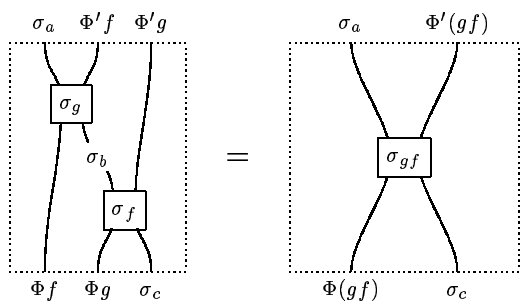
Definition 3.3.17 Suppose Φ and Φ' are 2-functors from a 2-category A to a 2-category B. A *lax natural transformation* $\sigma : \Phi \Longrightarrow \Phi'$ consists of a 1-cell $\sigma_a : \Phi a \longrightarrow \Phi' a$ for each 0-cell a of A, and a comparison 2-cell σ_f for each 1-cell $f : a \longrightarrow b$:



(3.26)

subject to the following conditions:

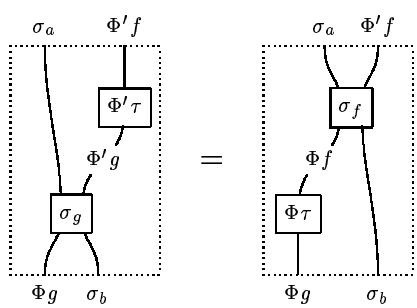
1. σ_{1_a} is the identity 2-cell on σ_a.
2. For any pair of 1-cells $f : a \longrightarrow b$, $g : b \longrightarrow c$, the following equality



(3.27)

holds.

3. For any pair of 1-cells $f, g : a \longrightarrow b$ and any 2-cell $\tau : f \Longrightarrow g$ the following equality



(3.28)

holds.

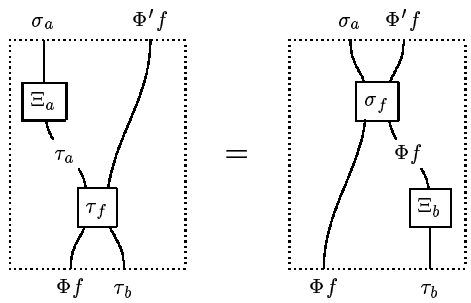
In case all comparison 2-cells σ's are isomorphisms, σ is called a *pseudonatural transformation.* In case all comparison 2-cells σ's are identities, σ is called a *2-natural transformation.*

Remark 3.3.18 It is typical that each 2-categorical notion has basically three variants: e.g. for a functor one has a *2-functor* (comparison 2-cells are identities), a *pseudofunctor* (comparison 2-cells are isomorphisms) and a *lax functor* (comparison 2-cells are arbitrary). Moreover, each lax notion has its dual, the so-called *op-lax* notion — reverse the comparison 2-cells.

More relaxed notions could have been introduced, e.g. that of a lax natural transformation between lax functors between bicategories. These notions are indeed studied in the literature, see e.g. [Gr74]. We, however, do not need such generality. □

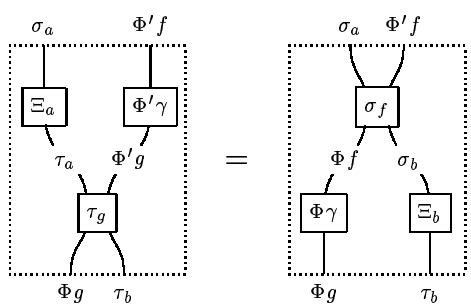
We need the notion of a morphism of lax natural transformations:

Definition 3.3.19 Suppose that $\sigma, \tau : \Phi \Longrightarrow \Phi'$ are lax natural transformations between 2-functors $\Phi, \Phi' : \mathsf{A} \longrightarrow \mathsf{A}$. A *modification* $\Xi : \alpha \rightsquigarrow \beta$ is a collection of 2-cells $\Xi_a : \sigma_a \Longrightarrow \tau_a$ indexed by 0-cells a of A such that the equality



(3.29)

holds for any 1-cell $f : a \longrightarrow b$ in A, and for any 2-cell $\gamma : f \Longrightarrow g$ in A the following equality



(3.30)

holds.

Remark 3.3.20 It is obvious that lax natural transformations can be composed and that for fixed 2-categories A, B we have the 2-quasicategory having all 2-functors from A to B as 0-cells, all lax natural transformations as 1-cells and all modifications as 2-cells. Also, we can define the 2-quasicategory with all 2-functors as from A to B 0-cells, all pseudonatural transformations as 1-cells and all modifications as 2-cells. □

In category theory, the notion of a (co)limit plays an important rôle. A (co)limit of a diagram $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{K}}$ in a category $\underline{\mathrm{K}}$ is a natural transformation between the constant functor and D satisfying the well-known (co)universal property. Obviously there are now several 2-categorical counterparts depending on the chosen concept of a functor and a natural transformation. Those concepts differ in the sense that different types

of (co)limits may be different for the same initial data (see [Bo94], Volume 1, Example 7.6.3). We need just a few of these concepts, namely, that of a *lax limit of a 2-functor between 2-categories*, that of a *bilimit of a pseudofunctor between 2-categories* and that of an *indexed limit*. The last notion is a proper notion of a limit in enriched category theory — a 2-category is precisely a category enriched over Cat. We will not need the full theory of enriched category theory.

Definition 3.3.21 Suppose that $D : \mathsf{D} \longrightarrow \mathsf{A}$ is a 2-functor between 2-categories. Let a be a 0-cell in A.

A *constant 2-functor* $const_a : \mathsf{D} \longrightarrow \mathsf{A}$ *with value* a sends each 0-cell to a, each 1-cell to 1_a and each 2-cell to i_{1_a}.

A *lax cone on* D *with vertex* a is a lax natural transformation $\lambda : const_a \Longrightarrow D$.

Denote by $\mathrm{LaxCone}(a, D)$ the category of lax cones on D with vertex a as objects and modifications as morphisms. Any lax cone $\lambda : const_a \Longrightarrow D$ induces a functor $\hat{\lambda}(b) : \mathsf{A}(b, a) \longrightarrow \mathrm{LaxCone}(b, D)$ by composition with λ.

We say that a lax cone $\lambda : const_a \Longrightarrow D$ is a *lax limit cone for* D, if the functor $\hat{\lambda}(b)$ is an isomorphism of categories for any 0-cell b. By the usual abuse of language we will sometimes call the vertex of a lax limit cone a lax limit.

As usual, lax limits (when they exist) are determined up to an isomorphism. This will not, however, be always precisely what we want. Sometimes we require a limit determined up to an equivalence. This is provided by the following notion of a bilimit.

Definition 3.3.22 Suppose that $D : \mathsf{D} \longrightarrow \mathsf{A}$ is a pseudofunctor between 2-categories. Let a be a 0-cell in A.

A *pseudocone on* D *with vertex* a is a pseudonatural transformation $\lambda : const_a \Longrightarrow D$.

Denote by $\mathrm{PseudoCone}(a, D)$ the category of pseudocones on D with vertex a as objects and modifications as morphisms. Any pseudocone $\lambda : const_a \Longrightarrow D$ induces a functor $\hat{\lambda}(b) : \mathsf{A}(b, a) \longrightarrow \mathrm{PseudoCone}(b, D)$ by composition with λ.

We say that a pseudocone $\lambda : const_a \Longrightarrow D$ is a *bilimit cone for* D, if the functor $\hat{\lambda}(b)$ is an equivalence of categories for any 0-cell b. By the usual abuse of language we will sometimes call the vertex of a bilimit cone a bilimit.

If a bilimit exists, then it is determined up to an equivalence, where two 0-cells a, b in a 2-category are said to be *equivalent*, provided there are 1-cells $f : a \longrightarrow b$, $g : b \longrightarrow a$ and isomorphism 2-cells $\alpha : i_a \Longrightarrow g \cdot f$, $\beta : f \cdot g \Longrightarrow i_b$.

Example 3.3.23 Suppose that $D : \mathsf{D} \longrightarrow \mathsf{CAT}$ is a small diagram. Regard the functor D as a pseudofunctor. Then a bilimit of D exists. A pseudococone consisting of a vertex $\underline{\mathrm{K}}$, functors $P_i : \underline{\mathrm{K}} \longrightarrow D(i)$ and comparison isomorphism natural transformations $\pi_u : D(u) \cdot P_i \Longrightarrow P_j$ which is a bilimit psudococone, can be described in the following way:

> $\underline{\mathrm{K}}$-objects are *compatible threads*. A compatible thread is a collection $\langle x_i, a_w \rangle$, where x_i is an object of $D(i)$ and $a_w : D(w)(x_{i_1}) \longrightarrow x_{i_2}$ is an isomorphism in $D(i_2)$ for a $\underline{\mathrm{D}}$-morphism $w : i_1 \longrightarrow i_2$ such that the following coherence conditions are satisfied:

$$a_{1_j} \quad = \quad 1_{x_j} \quad \text{for any } j \tag{3.31}$$

$$a_{w_2 \cdot w_1} \quad = \quad a_{w_2} \cdot D(w_2)(a_{w_1}) \quad \text{for any pair } w_1 : j_1 \longrightarrow j_2, w_2 : j_2 \longrightarrow j_3 \tag{3.32}$$

$\underline{\mathrm{K}}$-morphisms from $\langle x_i, a_w \rangle$ to $\langle y_i, b_w \rangle$ are collections $\langle f_i \rangle$, where each f_i is a $D(i)$-morphism from x_i to y_i and the following square commutes for any $\underline{\mathrm{D}}$-morphism $w : i \longrightarrow j$:

$$\begin{tikzcd} D(w)(x_i) \arrow[r, "D(w)(f_i)"] \arrow[d, "a_w"'] & D(w)(y_i) \arrow[d, "b_w"] \\ x_j \arrow[r, "f_j"'] & y_j \end{tikzcd} \tag{3.33}$$

Identities and composition in $\underline{\mathrm{K}}$ are defined in an obvious way.

Each functor P_i sends $\langle x_i, a_w \rangle$ to x_i and $\langle f_i \rangle$ to f_i.

The isomorphism comparison natural transformation $\pi_u : D(u) \cdot P_i \Longrightarrow P_j$ for $u : i \longrightarrow j$ has as its value at $\langle x_i, a_w \rangle$ the isomorphism $\pi_u(\langle x_i, a_w \rangle) = a_u$.

□

Definition 3.3.24 Suppose D is a small 2-category, A is a 2-category and $\Phi : \mathsf{D} \longrightarrow \mathsf{A}$, $W : \mathsf{D} \longrightarrow \mathsf{Cat}$ are 2-functors. Suppose a is a 0-cell in A.

A 2-natural transformation $\lambda : W \Longrightarrow \mathsf{Cat}(a, \Phi_-)$ is called a (W, A)-*cylinder over* Φ *with vertex* a.

Denote by $\mathrm{Cyl}(W, \mathsf{Cat}(a, \Phi_-))$ the category of (W, A)-cylinders over Φ with vertex a as objects and modifications as morphisms.

Any (W, A)-cylinder over Φ with vertex a $\lambda : W \Longrightarrow \mathsf{Cat}(a, \Phi_-)$ induces by composition a functor $\hat{\lambda}(b) : \mathsf{A}(b, a) \longrightarrow \mathrm{Cyl}(W, \mathsf{Cat}(a, \Phi_-))$.

We say that $\lambda : W \Longrightarrow \mathsf{Cat}(a, \Phi_-)$ is a *limit cone for* Φ *indexed by* W, if the functor $\hat{\lambda}(b)$ is an isomorphism of categories for any 0-cell b.

In this setting, the 2-functor W is called an *indexing type* and the vertex of a limit cylinder is called an *indexed limit of* Φ *indexed by* W.

If the indexing W is a 2-functor with the one-morphism category as its constant value, then we speak of a *conical indexed limit of* Φ.

We say that $\lambda : W \Longrightarrow \mathsf{Cat}(a, \Phi_-)$ is a *bilimit cone for* Φ *indexed by* W, if the functor $\hat{\lambda}(b)$ is an equivalence of categories for any 0-cell b.

Remark 3.3.25 The dual notions of a lax limit and an (indexed) bilimit are a *lax colimit* and an *(indexed) bicolimit*. The dual of an indexed limit (i.e. an *indexed colimit*) deserves an explicit formulation: a *colimit of* $\Phi : \mathsf{D} \longrightarrow \mathsf{A}$ *indexed by* $W : \mathsf{D}^{op} \longrightarrow \mathsf{Cat}$ is a 2-natural transformation (called a *colimit cocylinder*) $\lambda : W \Longrightarrow \mathsf{Cat}(\Phi_-, a)$ with the following universal property:

for any 0-cell b the functor $\hat{\lambda}(b) : \mathsf{A}(a,b) \longrightarrow \mathrm{CoCyl}(W, \mathsf{Cat}(\Phi_-, a))$ induced by composition is an isomorphism of categories.

(Where the definitions of cocylinders and of the category of cocylindres are straightforward.) Note that the indexing for colimits is now a contravariant functor!

Analogously one defines the notion of a *bicolimit of* Φ *indexed by* W. □

Example 3.3.26 The following is a useful instance of an indexed limit.

Recall the unit 2-category $\mathcal{I}$ of example 3.3.4. It is clear that a 2-functor Φ from $\mathcal{I}$ picks up a 0-cell in the codomain of Φ. Let A be a 2-category, denote by $\delta_a : \mathcal{I} \longrightarrow \mathsf{A}$ the 2-functor with a value a and by $\delta_{\underline{\mathrm{K}}} : \mathcal{I} \longrightarrow \mathsf{Cat}$ the 2-functor having a small category $\underline{\mathrm{K}}$ as its value. A limit of δ_a indexed by $\delta_{\underline{\mathrm{K}}}$ (if it exists) is called a *cotensor of* $\underline{\mathrm{K}}$ *and* a and the vertex of a limit cylinder is denoted by $[\underline{\mathrm{K}}, a]$.

It is worthwhile to realize that if $\mathsf{A} = \mathsf{CAT}$ then an indexed colimit of $\delta_{\underline{\mathrm{B}}}$ indexed by $\delta_{\underline{\mathrm{A}}}$ is the category $[\underline{\mathrm{A}}, \underline{\mathrm{B}}]$ having functors from $\underline{\mathrm{A}}$ to $\underline{\mathrm{B}}$ as objects and natural transformations between those functors as morphisms.

Thus, cotensors generalize powers. □

3.4 Ends, Kan Extensions

Definition 3.4.1 Let $F, G : \underline{\mathrm{A}}^{op} \times \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ be functors. A *dinatural transformation* $\tau : F \longrightarrow G$ from F to G is a collection (τ_a) of $\underline{\mathrm{B}}$-morphisms $\tau_a : F(a,a) \longrightarrow G(a,a)$ indexed by $\underline{\mathrm{A}}$-objects, such that for any $\underline{\mathrm{A}}$-morphism $f : a \longrightarrow a'$ the following diagram commutes:

$$\begin{tikzcd}
 & F(a,a) \arrow[rr, "\tau_a"] & & G(a,a) \arrow[dr, "G(1_a,f)"] & \\
F(a',a) \arrow[ur, "F(f,1_a)"] \arrow[dr, "F(1_{a'},f)"'] & & & & G(a,a') \\
 & F(a',a') \arrow[rr, "\tau_{a'}"'] & & G(a',a') \arrow[ur, "G(f,1_{a'})"'] &
\end{tikzcd} \tag{3.34}$$

A dinatural transformation from a constant functor $\underline{\mathrm{A}}^{op} \times \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ with value b to a functor F is called a *wedge* over F with vertex b. A universal wedge is called an *end* of F and the vertex of such a wedge is denoted by $\int_a F(a,a)$. By abuse of notation $\int_a F(a,a)$ will sometimes stand for the whole end wedge $\omega_a : \int_a F(a,a) \longrightarrow F(a,a)$.

Remark 3.4.2 End is a special kind of limit, therefore all facts about limits apply to ends. The dual notions are *cowedge* and *coend* (denoted by $\int^a F(a,a)$). Unless explicitly stated otherwise, when talking about (co)ends, the category $\underline{\mathrm{A}}$ will be considered small. □

Given two functors $F : \underline{A} \longrightarrow \underline{B}$, $G : \underline{A} \longrightarrow \underline{B}$ we denote by $\mathsf{Nat}(F, G)$ the *class of all natural transformations from F to G.*

Lemma 3.4.3 (End Formula for Nat) *Suppose $F : \underline{A} \longrightarrow \underline{B}$ and $G : \underline{A} \longrightarrow \underline{B}$ are functors. Then end of the functor $\underline{B}(F_-, G_-) : \underline{A}^{op} \times \underline{A} \longrightarrow \underline{\mathrm{Set}}$ exists and the sets $\mathsf{Nat}(F, G)$ and $\int_a \underline{B}(Fa, Ga)$ are isomorphic.*

Proof. See e.g. [ML71], Section IX.5. □

Definition 3.4.4 A *left Kan extension* of a functor $F : \underline{A} \longrightarrow \underline{C}$ along a functor $G : \underline{A} \longrightarrow \underline{B}$ is a functor $Lan_G(F) : \underline{B} \longrightarrow \underline{C}$ together with a natural transformation (called a *unit of the left Kan extension*) $\eta : F \Longrightarrow (Lan_G(F)) \cdot G$, such that the data $(Lan_G(F), \eta)$ are universal in the following sense: given any other pair (H, τ), where $H : \underline{B} \longrightarrow \underline{C}$ is a functor and $\tau : F \Longrightarrow H \cdot G$ is a natural transformation, then there is a unique natural transformation $\mu : Lan_G(F) \Longrightarrow H$, such that the diagram

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr, "\tau"'] & (Lan_G(F)) \cdot G \arrow[d, "\mu G"] \\ & H \cdot G \end{tikzcd} \tag{3.35}$$

commutes.

Remark 3.4.5 The dual notion are: *right Kan extension*, *counit of the right Kan extension*. Unless explicitly otherwise stated, when talking about Kan extensions, the category $\underline{A}$ will be considered small. □

Coends and left Kan extensions are tied together by the following result.

Lemma 3.4.6 *If all copowers $\coprod_{\underline{B}(Ga,b)} Fa'$ exist and if a coend $Lb = \int^a \coprod_{\underline{B}(Ga,b)} Fa$ exists for all b, then $b \mapsto Lb$ is an object function of $Lan_G(F)$.*

Proof. See [ML71], Section X.4, Theorem 1. □

3.5 Accessible Categories

This section is devoted to fixing the notation for accessible categories. For motivation, many examples and other results we refer to the books [MP89] and [AR94].

In this section λ is a regular cardinal — i.e. an infinite cardinal of cofinality λ (see e.g. [Je78]).

Definition 3.5.1 A diagram $D : \underline{D} \longrightarrow \underline{K}$ is called:

1. *λ-directed*, if the category $\underline{D}$ is a λ-directed poset (that is, any subset of less than λ elements has an upper bound),

2. λ-*filtered*, if the category $\underline{\mathrm{D}}$ is λ-filtered (that is, any subcategory of less than λ morphisms has a compatible cocone).

The following notion is very useful:

Definition 3.5.2 A functor $F : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ is *cofinal*, if for any $\underline{\mathrm{B}}$-object b the following two conditions are satisfied:

1. There exists a morpism $f : b \longrightarrow Fa$ for some $\underline{\mathrm{A}}$-object a.
2. For any pair $f_1 : b \longrightarrow Fa_1$, $f_2 : b \longrightarrow Fa_2$ there exist $\underline{\mathrm{A}}$-morphisms $g_1 : a_1 \longrightarrow a$ and $g_2 : a_2 \longrightarrow a$ such that the equality $Fg_1 \cdot f_1 = Fg_2 \cdot f_2$ holds.

Given a diagram $D : \underline{\mathrm{B}} \longrightarrow \underline{\mathrm{K}}$ and a cofinal functor $F : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$, then there is a bijective correspondence between compatible cocones for D and $D \cdot H$ — see [AR94], 0.11 for details.

Remark 3.5.3 It is proved in [AR94], Theorem 1.5 and Remark 1.21 that:

1. A diagram is λ-filtered iff it has a cofinal λ-directed subdiagram.
2. A category has λ-filtered colimits iff it has λ-directed colimits.
3. A functor preserves λ-filtered colimits iff it preserves λ-directed colimits.

□

Definition 3.5.4 An object x of a category $\underline{\mathrm{K}}$ is called λ-*presentable*, if the functor $\underline{\mathrm{K}}(x, \_) : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{Set}}$ preserves λ-filtered colimits.

Definition 3.5.5 A category $\underline{\mathrm{K}}$ is λ-*accessible* iff

(a) it has λ-filtered colimits,

(b) the full subcategory of $\underline{\mathrm{K}}$ consisting of all λ-presentable objects is equivalent to a small category; a choice of such a small category is denoted by $\underline{\mathrm{K}}_\lambda$,

(c) for any $\underline{\mathrm{K}}$-object x the cocone $\underline{\mathrm{K}}_\lambda/x$ is a λ-filtered colimit cocone. (We will call the corresponding diagram with the scheme $\underline{\mathrm{K}}_\lambda/x$ a *canonical diagram* for x and denote it by C_x.)

Remark 3.5.6 An important class of λ-accessible categories are *locally λ-presentable categories*. A category is locally λ-presentable iff it is λ-accessible and (co)complete — see [AR94], Corollary 2.47. □

Definition 3.5.7 Let $\underline{\mathrm{K}}$, $\underline{\mathrm{L}}$ be λ-accessible categories. A functor $F : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{L}}$ is called *λ-accessible*, if it preserves λ-filtered colimits, i.e. if $F \operatorname{colim}_{\underline{\mathrm{K}}} D \cong \operatorname{colim}_{\underline{\mathrm{L}}} FD$ for any small λ-directed diagram $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{K}}$.

It is proved in [AR94], Theorem 2.26 that λ-accessible categories are, up to an equivalence of categories, precisely the free cocompletions of small categories w.r.t. λ-filtered colimits.

Definition 3.5.8 Let $\underline{\mathrm{X}}$ be any category, let $F : \underline{\mathrm{X}}^{op} \longrightarrow \underline{\mathrm{Set}}$ be a functor. The comma category $\underline{\mathbf{1}}/F$ is called the *category of elements of* F and is denoted by $\mathsf{Elts}(F)$. The obvious *projection functor* from $\mathsf{Elts}(F)$ to $\underline{\mathrm{X}}^{op}$ sending $f : \langle x, a\rangle \longrightarrow \langle y, b\rangle$ to $f : a \longrightarrow b$ is denoted by ∂_F.

Remark 3.5.9 It is convenient to regard $\mathsf{Elts}(F)$-objects as pairs $\langle x, a\rangle$, where a is an $\underline{\mathrm{X}}^{op}$-object and $x : \underline{\mathbf{1}} \longrightarrow Fa$ is a mapping (identified with an *element* $x \in Fa$). The set of $\mathsf{Elts}(F)$-morphisms from $\langle x, a\rangle$ to $\langle y, b\rangle$ is a set of those $\underline{\mathrm{X}}^{op}$-morphisms $f : a \longrightarrow b$ such that $y = Ff \cdot x$. □

Definition 3.5.10 A functor $F : \underline{\mathrm{A}}^{op} \longrightarrow \underline{\mathrm{Set}}$, where $\underline{\mathrm{A}}$ is a small category, is called *λ-flat*, if $\mathsf{Elts}(F)$ is a λ-cofiltered category, i.e. if $\mathsf{Elts}(F)^{op}$ is λ-filtered. We denote by $\mathsf{Flat}_\lambda(\underline{\mathrm{A}})$ the *category of all λ-flat functors $F : \underline{\mathrm{A}}^{op} \longrightarrow \underline{\mathrm{Set}}$ and all natural transformations*, $I_{\underline{\mathrm{A}}} : \underline{\mathrm{A}} \longrightarrow \mathsf{Flat}_\lambda(\underline{\mathrm{A}})$ denotes the full embedding of $\underline{\mathrm{A}}$ in $\mathsf{Flat}_\lambda(\underline{\mathrm{A}})$ given on objects by $a \mapsto \underline{\mathrm{A}}(\_, a)$.

Definition 3.5.11 A *sketch* is a triple $\mathbb{S} = (\underline{\mathrm{S}}, P, Q)$, where $\underline{\mathrm{S}}$ is a small category, P is a set of cones in $\underline{\mathrm{S}}$ and Q is a set of cocones in $\underline{\mathrm{S}}$. Given two sketches $\mathbb{S}_1 = (\underline{\mathrm{S}}_1, P_1, Q_1)$ and $\mathbb{S}_2 = (\underline{\mathrm{S}}_2, P_2, Q_2)$, a *sketch morphism* is a functor $F : \underline{\mathrm{S}}_1 \longrightarrow \underline{\mathrm{S}}_2$ which maps each cone of P_1 to a cone in P_2 and each cocone in Q_1 to a cocone in Q_2.

Sketches, their morphisms and natural transformations obviously form a 2-category, denote it by Sketch.

Definition 3.5.12 Given a sketch $\mathbb{S} = (\underline{\mathrm{S}}, P, Q)$, then its $\underline{\mathrm{Set}}$-*model* is a functor $F : \underline{\mathrm{S}} \longrightarrow \underline{\mathrm{Set}}$ which sends each cone in P to a limit and each cocone in Q to a colimit in $\underline{\mathrm{Set}}$.

Obviously, models of a sketch form a category, if we take natural transformations as morphisms between $\underline{\mathrm{Set}}$-models. Denote this category by $\mathsf{MOD}(\mathbb{S})$.

An important feature of accessible categories is that they can be given by essentially small data. A λ-accessible category can either be described by its (essentially) small category of λ-presentable objects or as a category of models of a small sketch. More precisely:

Theorem 3.5.13 *The following are equivalent for a category* $\underline{\mathrm{K}}$*:*

1. $\underline{\mathrm{K}}$ *is λ-accessible for some λ.*

2. $\underline{\mathrm{K}}$ *is equivalent to the category of the form* $\mathsf{Flat}_\lambda(\underline{\mathrm{A}})$ *for some* λ *and some small category* $\underline{\mathrm{A}}$.

3. $\underline{\mathrm{K}}$ *is equivalent to the category of* $\underline{\mathrm{Set}}$*-models of a sketch.*

3.6 Notation for the Finitary Case

The following text is essentially on $\aleph_0$-accessible categories. Almost everything, however, can be generalized to λ-accessible categories. We have restricted ourselves to the cardinal $\aleph_0$ to keep an analogy with Domain Theory. The generalization to λ-accessible categories is always straightforward, and we believe that the results in case $\lambda = \aleph_0$ are sufficiently typical.

Remark 3.6.1 We adopt the following shortened expressions:

- We say *filtered* and *flat* instead of $\aleph_0$-filtered and $\aleph_0$-flat. Instead of $\mathsf{Flat}_{\aleph_0}(\underline{\mathrm{A}})$ we write $\underline{\mathrm{A}}^*$.
- We say *finitely presentable* instead of $\aleph_0$-presentable, an *accessible category* instead of an $\aleph_0$-accessible category. Instead of $\underline{\mathrm{K}}_{\aleph_0}$ we write $\underline{\mathrm{K}}_{fin}$ for the essentially small category of finitely presentable objects in $\underline{\mathrm{K}}$.
- We say a *finitary functor* instead of an $\aleph_0$-accessible functor.

□

We will explicitly emphasise the situations where generalizations to uncountable regular cardinals will not be possible.

Chapter 4

Categorical Generalizations of Domains

The idea of using categories as domains is advocated in the work of Daniel J. Lehmann:

> ...when considering non-deterministic programs the notions of complete partial order, least fixpoints of continuous functions and domain equations have to be generalized. It is not sufficient any more, when considering the process of successive approximations converging to the final value, to look at the sequence of objects but it is also necessary to consider the way in which each approximation is related to the preceding one, thus replacing a partial order by a category and a least upper bound by a colimit.
>
> [Leh76], p. 4

Lehmann also attributes the idea of having categories as domains to H. Egli. Domains in [Leh76] are categories having colimits of countable chains (called ω-categories) and domain morphisms are functors preserving these colimits (ω-functors). Lehmann in [Leh76] develops a fixed point calculus for this notion of a domain.

We choose a category having colimits of small filtered diagrams as a basic concept. The reasons are:

- Filtered diagrams can be thought of as of a piece of information, which is "consistent on finite subsets".
- Filtered diagrams appear quite naturally in category theory and one can use a link to the theory of accessible categories.

Besides of generalizing classical notions, we believe that viewing domains as categories rather than partially ordered sets helps to clarify conceptually what is going on in Domain Theory.

This chapter presents the basic generalizations of the classical definitions of Domain Theory. Thus we provide here generalizations of

- (pointed) DCPOs — Remark 4.4.12,
- pointed algebraic DCPOs — Definition 4.1.1,

- Scott domains — Definition 4.2.1,

and generalizations of morphisms between domains

- continuous maps — Definition 4.1.4,
- embedding-projection pairs — Definition 4.1.6.

We also give generalizations of the concepts of finitary closure and projection on a domain — Definition 4.5.3.

4.1 A Generalization of Pointed Algebraic DCPOs

Definition 4.1.1 A category $\underline{\mathrm{K}}$ which is finitely accessible and has an initial object is called a *generalized domain*.

Remark 4.1.2 The definition of a generalized domain is a translation of a notion of pointed algebraic DCPO from partially ordered sets to categories:

1. The existence of a least element has been generalized to the existence of an initial object.
2. Compact elements generalize to finitely presentable objects.
3. The existence of directed sups and the possibility to express each element by the sup of compact elements below it has been generalized to the requirement of finite accessibility.

In fact, the notion of a finitely accessible category is a translation of a notion of an algebraic DCPO to category theory. We need the existence of an initial object for technical reasons. □

Remark 4.1.3 Let $\underline{\mathrm{K}}$ be a generalized domain. Then it is easy to see that the small category $\underline{\mathrm{K}}_{fin}$ of representatives of finitely presentable objects in $\underline{\mathrm{K}}$ has an initial object. Conversely, if $\underline{\mathrm{A}}$ is a small category with initial object, then its free cocompletion w.r.t. small filtered colimits $\underline{\mathrm{A}}^*$ is a generalized domain (see e.g. [AR94], Theorem 2.26). □

As usual in Domain Theory, we introduce two notions of a morphism between generalized domains. First, an obvious generalization of continuous maps.

Definition 4.1.4 Let $\underline{\mathrm{K}}$ and $\underline{\mathrm{L}}$ be generalized domains. A functor $F : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{L}}$ is called *finitary*, if it preserves filtered colimits.

Lemma 4.1.5 *Identity functors are finitary. The composition of finitary functors is a finitary functor.*

The proof is trivial. The above lemma allows us to define the *2-quasicategory* GDOM of all generalized domains as 0-cells, all finitary functors as 1-cells and all natural transformations between finitary functors as 2-cells.

Definition 4.1.6 Let $\underline{\mathrm{K}}$ and $\underline{\mathrm{L}}$ be generalized domains. A pair of finitary functors $L : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{L}}$ and $R : \underline{\mathrm{L}} \longrightarrow \underline{\mathrm{K}}$ such that $L \dashv R$ is called a *finitary adjunction* from $\underline{\mathrm{K}}$ to $\underline{\mathrm{L}}$.

This may seem too general: a natural generalization of the notion of an embedding-projection pair is a notion of a finitary coreflection of a full subcategory.

Definition 4.1.7 A finitary adjunction $L \dashv R$ is called an *(e,p)-adjunction* if the unit of the adjunction is an isomorphism.

Remark 4.1.8 Let us remark that our concept of an (e,p)-adjunction is *broader* than the notion of an (e,p)-adjunction defined in [Ad97]. Adámek in [Ad97] requires the unit of an (e,p)-adjunction to be the identity natural transformation. Our definition, however, simplifies the reasoning about (e,p)-adjunctions.

In the context of categories the general notion of adjunction is, however, more natural, easy to work with and there is a symmetry which the notion of an (e,p)-adjunction lacks. We will show that much of the results known for embedding-projection pairs in Domain Theory hold for finitary adjunctions too. □

Lemma 4.1.9 *An identity functor has a finitary right adjoint. The composition of functors with finitary right adjoints is a functor with a finitary right adjoint.*

The proof follows from the well-known fact that left adjoints can be composed. The above lemma allows us to define a *2-quasicategory* GDOM^l of all generalized domains as 0-cells, all functors with finitary right adjoints as 1-cells and all natural transformations between them as 2-cells.

4.2 A Generalization of Scott Domains

In his paper [Ad97] Jiří Adámek introduced a categorical generalization of a Scott domain, namely the *Scott complete category*, and he proved that Scott complete categories possess properties which enable them to serve as a basis for computer language semantics.

Definition 4.2.1 ([Ad97]) A generalized domain $\underline{\mathrm{K}}$ which is boundedly cocomplete, i.e. each diagram with a compatible cocone has a colimit, is called a *Scott complete category* (SC category for short).

Remark 4.2.2 It should be mentioned that the definition of SC category is just a translation of the definition of a Scott domain to category theory:

1. The existence of a least element has been generalized to the existence of an initial object.

2. Compact elements generalize to finitely presentable objects.

3. The existence of directed sups and the possibility to express each element by the sup of compact elements below it has been generalized to the requirement of finite accessibility.

4. Subsets bounded from above have been generalized to *consistent diagrams*, i.e. diagrams having a compatible cocone. Instead of "diagram consistent in the category $\underline{K}$" we say "$\underline{K}$-consistent diagram".

□

Remark 4.2.3 Let $\underline{K}$ be an SC category. Then it is easy to see that the small category $\underline{K}_{fin}$ of representatives of finitely presentable objects in $\underline{K}$ has:

1. an initial object (inherited from $\underline{K}$),

2. colimits of finite consistent diagrams.

Conversely, the free cocompletion w.r.t. filtered colimits of any small category $\underline{A}$ with initial object and colimits of finite consistent diagrams is an SC category (Example 1.3 in [Ad97]). This generalizes the well-known fact that any Scott domain is the ideal completion of its complete upper semilattice of compact elements. □

Examples 4.2.4 There is a wealth of examples of SC categories:

1. Any Scott domain is an SC category.

2. Any locally finitely presentable category (see Remark 3.5.6) is an SC category. In particular, any coherent Grothendieck topos (see [Bo94], Vol. 3, Chapter 6.5) is an SC category.

□

Lemma 4.2.5 *Let $\underline{K}$ be a generalized domain. The following are equivalent:*

1. *$\underline{K}$ is boundedly cocomplete.*

2. *$\underline{K}$ is boundedly complete (each diagram with a compatible cone has a limit).*

Proof. See [Ad97], Theorem 1. □

In fact a Scott complete category is "not very far" from being locally finitely presentable:

Lemma 4.2.6 *Suppose that $\underline{K}$ is a Scott complete category. Then the following are equivalent:*

1. *$\underline{K}$ has a terminal object.*

2. $\underline{\mathrm{K}}$ *is locally finitely presentable.*

Proof. Trivial. □

Remark 4.2.7 Let us say explicitly that due to being boundedly cocomplete and due to the existence of initial object $\perp$, every SC category $\underline{\mathrm{K}}$ has small copowers of any object. Thus, for any $\underline{\mathrm{K}}$-object a we always have a functor $a \otimes \_ : \underline{\mathrm{Set}} \longrightarrow \underline{\mathrm{K}}$ defined on objects as

$$a \otimes x = \begin{cases} \coprod_x a, & \text{in case } x \neq \emptyset \\ \perp, & \text{otherwise} \end{cases}$$

and defined on $\underline{\mathrm{Set}}$-morphisms in obvious way, such that we have an adjunction $a \otimes \_ \dashv \underline{\mathrm{K}}(a, \_)$.

Thus SC categories are never small unless they are Scott domains. In fact, let a be an object of an SC category $\underline{\mathrm{K}}$ and let κ be an arbitrary cardinal number. Then the diagram D_κ repeating a κ-times has a compatible cocone, hence a colimit, say, c. If there are two distinct parallel morphisms in $\underline{\mathrm{K}}$ starting in a and terminating, say, in b, then there are 2^κ distinct cocones on D_κ, hence $\underline{\mathrm{K}}$ contains at least 2^κ distinct morphisms from c into b. Thus $\underline{\mathrm{K}}$ cannot be small. □

The following notion is inspired by topos theory (see e.g. [Joh77], Definition 7.11).

Definition 4.2.8 Let $\underline{\mathrm{K}}$ be a generalized domain. Any finitary adjunction $L \dashv R$ from $\underline{\mathrm{Set}}$ to $\underline{\mathrm{K}}$ is called a *point of* $\underline{\mathrm{K}}$. $\underline{\mathrm{K}}$ is said to have *enough points*, if there is a small set $\{L_i \dashv R_i \mid i \in I\}$ of points of $\underline{\mathrm{K}}$ such that the collection $\{R_i \mid i \in I\}$ jointly reflects isomorphisms.

Example 4.2.9 Any SC category $\underline{\mathrm{K}}$ has enough points: By Remark 4.2.7 for any finitely presentable object a the functor $R_a = \underline{\mathrm{K}}(a, \_)$ has a left adjoint $L_a = a \otimes \_$. Then $L_a \dashv R_a$ is a point of $\underline{\mathrm{K}}$ and the collection $\{R_a \mid a \text{ is a } \underline{\mathrm{K}}_{fin}\text{-object}\}$ jointly reflects isomorphisms. □

Lemma 4.2.10 *Let* $\underline{\mathrm{K}}$ *be a generalized domain. If* $L \dashv R$ *is a point of* $\underline{\mathrm{K}}$*, then* R *is isomorphic to* $\underline{\mathrm{K}}(a, \_)$ *for some finitely presentable object* a *having copowers.*

Proof. Any right adjoint R to $\underline{\mathrm{Set}}$ is (isomorphic to) a representable functor. The representing object must be finitely presentable, since R is supposed to preserve filtered colimits. □

Lemma 4.2.11 *Let* $\underline{\mathrm{K}}$ *have enough points. Suppose* $D : \underline{\mathrm{F}} \times \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{K}}$ *is a diagram where* $\underline{\mathrm{F}}$ *is finite and* $\underline{\mathrm{D}}$ *is filtered. Suppose that* $\lim_f D(f, d)$ *exists for any* d*. Then if* $\lim_f \operatorname{colim}_d D(f, d)$ *exists, it is isomorphic to* $\operatorname{colim}_d \lim_f D(f, d)$.

Proof. We can assume that the collection $\{\underline{\mathrm{K}}(a, \_) \mid a \text{ is a } \underline{\mathrm{K}}_{fin}\text{-object}\}$ jointly reflects isomorphisms. Since any functor $\underline{\mathrm{K}}(a, \_)$ is flat we have

$$\begin{aligned} \underline{\mathrm{K}}(a, \lim_f \operatorname{colim}_d D(f,d)) &\cong \lim_f \underline{\mathrm{K}}(a, \operatorname{colim}_d D(f,d)) \cong \lim_f \operatorname{colim}_d \underline{\mathrm{K}}(a, D(f,d)) \cong \\ &\cong \operatorname{colim}_d \lim_f \underline{\mathrm{K}}(a, D(f,d)) \cong \underline{\mathrm{K}}(a, \operatorname{colim}_d \lim_f D(f,d)) \end{aligned}$$

because finite limits commute with filtered colimits in $\underline{\mathrm{Set}}$. □

Thus we obtain a generalization of the well-known fact that finite infs commute with directed sups in Scott domains.

Corollary 4.2.12 *In any SC category (existing) finite limits commute with filtered colimits.*

It is obvious that all SC categories, all finitary functors and all natural transformations between them form a 2-quasicategory, let us denote it by SC.

It is proved in [Ad97], Theorem 3 that SC_o — the underlying quasicategory of SC — is cartesian closed. Thus, given SC categories $\underline{\mathrm{K}}$, $\underline{\mathrm{L}}$, the quasicategory $\mathsf{SC}(\underline{\mathrm{K}}, \underline{\mathrm{L}})$ having all finitary functors from $\underline{\mathrm{K}}$ to $\underline{\mathrm{L}}$ as objects and all natural transformations as morphisms is an SC category.

We also define SC^l as the 2-quasicategory of all SC categories, all left adjoints which have finitary right adjoints and all natural transformations between them.

Lemma 4.2.13 *Let $\underline{\mathrm{K}}$, $\underline{\mathrm{L}}$ be SC categories, x, resp. y objects of $\underline{\mathrm{K}}$, resp. $\underline{\mathrm{L}}$. Then the functor $\langle x, y\rangle : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{L}}$ defined by the composition $y \otimes \_ \cdot \underline{\mathrm{K}}(x, \_)$ is a finitely presentable object of $\mathsf{SC}(\underline{\mathrm{K}}, \underline{\mathrm{L}})$. Moreover, every finitely presentable object of $\mathsf{SC}(\underline{\mathrm{K}}, \underline{\mathrm{L}})$ is a finite colimit of a diagram on functors of the form $\langle x, y\rangle$.*

Proof. See [Ad97], Lemma 1. □

Theorem 4.2.14 (Yoneda Lemma for SC) *Let $\underline{\mathrm{K}}$, $\underline{\mathrm{L}}$ be SC categories, $F : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{L}}$ a functor. Then there is a bijection of hom-sets $\mathsf{SC}(\langle x, y\rangle, F)$ and $\underline{\mathrm{L}}(y, Fx)$.*

Proof. See [Ad97], Lemma 1. □

4.3 Representations of Finitary Adjunctions

In this section we will represent the 2-quasicategory GDOM^l by a certain legitimate 2-category. Recall the notion of a finitary adjunction from 4.1.6.

Lemma 4.3.1 *Let $\underline{\mathrm{K}}$ and $\underline{\mathrm{L}}$ be generalized domains. Let $L \dashv R$, $L : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{L}}$ and $R : \underline{\mathrm{L}} \longrightarrow \underline{\mathrm{K}}$ be a finitary adjunction. Let $\underline{\mathrm{K}}_{fin}$ be a small category representing all finitely presentable objects in $\underline{\mathrm{K}}$, let $I : \underline{\mathrm{K}}_{fin} \longrightarrow \underline{\mathrm{K}}$ be the inclusion. Then there is a choice of a small category $\underline{\mathrm{L}}_{fin}$ and a functor $F : \underline{\mathrm{K}}_{fin} \longrightarrow \underline{\mathrm{L}}_{fin}$ such that the following hold:*

1. *The equality $LI = JF$ holds, where $J : \underline{\mathrm{L}}_{fin} \longrightarrow \underline{\mathrm{L}}$ is the inclusion.*
2. *F preserves initial object.*

3. The comma category F/y is filtered for any $\underline{\mathrm{L}}_{fin}$-object y.

Proof. Since $L \dashv R$ is a finitary adjunction, the functor L clearly preserves finitely presentable objects. Therefore one can choose a small category $\underline{\mathrm{L}}_{fin}$ in such a way that the image of LI is contained in $\underline{\mathrm{L}}_{fin}$. Define the functor F as the domain-codomain restriction of L and then Conditions 1. and 2. of the theorem clearly hold.

To verify Condition 3. take any $\underline{\mathrm{L}}_{fin}$-object y. Since J is a full embedding, $LI = JF$ and $L \dashv R$, the categories F/y and I/RJy are isomorphic. The latter category is trivially filtered. □

Lemma 4.3.1 motivates the following definition:

Definition 4.3.2 Suppose that $\underline{\mathrm{A}}$ and $\underline{\mathrm{B}}$ are categories having initial object. A functor $F : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ is called *normal* if it preserves initial object and the category F/b is filtered for any $\underline{\mathrm{B}}$-object b. A functor which is simultaneously normal and a full embedding will be called a *normal embedding.*

Lemma 4.3.3 *The identity functors are normal. The composition of normal functors is normal.*

The proof is trivial. The above lemma allows us to define a *2-quasicategory* NORM of all categories with initial object as 0-cells, all normal functors as 1-cells and all natural transformations as 2-cells. The legitimate *2-category* of all small categories with initial object as 0-cells, all normal functors as 1-cells and all natural transformations as 2-cells will be denoted by norm. Analogously we define the *2-quasicategory* $\mathsf{NORM}_{\mathrm{e}}$ with all normal embeddings as 1-cells and the legitimate *2-category* $\mathsf{norm}_{\mathrm{e}}$.

Normal functors induce finitary adjunctions in the following sense. Recall that $\underline{\mathrm{A}}^*$ denotes the category of all flat functors from $\underline{\mathrm{A}}^{op}$ to $\underline{\mathrm{Set}}$ and all natural transformations (Definition 3.5.10).

Lemma 4.3.4 *Let $\underline{\mathrm{A}}$ and $\underline{\mathrm{B}}$ be small categories having initial object, let $F : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ be a normal functor. Then F can be extended to a finitary functor $L : \underline{\mathrm{A}}^* \longrightarrow \underline{\mathrm{B}}^*$ (i.e. L fulfills $LI_{\underline{\mathrm{A}}} = I_{\underline{\mathrm{B}}}F$) such that L has a finitary right adjoint $R : \underline{\mathrm{B}}^* \longrightarrow \underline{\mathrm{A}}^*$.*

Proof. Define L as $Lan_{I_{\underline{\mathrm{A}}}}(I_{\underline{\mathrm{B}}}F)$ and R as $Lan_{I_{\underline{\mathrm{B}}}F}(I_{\underline{\mathrm{A}}})$. Both of these Kan extensions exist, since they can be given pointwise as filtered colimits:

For any $\underline{\mathrm{A}}^*$-object x define the functor $\mathcal{L}_x : I_{\underline{\mathrm{A}}}/x \longrightarrow \underline{\mathrm{B}}^*$ as follows:

the functor $\mathcal{L}_x$ assigns the $\underline{\mathrm{B}}^*$-object $I_{\underline{\mathrm{B}}}Fa$ to every $I_{\underline{\mathrm{A}}}/x$-object $h : I_{\underline{\mathrm{A}}}a \longrightarrow x$,

the functor $\mathcal{L}_x$ assigns the $\underline{\mathrm{B}}^*$-morphism $I_{\underline{\mathrm{B}}}Fg : I_{\underline{\mathrm{B}}}Fa \longrightarrow I_{\underline{\mathrm{B}}}Fa'$ to every $I_{\underline{\mathrm{A}}}/x$-morphism $I_{\underline{\mathrm{A}}}g : (h : I_{\underline{\mathrm{A}}}a \longrightarrow x) \longrightarrow (h' : I_{\underline{\mathrm{A}}}a' \longrightarrow x)$.

Then Lx is a colimit of $\mathcal{L}_x$. Denote the colimit cocone by $\lambda^{(x)}$. For any $\underline{\mathrm{A}}^*$-morphism $f : x \longrightarrow x'$, Lf is the unique $\underline{\mathrm{B}}^*$-morphism such that for any $I_{\underline{\mathrm{A}}}/x$-object $h : I_{\underline{\mathrm{A}}}a \longrightarrow x$

the triangle

$$\begin{tikzcd} \mathcal{L}_x(h) \arrow[rr, "\lambda^{(x)}(h)"] \arrow[dr, "\lambda^{(x')}(fh)"'] & & Lx \arrow[dl, "Lf"] \\ & Lx' & \end{tikzcd} \tag{4.1}$$

commutes.

For any $\underline{\mathrm{B}}^*$-object y define the functor $\mathcal{R}_y : I_{\underline{\mathrm{B}}}F/y \longrightarrow \underline{\mathrm{A}}^*$ as follows:

the functor $\mathcal{R}_x$ assigns the $\underline{\mathrm{A}}^*$-object $I_{\underline{\mathrm{A}}}a$ to every $I_{\underline{\mathrm{B}}}F/y$-object $h : I_{\underline{\mathrm{B}}}Fa \longrightarrow y$,

the functor $\mathcal{R}_x$ assigns the $\underline{\mathrm{A}}^*$-morphism $I_{\underline{\mathrm{A}}}g : I_{\underline{\mathrm{A}}}a \longrightarrow I_{\underline{\mathrm{A}}}a'$ to every $I_{\underline{\mathrm{B}}}F/y$-morphism $I_{\underline{\mathrm{B}}}Fg : (h : I_{\underline{\mathrm{B}}}Fa \longrightarrow y) \longrightarrow (h' : I_{\underline{\mathrm{A}}}a' \longrightarrow y)$.

Since F is normal, $I_{\underline{\mathrm{B}}}F/y$ is filtered and Ry is a colimit of $\mathcal{R}_y$. Denote the colimit cocone by $\rho^{(y)}$. For any $\underline{\mathrm{B}}^*$-morphism $f : y \longrightarrow y'$, Rf is the unique $\underline{\mathrm{A}}^*$-morphism such that for any $I_{\underline{\mathrm{B}}}F/y$-object $h : I_{\underline{\mathrm{B}}}Fa \longrightarrow y$ the triangle

$$\begin{tikzcd} \mathcal{R}_y(h) \arrow[rr, "\rho^{(y)}(h)"] \arrow[dr, "\rho^{(y')}(fh)"'] & & Ry \arrow[dl, "Rf"] \\ & Ry' & \end{tikzcd} \tag{4.2}$$

commutes.

Both L and R preserve filtered colimits by definition. It remains to prove that $L \dashv R$. To do so we will define natural transformations $\eta : 1_{\underline{\mathrm{A}}^*} \Longrightarrow RL$ and $\varepsilon : LR \Longrightarrow 1_{\underline{\mathrm{B}}^*}$ and we will verify the triangle identities for them.

Definition of $\eta_x : x \longrightarrow RLx$. Recall that x is a filtered colimit of a canonical diagram $C_x : I_{\underline{\mathrm{A}}}/x \longrightarrow \underline{\mathrm{A}}^*$. Denote the colimit cocone by $\kappa^{(x)} : C_x \longrightarrow x$. We will use the fact that $\kappa^{(x)}(h) = h$ for any $I_{\underline{\mathrm{A}}}/a$-object $h : I_{\underline{\mathrm{A}}}a \longrightarrow x$. Define η_x as the unique morphism such that for any $I_{\underline{\mathrm{A}}}/a$-object $h : I_{\underline{\mathrm{A}}}a \longrightarrow x$ the triangle

$$\begin{tikzcd} C_x(h) \arrow[rr, "\kappa^{(x)}(h)"] \arrow[dr, "\rho^{(Lx)}(Lh)"'] & & x \arrow[dl, "\eta_x"] \\ & RLx & \end{tikzcd} \tag{4.3}$$

commutes. We have made use of the fact that $C_x(h) = I_{\underline{\mathrm{A}}}a = \mathcal{R}_{Lx}(Lh)$.

The collection (η_x) forms a natural transformation: take any $\underline{\mathrm{A}}^*$-morphism $f : x \longrightarrow x'$. It suffices to prove that

$$\eta_{x'} \cdot f \cdot \kappa^{(x)}(h) = RLf \cdot \eta_x \cdot \kappa^{(x)}(h)$$

for any $I_{\underline{\mathrm{A}}}/x$-object $h : I_{\underline{\mathrm{A}}}a \longrightarrow x$. This holds due to the above definitions:

$$RLf \cdot \eta_x \cdot \kappa^{(x)}(h) = RLf \cdot \rho^{(Lx)}(Lh) = \rho^{(Lx')}(Lf \cdot Lh) = \eta_{x'} \cdot \kappa^{(x')}(f \cdot h) = \eta_{x'} \cdot f \cdot \kappa^{(x)}(h).$$

Definition of $\varepsilon_y : LRy \longrightarrow y$. Recall that Ry is a filtered colimit of the diagram $\mathcal{R}_y$ and that L preserves that colimit. Define ε_y as the unique morphism such that for any $I_{\underline{\mathrm{B}}}F/y$-object $h : I_{\underline{\mathrm{B}}}Fa \longrightarrow y$ the triangle

$$\begin{tikzcd} L\mathcal{R}_y(h) \arrow[rr, "L\rho^{(y)}(h)"] \arrow[dr, "h"'] & & LRy \arrow[dl, "\varepsilon_y"] \\ & y & \end{tikzcd} \tag{4.4}$$

commutes. We have made use of the fact that for any $I_{\underline{\mathrm{B}}}F/y$-morphism $I_{\underline{\mathrm{B}}}Fg : (h : I_{\underline{\mathrm{B}}}Fa \longrightarrow y) \longrightarrow (h' : I_{\underline{\mathrm{B}}}Fa' \longrightarrow y)$ the equalities $L\mathcal{R}_y(g) = LI_{\underline{\mathrm{A}}}g = I_{\underline{\mathrm{B}}}Fg$ hold.

The collection (ε_y) forms a natural transformation: take any $\underline{\mathrm{B}}^*$-morphism $f : y \longrightarrow y'$. It suffices to prove that

$$\varepsilon_{y'} \cdot LRf \cdot L\rho^{(y)}(h) = f \cdot \varepsilon_y \cdot L\rho^{(y)}(h)$$

for any $I_{\underline{\mathrm{B}}}F/y$-object $h : I_{\underline{\mathrm{B}}}Fa \longrightarrow y$. This holds due to the above definitions:

$$\varepsilon_{y'} \cdot LRf \cdot L\rho^{(y)}(h) = \varepsilon_{y'} \cdot L\rho^{(y')}(f \cdot h) = f \cdot h = f \cdot \varepsilon_y \cdot L\rho^{(y)}(h).$$

The triangle equality $\varepsilon_L \cdot L\eta = 1_L$. Let x be any $\underline{\mathrm{A}}^*$-object. It suffices to prove that

$$\varepsilon_{Lx} \cdot L\eta_x \cdot Lh = Lh$$

for any $I_{\underline{\mathrm{A}}}/a$-object $h : I_{\underline{\mathrm{A}}}a \longrightarrow x$, since $Lh = L\kappa^{(x)}(h)$ is a colimit cocone. This is true due to the above definitions: $\varepsilon_{Lx} \cdot (L\eta_x \cdot Lh) = \varepsilon_{Lx} \cdot L\rho^{(Lx)}(Lh) = Lh$.

Before proving the second triangle identity we will verify $\eta_{I_{\underline{\mathrm{A}}}a} = \rho^{(LI_{\underline{\mathrm{A}}}a)}(1_{I_{\underline{\mathrm{B}}}Fa})$ for any $\underline{\mathrm{A}}$-object a. Denote $\rho^{(LI_{\underline{\mathrm{A}}}a)}(1_{I_{\underline{\mathrm{B}}}Fa})$ as r_a. Recall that $\eta_{I_{\underline{\mathrm{A}}}a}$ is the unique morphism such that for any $I_{\underline{\mathrm{A}}}/I_{\underline{\mathrm{A}}}a$-object $h' : I_{\underline{\mathrm{A}}}a \longrightarrow I_{\underline{\mathrm{A}}}a$ the triangle

$$\begin{tikzcd} C_{I_{\underline{\mathrm{A}}}a}(h') \arrow[rr, "\kappa^{(I_{\underline{\mathrm{A}}}a)}(h')"] \arrow[dr, "\rho^{LI_{\underline{\mathrm{A}}}a}(Lh')"'] & & I_{\underline{\mathrm{A}}}a \arrow[dl, "\eta_{I_{\underline{\mathrm{A}}}a}"] \\ & RLI_{\underline{\mathrm{A}}}a & \end{tikzcd} \tag{4.5}$$

commutes. Therefore it suffices to prove that the triangle

$$\begin{tikzcd} I_{\underline{\mathrm{A}}}a' \arrow[rr, "h'"] \arrow[dr, "RLh' \cdot r_{a'}"'] & & I_{\underline{\mathrm{A}}}a \arrow[dl, "r_a"] \\ & RLIa & \end{tikzcd} \tag{4.6}$$

commutes. Since $I_{\underline{\mathrm{A}}}$ is full, $h' = I_{\underline{\mathrm{A}}}k$ for some $k : a' \longrightarrow a$. Then $I_{\underline{\mathrm{B}}}Fk : (I_{\underline{\mathrm{B}}}Fk : I_{\underline{\mathrm{B}}}Fa' \longrightarrow I_{\underline{\mathrm{B}}}Fa) \longrightarrow (1_{I_{\underline{\mathrm{B}}}Fa} : I_{\underline{\mathrm{B}}}Fa \longrightarrow I_{\underline{\mathrm{B}}}Fa)$ is a $I_{\underline{\mathrm{B}}}F/I_{\underline{\mathrm{B}}}Fa$-morphism, therefore the following triangle

$$\begin{tikzcd} \mathcal{R}_{I_{\underline{\mathrm{B}}}Fa}(I_{\underline{\mathrm{B}}}Fk) \arrow[rr, "h'"] \arrow[dr, "\rho^{(I_{\underline{\mathrm{B}}}Fa)}(I_{\underline{\mathrm{B}}}Fk)"'] & & \mathcal{R}_{I_{\underline{\mathrm{B}}}Fa}(1_{I_{\underline{\mathrm{B}}}Fa}) \arrow[dl, "r_a"] \\ & RI_{\underline{\mathrm{B}}}Fa & \end{tikzcd} \tag{4.7}$$

commutes. Since $\mathcal{R}_{I_{\underline{\mathrm{B}}}Fa}(I_{\underline{\mathrm{B}}}Fk) = I_{\underline{\mathrm{A}}}a' = \mathcal{R}_{I_{\underline{\mathrm{B}}}Fa'}(1_{I_{\underline{\mathrm{B}}}Fa'})$, the following triangle

$$\begin{tikzcd} \mathcal{R}_{I_{\underline{\mathrm{B}}}Fa'}(1_{I_{\underline{\mathrm{B}}}Fa'}) \arrow[rr, "r_{a'}"] \arrow[dr, "\rho^{(I_{\underline{\mathrm{B}}}Fa)}(I_{\underline{\mathrm{B}}}Fk)"'] & & RI_{\underline{\mathrm{B}}}Fa' \arrow[dl, "RLh'"] \\ & RI_{\underline{\mathrm{B}}}Fa & \end{tikzcd} \tag{4.8}$$

commutes by the definition of R on morphisms. Putting (4.7) and (4.8) together proves that $\eta_{Ia} = r_a$.

The triangle equality $R\varepsilon \cdot \eta_R = 1_R$. It suffices to show that

$$R\varepsilon_y \cdot \eta_{Ry} \cdot \rho^{(y)}(h) = \rho^{(y)}(h)$$

for any $I_{\underline{\mathrm{B}}}F/y$-object $h : I_{\underline{\mathrm{B}}}Fa \longrightarrow y$. The equality $\eta_{Ry} \cdot \rho^{(y)}(h) = RL\rho^{(y)} \cdot \eta_{I_{\underline{\mathrm{A}}}a}$ holds due to the naturality of η. The equality $R\varepsilon_y \cdot RL\rho^{(y)}(h) = Rh$ holds by the definition of ε. Using the fact that $\eta_{I_{\underline{\mathrm{A}}}a} = r_a$, and the definition of R on morphisms we obtain that $Rh \cdot \eta_{I_{\underline{\mathrm{A}}}a} = \rho^{(y)}(h)$. The equality $R\varepsilon_y \cdot \eta_{Ry} \cdot \rho^{(y)}(h) = \rho^{(y)}(h)$ now follows. □

Remark 4.3.5 The proofs of Lemmas 4.3.1 and 4.3.4 could have been used verbatim with "filtered" replaced by "λ-filtered" for any regular cardinal λ. □

Thus we obtain the following corollary:

Corollary 4.3.6 (Adjoint Functor Theorem for λ-accessible Categories) *Let λ be a regular cardinal. Suppose that $\underline{\mathrm{K}}$ and $\underline{\mathrm{L}}$ are λ-accesible categories having initial object. Let $L : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{L}}$ be a λ-accessible functor. Then the following are equivalent:*

1. *L has a λ-accessible right adjoint.*
2. *L preserves λ-presentable objects and if we denote by $F : \underline{\mathrm{K}}_\lambda \longrightarrow \underline{\mathrm{L}}_\lambda$ the domain-codomain restriction of L, then the comma category F/b is λ-filtered for any $\underline{\mathrm{L}}_\lambda$-object b.*

We now show how (the finitary version of) the preceding corollary modifies for finitary adjunctions between *Scott complete categories.*

Lemma 4.3.7 *Let $\underline{\mathrm{K}}$ and $\underline{\mathrm{L}}$ be SC categories. Let $L \dashv R$, $L : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{L}}$ and $R : \underline{\mathrm{L}} \longrightarrow \underline{\mathrm{K}}$ be a finitary adjunction from $\underline{\mathrm{K}}$ to $\underline{\mathrm{L}}$. Let $\underline{\mathrm{K}}_{fin}$ be a small category representing all finitely presentable objects in $\underline{\mathrm{K}}$, let $I : \underline{\mathrm{K}}_{fin} \longrightarrow \underline{\mathrm{K}}$ be the inclusion. Then there is a choice of a small category $\underline{\mathrm{L}}_{fin}$ and a functor $F : \underline{\mathrm{K}}_{fin} \longrightarrow \underline{\mathrm{L}}_{fin}$ such that the following hold:*

1. *The equality $LI = JF$ holds, where $J : \underline{\mathrm{L}}_{fin} \longrightarrow \underline{\mathrm{L}}$ is the inclusion.*
2. *F preserves initial object.*
3. *If $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{K}}_{fin}$ is a finite non-empty diagram s.t. FD is consistent, then D is consistent and $F(\operatorname{colim} D) \cong \operatorname{colim}(FD)$.*

Proof. Since $L \dashv R$ is a finitary adjunction, the functor L clearly preserves finitely presentable objects. Therefore one can choose a small category $\underline{\mathrm{L}}_{fin}$ in such a way that the image of LI is contained in $\underline{\mathrm{L}}_{fin}$. Define the functor F as the domain-codomain restriction of L and then Conditions 1. and 2. of the theorem clearly hold.

To verify Condition 3. take any finite non-empty diagram $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{K}}_{fin}$ such that FD is consistent. Let $\mu : FD \Longrightarrow b$ be a compatible cocone for FD. Then D and μ determine a finite non-empty diagram $H : \underline{\mathrm{D}} \longrightarrow F/b$ defined by

- $\mu_d : FDd \longrightarrow b$ on $\underline{\mathrm{D}}$-objects d,
- $FD\delta : (\mu_d : FDd \longrightarrow b) \longrightarrow (\mu_{d'} : FDd' \longrightarrow b)$ on $\underline{\mathrm{D}}$-morphisms $\delta : d \longrightarrow d'$.

By Lemma 4.3.1 we know that F/b is filtered, thus there is a compatible cocone on H yielding a compatible cocone on D. Therefore the colimit of D exists in $\underline{\mathrm{K}}_{fin}$ and it clearly is preserved by F. □

This motivates the following definition:

Definition 4.3.8 A category is said to be *finitely consistently cocomplete* (FCC category for short) if it has colimits of finite consistent diagrams (including the empty diagram). A functor $F : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ between FCC categories is called *finitely consistently cocontinuous* (FCC functor) if it fulfills the following condition:

> given a finite diagram $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{A}}$ s.t. FD is consistent, then D is consistent and $F(\operatorname{colim} D) \cong \operatorname{colim}(FD)$.

An FCC functor which is a full embedding is called an *FCC embedding.*

Lemma 4.3.9 *The identity functors are FCC functors. The composition of FCC functors is an FCC functor.*

The proof is trivial. The above lemma allows us to define a *2-quasicategory* FCC of all FCC categories as 0-cells, all FCC functors as 1-cells and all natural transformations as 2-cells. The legitimate *2-category* of all small FCC categories as 0-cells will be denoted by fcc. Analogously we define the *2-quasicategory* $\mathsf{FCC}_{\mathrm{e}}$ with all FCC embeddings as 1-cells. The legitimate *2-category* of all small FCC categories, all FCC embeddings and all natural transformations will be denoted by $\mathsf{fcc}_{\mathrm{e}}$.

Lemma 4.3.10 *Every FCC functor between FCC categories is normal.*

Proof. Let $F : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ be an FCC functor between FCC categories. Since F preserves initial objects, it suffices to prove that the comma category F/b is filtered for any $\underline{\mathrm{B}}$-object b.

F/b is nonempty, since it contains the object $F\bot \longrightarrow b$. Take any finite non-empty diagram $D : \underline{\mathrm{D}} \longrightarrow F/b$. For any $\underline{\mathrm{D}}$-object d we have an F/b-object $f_d : Fa_d \longrightarrow b$ and for any $\underline{\mathrm{D}}$-morphism $\delta : d \longrightarrow d'$, $D\delta : f_d \longrightarrow f_{d'}$ is an $\underline{\mathrm{A}}$-morphism $D\delta : a_d \longrightarrow a_{d'}$ such

that the triangle

$$\begin{tikzcd} Fa_d \arrow[rr, "FD\delta"] \arrow[dr, "f_d"'] & & Fa_{d'} \arrow[dl, "f_{d'}"] \\ & b & \end{tikzcd} \tag{4.9}$$

commutes. Then D induces a finite non-empty diagram $C : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{A}}$ defined by $Cd = a_d$ and $C\delta = D\delta$ for any $\underline{\mathrm{D}}$-object d and any $\underline{\mathrm{D}}$-morphism $\delta : d \longrightarrow d'$. Moreover, the collection (f_d) forms a natural transformation $\gamma : FC \Longrightarrow b$, i.e. FC is $\underline{\mathrm{B}}$-consistent. Then C has a colimit $\kappa : C \Longrightarrow a$ and F preserves it. Therefore there is a unique $f : Fa \longrightarrow b$ such that the triangle

$$\begin{tikzcd} Fa_d \arrow[rr, "Fh_d"] \arrow[dr, "f_d"'] & & Fa \arrow[dl, "f"] \\ & b & \end{tikzcd} \tag{4.10}$$

commutes for any $\underline{\mathrm{D}}$-object d. We have proved that F/b is filtered. □

Corollary 4.3.11 *Let $\underline{\mathrm{A}}$ and $\underline{\mathrm{B}}$ be small FCC categories, $F : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ be FCC functor. Then F induces a finitary adjunction $L \dashv R$, $L : \underline{\mathrm{A}}^* \longrightarrow \underline{\mathrm{B}}^*$, $R : \underline{\mathrm{B}}^* \longrightarrow \underline{\mathrm{A}}^*$ from $\underline{\mathrm{A}}^*$ to $\underline{\mathrm{B}}^*$.*

Proof. Use Lemmas 4.3.10 and 4.3.4. □

It follows from considerations in [AP96] that GDOM^l cannot be equivalent to any category. We will repeat the argument here (see Lemma 4 of [AP96]): A category is required to have only a set of morphisms between any two of its objects. However, there are generalized domains $\underline{\mathrm{K}}$, $\underline{\mathrm{L}}$ such that $\mathsf{GDOM}^l(\underline{\mathrm{K}}, \underline{\mathrm{L}})$ is as large as the collection of all subclasses of the class of all sets: put $\underline{\mathrm{K}} = \underline{\mathrm{L}} = \underline{\mathrm{Set}}$, then for each class $\mathcal{C}$ of sets there exists a functor

$$F_{\mathcal{C}} : \underline{\mathrm{Set}} \longrightarrow \underline{\mathrm{Set}} \quad \text{with } F_{\mathcal{C}} \cong 1_{\underline{\mathrm{Set}}} \text{ and } F_{\mathcal{C}}x = x \text{ iff } x \in \mathcal{C}.$$

The functor $F_{\mathcal{C}}$ is clearly a finitary right adjoint. The collection of all subclasses of the class of all sets forms a conglomerate which is not a class. Therefore there cannot be an equivalence functor from GDOM^l to any category.

We will show that however, GDOM^l is *biequivalent* to the 2-category norm (recall Definition 3.3.12).

Theorem 4.3.12 (Representation of Finitary Adjunctions)
norm *and* GDOM^l *are biequivalent.*

Proof. Define a pseudofunctor $\Phi : \mathsf{norm} \longrightarrow \mathsf{GDOM}^l$ on 0-cells by $\Phi(\underline{\mathrm{A}}) = \underline{\mathrm{A}}^*$. The definition of

$$\Phi_{\underline{\mathrm{A}},\underline{\mathrm{B}}} : \mathsf{norm}(\underline{\mathrm{A}}, \underline{\mathrm{B}}) \longrightarrow \mathsf{GDOM}^l(\Phi\underline{\mathrm{A}}, \Phi\underline{\mathrm{B}})$$

is as follows:

$$\Phi_{\underline{\mathrm{A}},\underline{\mathrm{B}}}(\tau : F \Longrightarrow G) = \tau^* : F^* \Longrightarrow G^*$$

where $F^* = Lan_{I_{\underline{A}}}(I_{\underline{B}}F)$, $G^* = Lan_{I_{\underline{A}}}(I_{\underline{B}}G)$ and τ^* is the unique natural transformation with $\tau^* I_{\underline{A}} = I_{\underline{B}}\tau$. (Use the definition of a left Kan extension and the fact that $I_{\underline{A}}$ and $I_{\underline{B}}$ are full embeddings.)

The assignment $\Phi_{\underline{A},\underline{B}}$ is a functor: given $\tau : F \Longrightarrow G$, $\sigma : G \Longrightarrow H$ in $\mathsf{norm}(\underline{A}, \underline{B})$, we have that $(\sigma\tau)^* = \sigma^*\tau^*$, since $(I_{\underline{B}}\sigma) \cdot (I_{\underline{B}}\tau) = I_{\underline{B}}(\sigma \cdot \tau)$ and $(\sigma^* I_{\underline{A}}) \cdot (\tau^* I_{\underline{A}}) = (\sigma^*\tau^*)I_{\underline{A}}$ by the interchange law for horizontal composition. Analogously one verifies the preservation of identities.

We will show that Φ bears a structure of a pseudofunctor.

1. Given $F : \underline{A} \longrightarrow \underline{B}$ and $G : \underline{B} \longrightarrow \underline{C}$ in norm, denote by

$$\varphi_{F,G} : \Phi_{\underline{B},\underline{C}}(G)\Phi_{\underline{A},\underline{B}}(F) \Longrightarrow \Phi_{\underline{A},\underline{C}}(GF)$$

the unique natural transformation with $\varphi_{F,G} I_{\underline{A}} = 1_{(GF)^* I_{\underline{A}}}$ (which exists since G^*, being a left adjoint, preserves left Kan extensions — see [ML71], Section X.5, Theorem 1). It is clear that $\varphi_{F,G}$ is a natural isomorphism.

2. Given a 0-cell $\underline{A}$ in norm, denote by

$$\psi_{\underline{A}} : \Phi_{\underline{A},\underline{A}}(1_{\underline{A}}) \Longrightarrow 1_{\Phi(\underline{A})}$$

the unique natural transformation with $\psi_{\underline{A}} I_{\underline{A}} = 1_{I_{\underline{A}} \cdot 1_{\underline{A}}}$. It is clear that $\psi_{\underline{A}}$ is a natural isomorphism.

3. We will verify the coherence conditions which will show that the triple (Φ, φ, ψ) is a pseudofunctor.

Associativity coherence (cf. (3.18)): given 1-cells $F : \underline{A} \longrightarrow \underline{B}$, $G : \underline{B} \longrightarrow \underline{C}$ and $H : \underline{C} \longrightarrow \underline{D}$ in norm we are to prove that the diagram

$$\begin{tikzcd}
H^*(G^*F^*) = (H^*G^*)F^* \arrow[r, "\varphi_{G,H}F^*"] \arrow[d, "H^*\varphi_{F,G}"'] & (HG)^*F^* \arrow[d, "\varphi_{F,HG}"] \\
H^*(GF)^* \arrow[r, "\varphi_{GF,H}"'] & (HGF)^*
\end{tikzcd} \tag{4.11}$$

commutes. It suffices to show that for any $\underline{A}^*$-object x and any $I_{\underline{A}}/x$-object $h : I_{\underline{A}}a \longrightarrow x$ it holds that

$$(\varphi_{GF,H})_x \cdot (H^*\varphi_{F,G})_x \cdot H^*G^*F^*h = (\varphi_{F,GH})_x \cdot (\varphi_{G,H}F^*)_x \cdot H^*G^*F^*h.$$

Due to the naturality of φ's it suffices to show that

$$H^*G^*F^*h \cdot (\varphi_{F,GH}I_{\underline{A}})_a \cdot (\varphi_{G,H}I_{\underline{B}}F)_a = H^*G^*F^*h \cdot (\varphi_{GF,H}I_{\underline{A}})_a \cdot (H^*\varphi_{F,G}I_{\underline{A}})_a.$$

This is true, since all φ's are identities when evaluated at finitely presentable objects.

Analogously one verifies the identity coherences (cf. (3.19)):

$$\begin{tikzcd} F^* \cdot 1_{\underline{A}} \arrow[r, "F^*\psi_{\underline{A}}"] \arrow[d, "\varphi_{1_{\underline{A}},F}"'] & F^* \cdot 1_{\underline{A}^*} \arrow[d, "1"] \\ (F \cdot 1_{\underline{A}})^* \arrow[r, "1"'] & F^* \end{tikzcd} \qquad \begin{tikzcd} 1_{\underline{B}} \cdot F^* \arrow[r, "\psi_{\underline{B}}F^*"] \arrow[d, "\varphi_{F,1_{\underline{B}}}"'] & 1_{\underline{B}^*} \cdot F^* \arrow[d, "1"] \\ (1_{\underline{B}} \cdot F)^* \arrow[r, "1"'] & F^* \end{tikzcd} \tag{4.12}$$

We use the fact that all ψ's evaluated at finitely presentable objects are identities.

The pseudofunctor (Φ, φ, ψ) is a biequivalence.
In fact, Condition (a) in the definition of biequivalence requires any generalized domain $\underline{K}$ to be equivalent to a category of the form $\underline{A}^*$ for some norm-object $\underline{A}$, see 3.5.13.
Condition (b) in the definition of biequivalence requires $\Phi_{\underline{A},\underline{B}}$ to be full and faithful and essentially onto on objects. $\Phi_{\underline{A},\underline{B}}$ is full and fathful by definition and any 1-cell $L : \underline{A}^* \longrightarrow \underline{B}^*$ in GDOM^l is isomorphic to F^* for a 1-cell $F : \underline{A} \longrightarrow \underline{B}$ in norm, where F is a domain-codomain restriction of L. □

Corollary 4.3.13 *The biequivalence Φ of Theorem 4.3.12 can be restricted to a biequivalence of SC^l and fcc.*

The biequivalence Φ of Theorem 4.3.12 also allows us to define an interesting class of generalized domains.

In Domain Theory there is an important class of domains — the so called *SFP domains* (or *bifinite domains*). SFP domains (SFP stands for "sequence of finite posets") were originally defined as colimits of countable chains of embedding parts of embedding-projection pairs between finite posets with least elements (see e.g. [GS90] for details).

Countable posets $\langle D, \sqsubseteq \rangle$ arising as posets of compact elements of SFP domains can be characterized by the following property:

> if $X \subseteq D$ is a finite set, then there is a finite set A, such that $X \subseteq A$ and the embedding of $\langle A, \sqsubseteq \rangle$ in $\langle D, \sqsubseteq \rangle$ is a normal embedding.

Posets having the above property are called *Plotkin posets* in [GS90]. We propose the name an $\aleph_0$-*Plotkin category* for their obvious categorical generalization.

Recall from 3.1.1 that an $\aleph_0$-category is a category with countably many objects and finite hom-sets.

Definition 4.3.14 A $\aleph_0$-category $\underline{K}$ with an initial object is called an $\aleph_0$-*Plotkin category* if each finite diagram $D : \underline{D} \longrightarrow \underline{K}$ factors through a normal embedding $F : \underline{A} \longrightarrow \underline{K}$ where $\underline{A}$ is a finite category with an initial object.

We need the following easy lemma:

Lemma 4.3.15 *Suppose that the following triangle*

$$\begin{tikzcd} \underline{A} \arrow[rr, "F"] \arrow[dr, "H"'] & & \underline{B} \arrow[dl, "G"] \\ & \underline{C} & \end{tikzcd}$$

of full embeddings commutes and that categories $\underline{A}$, $\underline{B}$, $\underline{C}$ *have initial objects. If both* H *and* G *are normal embeddings, then so is* F.

Lemma 4.3.16 *Let* $\underline{K}$ *be an* $\aleph_0$*-category with an initial object. Then the following are equivalent:*

1. $\underline{K}$ *is an* $\aleph_0$*-Plotkin category.*
2. $\underline{K}$ *is a colimit of a countable chain formed by finite* $\aleph_0$*-Plotkin categories and by normal embeddings.*

Proof. 1⇒2: This is clear, if the category $\underline{K}$ is finite. For $\underline{K}$ infinite, enumerate $\underline{K}$-objects as $\{a_0, a_1, \ldots\}$ such that $a_0 = \bot$. The chain of normal embeddings

$$(F_{i,i+1} : \underline{K}_i \longrightarrow \underline{K}_{i+1} \mid i \geq 0)$$

between finite $\aleph_0$-Plotkin categories is defined as follows:

$\underline{K}_0$ is the full subcategory of $\underline{K}$ on the object a_0. The full embedding $E_0 : \underline{K}_0 \longrightarrow \underline{K}$ is clearly normal.

Suppose we have defined $\underline{K}_i$ and a normal embedding $E_i : \underline{K}_i \longrightarrow \underline{K}$. Pick up the least number n such that a_n is not a $\underline{K}_i$-object. Define $\underline{L}$ as the full subcategory of $\underline{K}$ on all $\underline{K}_i$-objects and a_n. Denote by $E : \underline{L} \longrightarrow \underline{K}$ the full embedding. Since $\underline{K}$ is an $\aleph_0$-Plotkin category, E factors through a normal embedding $E_{i+1} : \underline{K}_{i+1} \longrightarrow \underline{K}$. Define $F_{i,i+1} : \underline{K}_i \longrightarrow \underline{K}_{i+1}$ as a full embedding. The functor $F_{i,i+1}$ is a normal embedding by Lemma 4.3.15.

It is clear that $(E_i : \underline{K}_i \longrightarrow \underline{K} \mid i \geq 0)$ is a colimit cone of $(F_{i,i+1} : \underline{K}_i \longrightarrow \underline{K}_{i+1} \mid i \geq 0)$.

2⇒1: Trivial. □

4.4 Representations of Finitary Functors

In this section we intend to generalize the correspondence between

continuous functions between (Scott) domains, and

approximable relations of subposets of compact elements of (Scott) domains.

In Domain Theory, given two (Scott) domains $\langle D, \sqsubseteq_1 \rangle$, $\langle E, \sqsubseteq_2 \rangle$, then a continuous mapping (i.e. a finitary functor) $f : D \longrightarrow E$ is fully described by the following binary relation $R \subseteq A \times B$: $A = D_{fin}$, $B = E_{fin}$ are the subposets of compact elements and aRb iff $b \sqsubseteq_2 f(a)$. The relation R satisfies the following conditions:

1. For each $a \in A$ there exists $b \in B$ such that aRb.
2. If aRb, $a \sqsubseteq_1 a'$ and $b' \sqsubseteq_2 b$, then $a'Rb'$.
3. If aRb_1 and aRb_2, then there exists $b \in B$ such that $b_1 \sqsubseteq_2 b$, $b_2 \sqsubseteq_2 b$ and aRb.

Given two posets $\langle A, \sqsubseteq_1 \rangle$, $\langle B, \sqsubseteq_2 \rangle$ having least element, any relation R satisfying the above conditions 1.–3. is called an *approximable relation from A to B*. This gives rise to a continuous map $f : A^* \longrightarrow B^*$ between the ideal completions A^* of A and B^* of B defined by $f(I) = \{b \in B \mid aRb \text{ for some } a \in I\}$ (see [SLG94], Chapter 6 for details).

There is a categorical generalization of relations between (partially ordered) sets — the notion of a distributor. Recall ([Bo94], Volume 1, Section 7.8) that a *distributor*

$$\varphi : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$$

between small categories is a functor $\varphi : \underline{\mathrm{B}}^{op} \times \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{Set}}$. Given distributors $\varphi : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ and $\psi : \underline{\mathrm{B}} \longrightarrow \underline{\mathrm{C}}$, their *composition* is the distributor $\psi \cdot \varphi : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{C}}$ defined on objects as follows:

$$\psi \cdot \varphi(c, a) = \left(\coprod_b \psi(c, b) \times \varphi(b, a) \right) / \sim \tag{4.13}$$

where $\sim$ is the least equivalence with

$$(\psi(1_c, f)(x), y)) \sim (x, \varphi(f, 1_a)(y)) \tag{4.14}$$

for $x \in \psi(c, b)$, $y \in \varphi(b', a)$, $f : b \longrightarrow b'$. We denote the equivalence class of (u, v) by $[u, v]$. The distributor $\psi \cdot \varphi$ is defined on morphisms as follows:

$$\begin{array}{rcl} \psi \cdot \varphi(f, g) : \psi \cdot \varphi(c, a) & \longrightarrow & \psi \cdot \varphi(c', a') \\ [u, v] & \mapsto & [\psi(f, 1_b)(u), \varphi(1_b, g)(v)] \,. \end{array}$$

This composition is associative up to isomorphism. If we define the distributor

$$i_{\underline{\mathrm{A}}} : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{A}} \tag{4.15}$$

to be the hom-functor, then $i_{\underline{\mathrm{A}}}$'s serve as identities (up to isomorphism) for composition of distributors. Thus we obtain a *biquasicategory* DIST of distributors with all small categories as 0-cells, all distributors as 1-cells and all *morphisms between distributors* (i.e. all natural transformations between the respective functors) as 2-cells (see [Bo94]).

Example 4.4.1 Suppose $\langle A, \sqsubseteq_1 \rangle$ and $\langle B, \sqsubseteq_2 \rangle$ are partially ordered sets having least elements. Let R be an approximable relation from A to B. Regard these partially ordered sets as categories and denote them by $\underline{\mathrm{A}}$ and $\underline{\mathrm{B}}$ and define

$$\rho(b, a) = \begin{cases} \{*\}, & \text{if } aRb \\ \emptyset, & \text{otherwise} \end{cases}$$

Condition 2. from the definition of an approximable relation says that ρ is a functor $\rho : \underline{\mathrm{B}}^{op} \times \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{Set}}$, in other words a distributor $\rho : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$. Conditions 1. and 3. of the definition of approximable relation can be summed up as follows: for any $\underline{\mathrm{A}}$-object a, the functor $\rho(\_, a) : \underline{\mathrm{B}}^{op} \longrightarrow \underline{\mathrm{Set}}$ is flat. $\square$

The above example motivates the following definition:

Definition 4.4.2 Let $\underline{A}$, $\underline{B}$ be small categories having initial objects. A distributor $\varphi : \underline{A} \longrightarrow\!\!\!\!\!\!\circ\ \underline{B}$ is called *flat*, if for each $\underline{A}$-object a the functor $\varphi(\_, a) : \underline{B}^{op} \longrightarrow \underline{\mathrm{Set}}$ is flat.

Remark 4.4.3 Flat distributors appear in [JW78] under the name *left flat profunctors.* □

Remark 4.4.4 The one-morphism category $\underline{\mathbf{1}}$ has an initial object and a distributor $\varphi : \underline{\mathbf{1}} \longrightarrow\!\!\!\!\!\!\circ\ \underline{A}$ is flat iff φ is (isomorphic to) a flat functor from $\underline{A}^{op}$ to $\underline{\mathrm{Set}}$. Thus, any generalized domain $\underline{K}$ is equivalent to a category of flat distributors from $\underline{\mathbf{1}}$ to $\underline{K}_{fin}$ and natural transformations between them. □

Lemma 4.4.5 *There is a biquasicategory whose 0-cells are all small categories having initial object, 1-cells are all flat distributors with composition given by (4.13) and identities (4.15) and 2-cells are all morphisms between flat distributors.*

Proof. Let us prove the appropriate axioms of a biquasicategory (cf. 3.3.9).

It is clear that the identity distributor $i_{\underline{A}} : \underline{A} \longrightarrow\!\!\!\!\!\!\circ\ \underline{A}$ is flat.

Let $\varphi : \underline{A} \longrightarrow\!\!\!\!\!\!\circ\ \underline{B}$ and $\psi : \underline{B} \longrightarrow\!\!\!\!\!\!\circ\ \underline{C}$ be flat distributors. Choose any $\underline{A}$-object a. We have to show that the category $\mathsf{Elts}(\psi \cdot \varphi(\_, a))$ is cofiltered.

- $\mathsf{Elts}(\psi \cdot \varphi(\_, a))$ is nonempty — this is clear.
- Let $[u_1, v_1] \in \psi \cdot \varphi(c_1, a)$ and $[u_2, v_2] \in \psi \cdot \varphi(c_2, a)$, that is $u_1 \in \psi(c_1, b_1)$, $v_1 \in \varphi(b_1, a)$ and $u_2 \in \psi(c_2, b_2)$, $v_2 \in \varphi(b_2, a)$ for some $\underline{B}$-objects b_1 and b_2. Since $\varphi(\_, a)$ is flat, there exist a $\underline{B}$-object b, an element $v \in \varphi(b, a)$ and $\underline{B}$-morphisms $f_1 : b_1 \longrightarrow b$ $f_2 : b_2 \longrightarrow b$ such that $v_1 = \varphi(f_1, 1_a)(v)$ and $v_2 = \varphi(f_2, 1_a)(v)$. The following identities hold by (4.14):

$$\begin{aligned} [u_1, v_1] &= [u_1, \varphi(f_1, 1_a)(v)] = [\psi(1_{c_1}, f_1)(u_1), v]\,, \\ [u_2, v_2] &= [u_2, \varphi(f_2, 1_a)(v)] = [\psi(1_{c_2}, f_2)(u_2), v]\,. \end{aligned}$$

  Put $\bar{u}_1 = \psi(1_{c_1}, f_1)(u_1) \in \psi(c_1, b)$ and $\bar{u}_2 = \psi(1_{c_2}, f_2)(u_2) \in \psi(c_2, b)$. Since ψ is flat, there exist a $\underline{C}$-object c, $u \in \psi(c, b)$ and $\underline{C}$-morphisms $g_1 : c_1 \longrightarrow c$ $g_2 : c_2 \longrightarrow c$ such that $\bar{u}_1 = \psi(g_1, 1_b)(u)$ and $\bar{u}_2 = \psi(g_2, 1_a)(u)$. Then $[u, v] \in \psi \cdot \varphi(c, a)$ is the vertex of a cone for $[u_1, v_1]$ and $[u_2, v_2]$ formed by $\psi \cdot \varphi(g_1, 1_a)$ and $\psi \cdot \varphi(g_2, 1_a)$.
- Suppose that $\psi \cdot \varphi(f_1, 1_a)$ and $\psi \cdot \varphi(f_2, 1_a)$ are parallel morphisms from $[u_1, v_1] \in \psi \cdot \varphi(c_1, a)$ to $[u_2, v_2] \in \psi \cdot \varphi(c_2, a)$, where $u_1 \in \psi(c_1, b_1)$, $v_1 \in \varphi(b_1, a)$ and $u_2 \in \psi(c_2, b_2)$, $v_2 \in \varphi(b_2, a)$ for some $\underline{B}$-objects b_1 and b_2.

  Since both $\psi \cdot \varphi(f_1, 1_a)$ and $\psi \cdot \varphi(f_2, 1_a)$ are parallel morphisms from $[u_1, v_1] \in \psi \cdot \varphi(c_1, a)$ to $[u_2, v_2] \in \psi \cdot \varphi(c_2, a)$ we have that

$$\psi \cdot \varphi(f_1, 1_a)\,[u_2, v_2] = \psi \cdot \varphi(f_2, 1_a)\,[u_2, v_2] = [u_1, v_1] \tag{4.16}$$

On the other hand, by (4.14) we have

$$\psi \cdot \varphi(f_1, 1_a)\,[u_2, v_2] = [\psi(f_1, 1_{b_2})(u_2), \varphi(1_{b_2}, 1_a)(v_2)] = [\psi(f_1, 1_{b_2})(u_2), v_2] \tag{4.17}$$

since $\varphi(1_{b_2}, 1_a)(v_2) = v_2$. Similarly the identities

$$\psi \cdot \varphi(f_2, 1_a)\,[u_2, v_2] = [\psi(f_2, 1_{b_2})(u_2), \varphi(1_{b_2}, 1_a)(v_2)] = [\psi(f_2, 1_{b_2})(u_2), v_2] \tag{4.18}$$

hold. Thus if we put $\bar{u}_1 = \psi(f_1, 1_{b_2})(u_2) \in \psi(c_1, b_2)$ and $\bar{u}_2 = \psi(f_2, 1_{b_2})(u_2) \in \psi(c_1, b_2)$, we have that $[\bar{u}_1, v_2] = [\bar{u}_2, v_2]$. By the definition of the equivalence $\sim$ in the composition of distributors there is a $k > 0$ such that there exist

- a finite zig-zag of $\underline{\mathrm{B}}$-morphisms $m_1 : x_0 \longrightarrow x_1$, $m_2 : x_2 \longrightarrow x_1$, $\ldots$, $m_{2k} : x_{2k} \longrightarrow x_{2k-1}$ with $x_0 = x_{2k} = b_2$,
- a k-tuple $g_i \in \varphi(x_i, a)$ with $g_0 = g_{2k} = v_2$,
- a k-tuple $h_i \in \varphi(c_1, x_i)$ with $h_0 = \bar{u}_1$, $h_{2k} = \bar{u}_2$,

such that the corresponding m_i's and g_i's form a finite non-empty diagram D in $\mathsf{Elts}(\varphi(\_, a))$ and the corresponding m_i's and h_i's form a finite non-empty diagram in $\mathsf{Elts}(\psi(c_1, \_))$.

Since the distributor φ is flat, the diagram D has a compatible cone in $\mathsf{Elts}(\varphi(\_, a))$. Explicitly, there are a $\underline{\mathrm{B}}$-object $\bar{b}$, an element $\bar{v} \in \varphi(\bar{b}, a)$ and a k-tuple of $\underline{\mathrm{B}}$-morphisms $\bar{g}_i : x_i \longrightarrow \bar{b}$ such that the corresponding m_i's and $\bar{g}_i$'s commute in $\underline{\mathrm{B}}$.

Since $g_0 = g_{2k} = v_2$, $\varphi(\bar{g}_0, 1_a)$ and $\varphi(\bar{g}_{2k}, 1_a)$ form a parallel pair of morphisms in $\mathsf{Elts}(\varphi(\_, a))$ from $\bar{v} \in \varphi(\bar{b}, a)$ to $v_2 \in \varphi(b_2, a)$. Since φ is flat, these two morphisms can be equalized in $\mathsf{Elts}(\varphi(\_, a))$. Explicitly, there are a $\underline{\mathrm{B}}$-object b, a $\underline{\mathrm{B}}$-morphism $g : \bar{b} \longrightarrow b$ and $v \in \varphi(b, a)$ such that

$$\bar{v} = \varphi(g, 1_a)(v) \tag{4.19}$$

and

$$g \cdot \bar{g}_0 = g \cdot \bar{g}_{2k} \tag{4.20}$$

Define

$$\hat{u} = \psi(1_{c_1}, g \cdot \bar{g}_0)(\bar{u}_1) = \psi(1_{c_1}, g \cdot \bar{g}_{2k})(\bar{u}_2) \in \psi(c_1, b) \tag{4.21}$$

and

$$\check{u} = \psi(1_{c_2}, g \cdot \bar{g}_0)(u_2) \in \psi(c_2, b) \tag{4.22}$$

By definition, $\psi(f_1, 1_b)$ and $\psi(f_2, 1_b)$ are parallel morphisms from $\hat{u} \in \psi(c_1, b)$ to $\check{u} \in \psi(c_2, b)$ in $\mathsf{Elts}(\psi(\_, b))$. Since ψ is flat, these two morphisms can be equalized in $\mathsf{Elts}(\psi(\_, b))$. Explicitly, there are a $\underline{\mathrm{C}}$-object c, a $\underline{\mathrm{C}}$-morphism $f : c_2 \longrightarrow c$ and $u \in \psi(c, b)$ such that

$$\check{u} = \psi(f, 1_b)(u) \tag{4.23}$$

and

$$f \cdot f_1 = f \cdot f_2 \tag{4.24}$$

We will prove that $\psi \cdot \varphi(f, 1_a)$ equalizes the pair $\psi \cdot \varphi(f_1, 1_a)$, $\psi \cdot \varphi(f_2, 1_a)$ in $\mathsf{Elts}(\psi \cdot \varphi(\_, a))$. It suffices to show that $\psi \cdot \varphi(f, 1_a)$ is a morphism from $[u, v] \in \psi \cdot \varphi(c, a)$ to $[u_2, v_2] \in \psi \cdot \varphi(c_2, a)$, i.e. that

$$\psi \cdot \varphi(f, 1_a)\,[u, v] = [u_2, v_2] \tag{4.25}$$

holds.

This is true, since $\psi \cdot \varphi(f, 1_a)\,[u, v] = [\psi(f, 1_b)(u), v]$ by (4.14). The equation $[\psi(f, 1_b)(u), v] = [\check{u}, v]$ holds by (4.23) and we have that $[\check{u}, v] = [\psi(1_{c_2}, g \cdot \bar{g}_0)(u_2), v]$ holds by (4.22). By (4.14) the identity $[\psi(1_{c_2}, g \cdot \bar{g}_0)(u_2), v] = [u_2, \varphi(g \cdot \bar{g}_0, 1_a)(v)]$ holds. Finally, by (4.19), we have $[u_2, \varphi(g \cdot \bar{g}_0, 1_a)(v)] = [u_2, v_2]$, thus (4.25) holds.

The associativity isomorphisms (cf. (3.13))

$$\alpha_{\varphi,\psi,\vartheta} : \vartheta \cdot (\psi \cdot \varphi) \longrightarrow (\vartheta \cdot \psi) \cdot \varphi \tag{4.26}$$

for flat distributors $\varphi : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$, $\psi : \underline{\mathrm{B}} \longrightarrow \underline{\mathrm{C}}$, $\vartheta : \underline{\mathrm{C}} \longrightarrow \underline{\mathrm{D}}$ are defined pointwise as follows: an element $[h, [g, f]] \in \vartheta \cdot (\psi \cdot \varphi)(d, a)$ is sent to $[[h, g], f] \in (\vartheta \cdot \psi) \cdot \varphi(d, a)$.

The right and left identity isomorphisms (cf. (3.14))

$$\rho_\varphi : \varphi \cdot i_{\underline{\mathrm{A}}} \longrightarrow \varphi \quad \text{and} \quad \lambda_\varphi : i_{\underline{\mathrm{B}}} \cdot \varphi \longrightarrow \varphi \tag{4.27}$$

for $\varphi : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ are defined pointwise as follows: ρ_φ sends $[g, f] \in \varphi \cdot i_{\underline{\mathrm{A}}}(b, a)$ to $g \cdot f \in \varphi(b, a)$. Analogously, λ_φ sends $[g, f] \in i_{\underline{\mathrm{B}}} \cdot \varphi(b, a)$ to $g \cdot f \in \varphi(b, a)$.

We omit the routine calculations which show that the equalities (3.13) and (3.14) hold for α, ρ and λ. □

The *biquasicategory of flat distributors and their morphisms* from the last lemma is denoted by FLAT.

Notation 4.4.6 Suppose that $\underline{\mathrm{K}}$, $\underline{\mathrm{L}}$ are generalized domains and that $F : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{L}}$ is a finitary functor. Then we denote by F_* the following flat distributor $F_* : \underline{\mathrm{K}}_{fin} \longrightarrow \underline{\mathrm{L}}_{fin}$:

Define $F_*(b, a) = \underline{\mathrm{L}}(b, Fa)$ on objects. F_* is defined on morphisms as follows: for $f : b' \longrightarrow b$ in $\underline{\mathrm{L}}_{fin}$, and $g : a \longrightarrow a'$ in $\underline{\mathrm{K}}_{fin}$, the function $F_*(f, g) : F_*(b, a) \longrightarrow F_*(b', a')$ assigns $Fg \cdot h \cdot f$ to $h \in F_*(b, a)$.

Since the category $\mathsf{Elts}(F_*(\_, a))^{op}$ is isomorphic to the filtered category J/Fa, the distributor F_* is flat. □

Notation 4.4.7 Suppose that $\underline{\mathrm{A}}$, $\underline{\mathrm{B}}$ are small categories having initial object and that $\varphi : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ is a flat distributor. Then we denote by F_φ a left Kan extension of the following functor $G : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}^*$ along the full embedding of $\underline{\mathrm{A}}$ into $\underline{\mathrm{A}}^*$:

$G(a) = \varphi(\_, a)$ on objects. Since φ is flat, $G(a)$ is a $\underline{\mathrm{B}}^*$-object. Any morphism $f : a \longrightarrow a'$ induces a natural transformation $Gf : Ga \longrightarrow Ga'$ defined pointwise as the map $(Gf)_b : \varphi(b, a) \longrightarrow \varphi(b, a')$ sending $x \in \varphi(b, a)$ to $\varphi(1_b, f)(x) \in \varphi(b, a')$. G is clearly a functor.

By definition, F_φ is a finitary functor. □

We will prove that the processes described in 4.4.6 and 4.4.7 are essentially inverse to each other. Recall that GDOM denotes the 2-quasicategory having all generalized domains as 0-cells, all finitary functors as 1-cells and all natural transformations as 2-cells.

Theorem 4.4.8 (Representation of Finitary Functors)
GDOM *and* FLAT *are biequivalent.*

Proof. Recall that we have chosen for each generalized domain $\underline{\mathrm{K}}$ a fixed small category $\underline{\mathrm{K}}_{fin}$ representing all finitely presentable objects of $\underline{\mathrm{K}}$.

Define a pseudofunctor $\Phi : \mathsf{GDOM} \longrightarrow \mathsf{FLAT}$ on 0-cells by $\Phi(\underline{\mathrm{K}}) = \underline{\mathrm{K}}_{fin}$.

The functor

$$\Phi_{\underline{\mathrm{K}},\underline{\mathrm{L}}} : \mathsf{GDOM}(\underline{\mathrm{K}}, \underline{\mathrm{L}}) \longrightarrow \mathsf{FLAT}(\Phi(\underline{\mathrm{K}}), \Phi(\underline{\mathrm{L}}))$$

is defined as follows: $\Phi_{\underline{\mathrm{K}},\underline{\mathrm{L}}}(\tau : F \Longrightarrow G) = \tau_* : F_* \Longrightarrow G_*$, where F_* and G_* are defined as in 4.4.6 and $\tau_*(b,a) : F_*(b,a) \longrightarrow G_*(b,a)$ is the map sending every $\underline{\mathrm{L}}$-morphism $f : b \longrightarrow Fa$ to $\tau_a \cdot f : b \longrightarrow Ga$. Then the collection $\tau_* = (\tau_*(b,a))$ is a natural transformation and the assignment $\Phi_{\underline{\mathrm{K}},\underline{\mathrm{L}}}$ is clearly a functor.

Next, we will show that Φ bears a structure of a pseudofunctor.

1. For any pair of 1-cells $F : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{L}}$ and $G : \underline{\mathrm{L}} \longrightarrow \underline{\mathrm{M}}$, the natural isomorphism

$$\varphi_{F,G} : \Phi_{\underline{\mathrm{L}},\underline{\mathrm{M}}}(G)\Phi_{\underline{\mathrm{K}},\underline{\mathrm{L}}}(F) \Longrightarrow \Phi_{\underline{\mathrm{K}},\underline{\mathrm{L}}}(GF)$$

   is defined as follows: $\varphi_{F,G}(c,a) : G_*F_*(c,a) \longrightarrow (GF)_*(c,a)$ is the map sending the equivalence class $[g : c \longrightarrow Gb, f : b \longrightarrow Fa]$ to $Gf \cdot g : c \longrightarrow GFa$. This is a bijection since any $h : c \longrightarrow GFa$ factors as $Gf \cdot g$ for some $g : c \longrightarrow Gb$ and $f : b \longrightarrow Fa$. Moreover, if h factors as $Gf_1 \cdot g_1$ for some $g_1 : c \longrightarrow Gb_1$ and $f_1 : b_1 \longrightarrow Fa$, then $[g_1, f_1] = [g, f]$, since the factorization is essentially unique. It is clear that the collection $(\varphi_{F,G}(c,a))$ constitutes a natural transformation.

2. For any 0-cell $\underline{\mathrm{K}}$ in, define the natural isomorphism

$$\psi_{\underline{\mathrm{K}}} : 1_{\Phi(\underline{\mathrm{K}})} \Longrightarrow \Phi(1_{\underline{\mathrm{K}}})$$

   pointwise as the identity morphism: $\psi_{\underline{\mathrm{K}}}(a,a') : \underline{\mathrm{K}}_{fin}(a,a') \longrightarrow \underline{\mathrm{K}}(a,a')$ (we regard the embedding of $\underline{\mathrm{K}}_{fin}$ in $\underline{\mathrm{K}}$ as an actual inclusion).

3. The coherence conditions for φ's and ψ's. Recall that FLAT is a biquasicategory and that in 4.4.5 we have defined

   the associativity isomorphisms (cf. (4.26)) $\alpha_{\varphi,\psi,\vartheta} : \vartheta \cdot (\psi \cdot \varphi) \longrightarrow (\vartheta \cdot \psi) \cdot \varphi$,

   the identity isomorphisms (cf. (4.27)) $\rho_\varphi : \varphi \cdot i_{\underline{\mathrm{A}}} \longrightarrow \varphi$ and $\lambda_\varphi : i_{\underline{\mathrm{B}}} \cdot \varphi \longrightarrow \varphi$.

   Associativity coherence: for any 1-cells $F : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{L}}$, $G : \underline{\mathrm{L}} \longrightarrow \underline{\mathrm{M}}$, $H : \underline{\mathrm{M}} \longrightarrow \underline{\mathrm{N}}$ in

GDOM we are to prove that the diagram (cf. (3.15))

$$\begin{tikzcd}
H_*(G_*F_*) \arrow[r, "H_*\varphi_{F,G}"] \arrow[d, "\alpha_{F_*,G_*,H_*}"'] & H_*(GF)_* \arrow[dd, "\varphi_{GF,H}"] \\
(H_*G_*)F_* \arrow[d, "\varphi_{G_*,H_*}F_*"'] & \\
(HG)_*F_* \arrow[r, "\varphi_{F,HG}"'] & (HGF)_*
\end{tikzcd} \tag{4.28}$$

commutes. This is straightforward: let $[h,[g,f]]$ be any element in $(H_*(G_*F_*))(d,a)$. By definitions, the following square:

$$\begin{tikzcd}
{[h,[g,f]]} \arrow[r, mapsto, "(H_*\varphi_{F,G})_{d,a}"] \arrow[d, mapsto, "(\alpha_{F_*,G_*,H_*})_{d,a}"'] & {[h,Gf\cdot g]} \arrow[dd, mapsto, "(\varphi_{GF,H})_{d,a}"] \\
{[[h,g],f]} \arrow[d, mapsto, "(\varphi_{G_*,H_*}F_*)_{d,a}"'] & \\
{[Hg\cdot h,f]} \arrow[r, mapsto, "(\varphi_{F,HG})_{d,a}"'] & HGf\cdot Hg\cdot h
\end{tikzcd} \tag{4.29}$$

commutes.

Identity coherences: we are to prove that the following diagrams

$$\begin{tikzcd}
F_*\cdot 1_{\underline{\mathrm{A}}_{fin}} \arrow[r, "F_*\psi_{\underline{\mathrm{K}}}"] \arrow[d, "\rho_{F_*}"'] & F_*\cdot(1_{\underline{\mathrm{K}}})_* \arrow[d, "\varphi_{1_{\underline{\mathrm{K}}},F}"] \\
F_* \arrow[r, "1"'] & (F\cdot 1_{\underline{\mathrm{K}}})_*
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{\underline{\mathrm{L}}_{fin}}\cdot F_* \arrow[r, "\psi_{\underline{\mathrm{L}}}F_*"] \arrow[d, "\lambda_{F_*}"'] & (1_{\underline{\mathrm{L}}})_*\cdot F_* \arrow[d, "\varphi_{F,1_{\underline{\mathrm{L}}}}"] \\
F_* \arrow[r, "1"'] & (1_{\underline{\mathrm{L}}}\cdot F)_*
\end{tikzcd} \tag{4.30}$$

commute for any 1-cell $F:\underline{\mathrm{K}}\longrightarrow\underline{\mathrm{L}}$ in GDOM. We will prove the right identity coherence, the left one is verified similarly. Pick up any element $[g,f]$ in $F_*\cdot i_{\underline{\mathrm{K}}_*}(b,a)$. Then the square

$$\begin{tikzcd}
{[g,f]} \arrow[r, mapsto, "(F_*\psi_{\underline{\mathrm{K}}})_{b,a}"] \arrow[d, mapsto, "(\rho_{F_*})_{b,a}"'] & {[g,f]} \arrow[d, mapsto, "(\varphi_{1_{\underline{\mathrm{K}}},F})_{b,a}"] \\
Ff\cdot g \arrow[r, equal] & Ff\cdot g
\end{tikzcd} \tag{4.31}$$

commutes by definitions.

It remains to prove that the triple (Φ,φ,ψ) is a biequivalence.
Clearly if we take any 0-cell $\underline{\mathrm{A}}$ in FLAT then it is equivalent to the category of the form $\underline{\mathrm{K}}_{fin}$, where $\underline{\mathrm{K}}=\underline{\mathrm{A}}^*$. Thus, Condition (a) in the definition of biequivalence is satisfied.

To prove that Condition (b) is satisfied, we have to show that $\Phi_{\underline{\mathrm{K}},\underline{\mathrm{L}}}$ is essentially onto on objects and that it is full and faithful.

Pick up any flat distributor $\varphi : \underline{\mathrm{K}}_{fin} \longrightarrow \underline{\mathrm{L}}_{fin}$. The process of 4.4.7 gives rise to a finitary functor $F_\varphi : (\underline{\mathrm{K}}_{fin})^* \longrightarrow (\underline{\mathrm{L}}_{fin})^*$. Then φ and $(F_\varphi)_*$ are isomorphic. In fact, by definition, $(F_\varphi)_*(b,a) = (\underline{\mathrm{L}}_{fin})^*(b, F_\varphi a)$. Define the map

$$\begin{array}{rcccl} \tau_{b,a} & : & (\underline{\mathrm{L}}_{fin})^*(b, F_\varphi a) & \longrightarrow & \varphi(b,a) \\ & & f : b \longrightarrow F_\varphi a & \mapsto & f_b(1_b) \end{array}$$

(we have made use of the fact that b is identified with a hom-functor $(\underline{\mathrm{L}}_{fin})^*(\_, b)$, $F_\varphi a = \varphi(\_, a)$ and that f is a natural transformation). Then $\tau_{b,a}$ is a bijection by the Yoneda lemma and the collection $\tau = (\tau_{b,a})$ is a natural transformation.

To prove that $\Phi_{\underline{\mathrm{K}},\underline{\mathrm{L}}}$ is full and faithful, it suffices to show that any 2-cell $\tau : F_* \longrightarrow G_*$ is of the form σ_* for a unique natural transformation $\sigma : F \longrightarrow G$. For any $\underline{\mathrm{K}}_{fin}$-object a, the morphism $\tau(\_, a)$ forms a natural transformation, which by the Yoneda lemma corresponds bijectively to a morphism $\sigma_a : Fa \longrightarrow Ga$. It is easy to see that the collection $\sigma = (\sigma_a)$ is a natural transformation and that $\sigma_* = \tau$. □

Corollary 4.4.9 *The biequivalence Ψ of Theorem 4.4.8 can be restricted to a biequivalence of* SC *and* FLAT_{FCC} *— the bicategory of all small FCC categories, all flat distributors and all morphisms between them.*

Remark 4.4.10 The proofs in this section do not use any special use of the cardinal $\aleph_0$. In fact, they can be used verbatim with $\aleph_0$ replaced by any regular cardinal λ. Thus, if we define:

> A distributor $\varphi : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ between small categories having initial object is *λ-flat* if the functor $\varphi(\_, a)$ is λ-flat for any $\underline{\mathrm{A}}$-object a.
>
> λ-FLAT is a *biquasicategory* of all λ-flat distributors.
>
> λ-$\mathsf{ACC}_\perp$ is the *2-quasicategory* of all λ-accessible categories having initial object, all λ-accessible functors and all natural transformations.

we obtain the following corollary. □

Corollary 4.4.11 *Let λ be a regular cardinal. Then λ-$\mathsf{ACC}_\perp$ is biequivalent to λ-FLAT.*

Remark 4.4.12 Let us make a few remarks on generalizations of DCPOs.

A straightforward categorical generalization of a DCPO is the notion of a category having small filtered colimits. It is easy to see that one can form a 2-quasicategory FILT with all categories having filtered colimits as 0-cells, all finitary functors as 1-cells and all natural transformations as 2-cells.

Analogously we can define FILT^l as the 2-quasicategory of all categories having small filtered colimits as 0-cells, all left adjoint functors with finitary right adjoints as 1-cells and all natural transformations as 2-cells.

A category having filtered colimits is a concept resembling the notion of an ω-category of Lehmann ([Leh76]). What is the relationship of these concepts?

Since any ω-chain is directed, any category with filtered colimits is an ω-category. The converse is not true: a category has filtered colimits iff it has colimits of chains of an arbitrary length ([AR94], Corollary 1.7).

Also, a generalization of a pointed DCPO is straightforward: a category which has filtered colimits and has an initial object is called an *inductive category*. (The name "inductive category" has been proposed by Paul Taylor in [Tay87].)

There is a lot of examples of inductive categories:

1. Any pointed DCPO is an inductive category.
2. Any generalized domain is inductive.
3. Any SC category is inductive.
4. Any locally finitely presentable category is inductive.
5. Any cocomplete category is inductive.

It is clear that one can define 2-quasicategories:

1. IND of all inductive categories, all finitary functors and all natural transformations,
2. IND^l of all inductive categories, all left adjoints with finitary right adjoints and all natural transformations,

in a similar way as has already been done for e.g. generalized domains. It should be emphasised, though, that there is *no* analogy to Theorems 4.3.12 and 4.4.8 here. □

4.5 Permanence Properties of Generalized Domains

In Domain Theory one is often interested in the following question: under which constructions for posets is the given category $\underline{\mathrm{D}}$ of domains closed? We call these results permanence properties.

In our setting domains are categories, thus it is natural to ask under which categorical constructions in the 2-quasicategory CAT a given 2-quasicategory of domains is closed. We are interested especially in limit constructions, since domains can behave rather badly w.r.t. colimits in CAT as the following example shows:

Example 4.5.1 A coproduct of two non-empty generalized domains is not a generalized domain. □

Another reason why we are interested in limit constructions is the fact that constructions using *finitary closures* and *finitary kernels* are also instances of a (rather general) notion of a limit. We first give classical definitions (cf. [CCL80], Chapter 0, Definition 3.8).

Definition 4.5.2 Suppose $\langle X, \sqsubseteq \rangle$ is a poset and $f : X \longrightarrow X$ is a monotone map with $f = f \cdot f$.

The mapping f is called a *finitary closure* if it preserves directed sups and $1_X \sqsubseteq f$ (pointwise).

The mapping f is called a *finitary kernel* if it preserves directed sups and $f \sqsubseteq 1_X$ (pointwise).

Given a domain $\langle X, \sqsubseteq \rangle$ and a finitary closure f on $\langle X, \sqsubseteq \rangle$ one can ask whether the image of f a domain again. Before we give a partial answer, we present a categorical generalization of finitary projections and finitary closures.

Definition 4.5.3 A monad $\mathbb{T} = (T, \eta, \mu)$ is called a *finitary* if T is a finitary functor. A comonad $\mathbb{G} = (G, \varepsilon, \delta)$ is called a *finitary* if G is a finitary functor.

The obvious generalizations of closures and projections are:

finitary closure	$\mapsto$	finitary monad
finitary kernel	$\mapsto$	finitary comonad

It is clear that the study of e.g. the image of a finitary closure on a poset generalizes to the study of the category of Eilenberg-Moore algebras of a finitary monad.

Since the categories of Eilenberg-Moore (co)algebras are a special instance of a very general limit construction, we are going to study the existence of certain limits (namely, indexed (bi)limits — recall 3.3.24) of generalized domains.

Recall the 2-category Sketch of all small sketches, all sketch morphisms and all natural transformations (Definition 3.5.11). Denote by $U : \mathsf{Sketch} \longrightarrow \mathsf{Cat}$ the obvious forgetful 2-functor.

Throughout this section, $\mathcal{S}$ is going to be any full sub-2-category of Sketch. The restriction of U to $\mathcal{S}$ is denoted by $U_{\mathcal{S}}$.

The following definitions are inspired by the concept of a *topological category* — see [AHS90], Chapter VI. In fact, much of the theory of topological categories over $\underline{\mathsf{Set}}$ can be generalized to "topological" 2-categories over Cat. We do not need the full strength of such a theory but it might be worthwhile to study topological categories in the enriched context. Compare the following definition with Definition 10.41 of [AHS90].

Definition 4.5.4 Suppose D is a small 2-category, $\Gamma : \mathsf{D} \longrightarrow \mathsf{Sketch}$, $W : \mathsf{D} \longrightarrow \mathsf{Cat}$ are 2-functors.

1. Any $(W, \underline{\mathbb{S}})$-cylinder
$$\gamma : W \Longrightarrow \mathsf{Cat}(U_{\mathcal{S}} \cdot \Gamma(\_), \underline{\mathbb{S}})$$
over $U_{\mathcal{S}} \cdot \Gamma$ is called a *$U_{\mathcal{S}}$-structured $(W, \underline{\mathbb{S}})$-cylinder over* Γ.

2. A $(W, \underline{\mathbb{S}})$-cylinder
$$\tau : W \Longrightarrow \mathcal{S}(\Gamma(\_), \mathbb{S})$$
over Γ is called an *$U_{\mathcal{S}}$-lift of* γ, if $U_{\mathcal{S}}(\mathbb{S}) = \underline{\mathbb{S}}$ and $U_{\mathcal{S}}(\tau) = \gamma$.

3. A $(W, \underline{\mathbb{S}})$-cylinder
$$\tau : W \Longrightarrow \mathcal{S}(\Gamma(\_), \mathbb{S})$$

over Γ is called an *$U_{\mathcal{S}}$-final lift of* γ, if τ is a $U_{\mathcal{S}}$-lift of γ and the following holds: a functor $F : U_{\mathcal{S}}(\mathbb{S}) \longrightarrow U_{\mathcal{S}}(\mathbb{S}')$ is of the form $U_{\mathcal{S}}(\bar{F})$ for a (neccessarily unique) functor $\bar{F} : \mathbb{S} \longrightarrow \mathbb{S}'$, whenever $F \cdot \gamma = U_{\mathcal{S}}$ for some $\sigma : W \Longrightarrow \mathcal{S}(\Gamma(\_), \mathbb{S}')$.

Lemma 4.5.5 Sketch *has unique U-final lifts of U-structured $(W, \underline{\mathrm{S}})$-cylinders over Γ.*

Proof. (essentially [MP89], Proposition 5.1.4) Suppose $\gamma : W \Longrightarrow \mathsf{Cat}(U \cdot \Gamma(\_), \underline{\mathrm{S}})$ is a U-structured $(W, \underline{\mathrm{S}})$-cylinder over Γ. Define the sketch $\mathbb{S} = (\underline{\mathrm{S}}, P_\gamma, Q_\gamma)$ as follows:

- cones in P_γ are images of distinguished cones in the sketch $\Gamma(D)$ under functors $\gamma_D(X) : U\Gamma(D) \longrightarrow \underline{\mathrm{S}}$ for all 0-cells D in D and all $W(D)$-objects X,
- the set of of cocones in Q_γ is defined similarly.

We have clearly defined a U-final lift τ of γ. □

Definition 4.5.6 A full sub-2-category $\mathcal{S}$ of Sketch is called *finally closed*, if U-final lifts of U-structured $(W, \underline{\mathrm{S}})$-cylinders over Γ have vertices in $\mathcal{S}$ for all small categories $\underline{\mathrm{S}}$ and all 2-functors $\Gamma : \mathsf{D} \longrightarrow \mathcal{S}$, $W : \mathsf{D} \longrightarrow \mathsf{Cat}$.

Lemma 4.5.7 *If $\mathcal{S}$ is finally closed in* Sketch*, then $\mathcal{S}$ has all small indexed colimits and $U_{\mathcal{S}}$ preserves them.*

Proof. See [MP89], Proposition 5.1.4. □

Let λ be a regular cardinal. Following [Ag88] we call $\mathcal{S}$ a *λ-variety of sketches*, if all distinguished cones of each sketch in $\mathcal{S}$ are of cardinality $< \lambda$. Denote by $\mathsf{MOD}(\mathcal{S})$ the full sub-2-quasicategory of λ-ACC, whose objects are all categories of models of a sketch in $\mathcal{S}$.

Lemma 4.5.8 *Suppose that $\mathcal{S}$ is a λ-variety of sketches finally closed in* Sketch*. Then* $\mathsf{MOD}(\mathcal{S})$ *is closed in* CAT *under small indexed bilimits.*

Example 4.5.9 It is straightforward that the following λ-varieties of sketches are finally closed in Sketch:

1. $\mathcal{S}_1$: sketches with no specified cocones. Then $\mathsf{MOD}(\mathcal{S}_1)$ is precisely the 2-quasicategory of all locally λ-presentable categories, all λ-accessible functors and all natural transformations.
2. $\mathcal{S}_2$: sketches with discrete specified cocones. Then $\mathsf{MOD}(\mathcal{S}_2)$ is precisely the 2-quasicategory of all locally λ-multipresentable categories (see [AR94], Definition 4.28), all λ-accessible functors and all natural transformations.
3. $\mathcal{S}_3$: sketches with specified finite cones and empty cocones. Then $\mathsf{MOD}(\mathcal{S}_3)$ is precisely the 2-quasicategory SC of all Scott complete categories, all finitary functors and all natural transformations.

□

Thus we obtain the following corollaries:

Corollary 4.5.10 *Each of the following 2-quasicategories is closed under indexed bilimits in* CAT*:*

1. *The 2-quasicategory of all locally λ-presentable categories, all λ-accessible functors and all natural transformations.*
2. *The 2-quasicategory of all locally λ-multipresentable categories, all λ-accessible functors and all natural transformations.*
3. *The 2-quasicategory* SC *of all Scott complete categories, all finitary functors and all natural transformations.*

Corollary 4.5.11 *Let* $\underline{\mathrm{K}}$ *be a category,* $\mathbb{T}$ *a monad on* $\underline{\mathrm{K}}$*,* $\mathbb{G}$ *a comonad on* $\underline{\mathrm{K}}$*. Then the following hold:*

1. *If* $\underline{\mathrm{K}}$ *is locally λ-presentable and* $\mathbb{T}$ *and* $\mathbb{G}$ *are λ-accessible, then both* $\underline{\mathrm{K}}^{\mathbb{T}}$ *and* $\underline{\mathrm{K}}_{\mathbb{G}}$ *are locally λ-presentable categories.*
2. *If* $\underline{\mathrm{K}}$ *is locally λ-multipresentable and* $\mathbb{T}$ *and* $\mathbb{G}$ *are λ-accessible, then both* $\underline{\mathrm{K}}^{\mathbb{T}}$ *and* $\underline{\mathrm{K}}_{\mathbb{G}}$ *are locally λ-multipresentable categories.*
3. *If* $\underline{\mathrm{K}}$ *is Scott complete,* $\mathbb{T}$ *and* $\mathbb{G}$ *are finitary and then both* $\underline{\mathrm{K}}^{\mathbb{T}}$ *and* $\underline{\mathrm{K}}_{\mathbb{G}}$ *are Scott complete categories.*

Chapter 5

Cocompletions of Categories

In Chapter 6 we are going to present a categorical generalization of another type of a domain — namely that of a *continuous domain*. To define the concept of a continuous category one has to extend the notion of a *free cocompletion of a category w.r.t. small filtered colimits*. Recall that so far we have worked with a free cocompletion $\underline{A}^*$ with respect to small filtered colimits of a *small* category $\underline{A}$. Categories equivalent to $\underline{A}^*$ for a small $\underline{A}$ are precisely the finitely accessible categories. In Section 5.1 we are going to generalize the notion of a free cocompletion in two directions:

1. We allow *arbitrary* categories, not just small categories, to be cocompleted.

2. We want certain colimits to be preserved by the cocompletion.

We show that every category can be cocompleted such that a prescribed class of colimits is preserved by the cocompletion — Theorem 5.1.12. We also show in Section 5.2 that cocompletions can be described in a way which resembles cocompletions of posets by means of ideals. Finally, in Section 5.4, we recall from Expose I of [SGA4] the description of a free cocompletion of a category w.r.t. small filtered colimits which we will use in Chapter 6.

5.1 Free $\mathcal{F}$-Conservative $\mathcal{C}$-Cocompletions

A free cocompletion of a category $\underline{X}$ w.r.t. a given class of colimits solves the problem of cocompleting that category; but existing colimits of a given type in $\underline{X}$ are "destroyed" in the free cocompletion. Our goal is to give a cocompletion which retains some class of existing colimits in the category $\underline{X}$.

Notation 5.1.1 Let $\mathcal{F}$ and $\mathcal{C}$ be classes of small categories. For every category $\underline{X}$ we denote by

$$I_{\underline{X}} : \underline{X} \longrightarrow \mathcal{C}\text{-}\mathsf{Cocompl}(\underline{X})_{\mathcal{F}}$$

a *free $\mathcal{F}$-conservative $\mathcal{C}$-cocompletion of* $\underline{X}$, i.e., a $\mathcal{C}$-cocomplete category $\mathcal{C}\text{-}\mathsf{Cocompl}(\underline{X})_{\mathcal{F}}$ and a full and faithful functor $I_{\underline{X}}$ preserving $\mathcal{F}$-colimits with the following universal property:

for every functor $H : \underline{\mathrm{X}} \longrightarrow \underline{\mathrm{Y}}$ to a $\mathcal{C}$-cocomplete category $\underline{\mathrm{Y}}$ and such that H preserves $\mathcal{F}$-colimits, there is a unique (up to a natural isomorphism) functor $H^+ : \mathcal{C}\text{-}\mathsf{Cocompl}(\underline{\mathrm{X}})_{\mathcal{F}} \longrightarrow \underline{\mathrm{Y}}$ which preserves $\mathcal{C}$-colimits and satisfies $H^+ \cdot I_{\underline{\mathrm{X}}} = H$.

In case when $\mathcal{F} = \emptyset$ we write $\mathcal{C}\text{-}\mathsf{Cocompl}(\underline{\mathrm{X}})$ instead of $\mathcal{C}\text{-}\mathsf{Cocomp}(\underline{\mathrm{X}})_{\emptyset}$ and call this category a *free $\mathcal{C}$-cocompletion of* $\underline{\mathrm{X}}$.

In case when $\mathcal{C}$ consists of all small categories we write $\mathsf{Cocompl}(\underline{\mathrm{X}})_{\mathcal{F}}$ instead of $\mathcal{C}\text{-}\mathsf{Cocompl}(\underline{\mathrm{X}})_{\mathcal{F}}$ and call this category a *free $\mathcal{F}$-conservative cocompletion of* $\underline{\mathrm{X}}$. □

Note that if a free conservative cocompletion exists, it is determined uniquely up to an equivalence of categories.

Example 5.1.2 An example of a cocompletion of small categories which retains a class of existing colimits is the concept of a *locally λ-presentable category*, where λ is a regular cardinal — see Remark 3.5.6 for a definition of a locally presentable category. Locally λ-presentable categories are precisely free $\mathcal{F}$-conservative cocompletions of small categories, where $\mathcal{F}$ consists of all categories which have less than λ-morphisms ([AR94], Theorem 1.46). See also item 3. of 5.1.14. □

The main result of this section (Theorem 5.1.12) is as follows:

Given a pair $\mathcal{C}$, $\mathcal{F}$ of classes of small categories, a free $\mathcal{F}$-conservative $\mathcal{C}$-cocompletion exists for any category $\underline{\mathrm{X}}$.

Essential ideas for the proof of this result are taken from [Er86], where cocompletions of posets by different kinds of *ideals* are studied. An ideal in a poset $\langle X, \sqsubseteq\rangle$ is a downward closed subset $I \subseteq X$, i.e. whenever $x \in I$ and $y \sqsubseteq x$, then $y \in I$. [1] The set of all ideals of $\langle X, \sqsubseteq\rangle$ ordered by inclusion forms a complete lattice L. Cocompletions of $\langle X, \sqsubseteq\rangle$ under certain types of colimits then appear as subposets of L (cf. [Er86]). It is easy to see that ideals of $\langle X, \sqsubseteq\rangle$ correspond bijectively to monotone maps from $\langle X, \sqsubseteq\rangle^{op}$ to the two-element chain. Thus, in the poset case, one can work with the complete lattice of certain monotone maps instead of the complete lattice of all ideals. This is the approach that we take for categories. That is, we start with a complete and cocomplete quasicategory of all $\underline{\mathrm{Set}}$-valued contravariant functors on $\underline{\mathrm{X}}$ and we find the desired cocompletion as a full legitimate subcategory. Later in this chapter we indicate how to generalize the notion of an ideal to categories (see Definition 5.2.1) and in Theorem 5.2.12 we give a description of a free $\mathcal{F}$-conservative $\mathcal{C}$-cocompletion of a category using this generalized notion of an ideal.

For any category $\underline{\mathrm{X}}$, let $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]$ denote the quasicategory of all functors $F : \underline{\mathrm{X}}^{op} \longrightarrow \underline{\mathrm{Set}}$ and all natural transformations between them. $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]$ is a legitimate category iff $\underline{\mathrm{X}}$ is (equivalent to) a small category (see [FS95]). In that case, $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]$ is a *free cocompletion of* $\underline{\mathrm{X}}$ *w.r.t. small colimits.* However, given any category $\underline{\mathrm{X}}$, the quasicategory $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]$ has small limits and small colimits, both computed pointwise.

The following notions are standard (see e.g. [AHS90], Section 13):

[1] The word "ideal" is overloaded. The notion of an ideal we use in the current chapter is *different* from the notion of an ideal we gave in 2.1.9.

Definition 5.1.3 Suppose a class $\mathcal{W}$ of small categories is given.

1. A category $\underline{\mathrm{X}}$ is said to be *$\mathcal{W}$-cocomplete* if it has $\mathcal{W}$-colimits, i.e. if for any $\underline{\mathrm{D}} \in \mathcal{W}$ and any diagram $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{X}}$ a colimit of D exists in $\underline{\mathrm{X}}$.

2. A functor $H : \underline{\mathrm{X}} \longrightarrow \underline{\mathrm{Y}}$ is said to *preserve $\mathcal{W}$-colimits* (or *$\mathcal{W}$-cocontinuous*) provided that the following holds for any $\underline{\mathrm{D}} \in \mathcal{W}$: whenever $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{X}}$ is a diagram and $\gamma : D \Longrightarrow x$ is a colimit of D in $\underline{\mathrm{X}}$, then $H(\gamma) : H \cdot D \Longrightarrow Hx$ is a colimit of $H \cdot D$ in $\underline{\mathrm{Y}}$.

Given $\mathcal{W}$, we denote by $\mathcal{W}^{op}$ the class of all small categories $\underline{\mathrm{D}}^{op}$ with $\underline{\mathrm{D}} \in \mathcal{W}$.

Remark 5.1.4 Definitions of dual notions, *$\mathcal{W}$-complete category*, *$\mathcal{W}$-continuous functor* are straightforward.

We will make use of the following notion: given $\mathcal{W}$, we say that a functor $F : \underline{\mathrm{X}} \longrightarrow \underline{\mathrm{Y}}$ *preserves $\mathcal{W}^{op}$-limits* if it preserves limits of all diagrams $D : \underline{\mathrm{D}}^{op} \longrightarrow \underline{\mathrm{X}}$ for all $\underline{\mathrm{D}} \in \mathcal{W}$. □

Denote by $\underline{\mathrm{Class}}$ the quasicategory of all classes and all mappings. For a category $\underline{\mathrm{X}}$ we denote by $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]$ the quasicategory of all functors from $\underline{\mathrm{X}}^{op}$ to $\underline{\mathrm{Class}}$ and all natural transformations.

Notation 5.1.5 For a class $\mathcal{W}$ of small categories and any category $\underline{\mathrm{X}}$ we denote by

$[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]_{\mathcal{W}}$ the full subquasicategory of $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]$ having as objects all functors $F : \underline{\mathrm{X}}^{op} \longrightarrow \underline{\mathrm{Set}}$ which preserve $\mathcal{W}$-limits,

$[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$ the full subquasicategory of $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]$ having as objects all functors $F : \underline{\mathrm{X}}^{op} \longrightarrow \underline{\mathrm{Class}}$ which preserve $\mathcal{W}$-limits.

In case $\mathcal{W}$ is the empty set, we write $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]$ and $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]$ instead of $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]_{\emptyset}$ and $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\emptyset}$ in accordance with our previous notation. □

The following result is due to Freyd, Kelly and Kennison:

Lemma 5.1.6 *Let $\underline{\mathrm{X}}$ be a small category. Let $\mathcal{W}$ be any class of small categories. Then $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]_{\mathcal{W}}$ is a full reflective subcategory of $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]$.*

Proof. See [GU71], Korollar 8.14. □

We will see later that in the construction of a conservative cocompletion of a *small* category $\underline{\mathrm{X}}$, all that is needed is the fact that $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]_{\mathcal{W}}$ is a cocomplete category. This is not straightforward in case of an *arbitrary* category $\underline{\mathrm{X}}$. Observe, however, that Lemma 5.1.6 can be "lifted" to a higher universe in the following sense.

Lemma 5.1.7 *Let $\underline{\mathrm{X}}$ be any category. Let $\mathcal{W}$ be any class of small categories. Then $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$ is a full reflective subquasicategory of $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]$ and therefore it has all (large) limits and colimits.*

Proof. The proof of the lemma (for a more general choice of $\mathcal{W}$) can be found in [Ke82], Section 3.11. □

Definition 5.1.8 Let $\mathcal{W}$ be any class of small categories. Let $\underline{\mathrm{X}}$ be any category considered as a full subcategory of $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$. We denote by $[\![\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]\!]_{\mathcal{W}}$ the closure of $\underline{\mathrm{X}}$ in $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$ under small colimits and we denote by

$$Y_{\underline{\mathrm{X}}} : \underline{\mathrm{X}} \longrightarrow [\![\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]\!]_{\mathcal{W}}$$

the codomain restriction of the Yoneda embedding of $\underline{\mathrm{X}}$ to $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$. Objects of $[\![\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]\!]_{\mathcal{W}}$ are called *$\mathcal{W}$-reachable functors.* In case $\mathcal{W} = \emptyset$, we write $[\![\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]\!]$ instead of $[\![\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]\!]_{\emptyset}$ and call such functors *reachable.*

Remark 5.1.9 Let us make a few comments on the previous definition.

1. Representable functors preserve all limits, therefore we can by the Yoneda lemma identify $\underline{\mathrm{X}}$ with its image $Y(\underline{\mathrm{X}})$ under the Yoneda embedding functor $Y : \underline{\mathrm{X}} \longrightarrow [\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]$ and regard thus $\underline{\mathrm{X}}$ as a full subcategory of $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$.

   By definition, $[\![\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]\!]_{\mathcal{W}}$ has all small colimits, since $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$ is cocomplete by Lemma 5.1.7. It is also clear that the inclusion of $[\![\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]\!]_{\mathcal{W}}$ in $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$ preserves all small colimits.

2. The closure of $\underline{\mathrm{X}}$ in $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$ under small colimits is defined as the smallest full isomorphism-closed subquasicategory of $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$ which contains $\underline{\mathrm{X}}$ and which is closed under small colimits in $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$.

   It follows that closure of $\underline{\mathrm{X}}$ in $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$ under small colimits exists and is uniquely determined:

   Consider all full isomorphism-closed subquasicategories $\underline{\mathrm{K}}$ of $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$ which contain $\underline{\mathrm{X}}$ and which are closed under small colimits in $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$. The conglomerate of all such $\underline{\mathrm{K}}$'s is nonempty and the desired closure is the intersection of this conglomerate.

3. We use the fact that one can construct a full subcategory $\underline{\mathrm{X}}_{\mathcal{W}}$ of $[\![\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]\!]_{\mathcal{W}}$ which is *equivalent* to $[\![\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]\!]_{\mathcal{W}}$ explicitly as a colimit of the following transfinite chain of full embeddings $E_{\alpha,\beta} : \underline{\mathrm{X}}_{\alpha} \longrightarrow \underline{\mathrm{X}}_{\beta}$ (where $\beta \geq \alpha$ are ordinal numbers) — see [Ke82], Section 3.5:

   Define $\underline{\mathrm{X}}_0$ to be the full subquasicategory of $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$ determined by all hom-functors $\underline{\mathrm{X}}(\_, x)$. Note that due to the Yoneda lemma $\underline{\mathrm{X}}_0$ is indeed a legitimate category.

   Suppose that $\alpha > 0$ is an ordinal number and that for all $\beta < \alpha$ the quasicategories $\underline{\mathrm{X}}_{\beta}$ have been defined. For a successor ordinal $\alpha = \gamma + 1$, the quasicategory $\underline{\mathrm{X}}_{\alpha}$ is going to be a full subquasicategory of $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$ determined by all $\underline{\mathrm{X}}_{\gamma}$-objects and *chosen* colimits F_D for all functors $D : \underline{\mathrm{D}} \longrightarrow [\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$ with $\underline{\mathrm{D}}$ a small category, which factor through $\underline{\mathrm{X}}_{\gamma}$. The functors $E_{\beta,\alpha} : \underline{\mathrm{X}}_{\beta} \longrightarrow \underline{\mathrm{X}}_{\alpha}$ for $\beta \leq \alpha$ are

the obvious embeddings. For a limit ordinal α, put $\underline{X}_\alpha$ to be the union of all $\underline{X}_\beta$'s for $\beta < \alpha$. Full embeddings $E_{\alpha,\beta} : \underline{X}_\alpha \longrightarrow \underline{X}_\beta$ are defined in the obvious way.

Define the quasicategory $\underline{X}_{\mathcal{W}}$ as a union of all $\underline{X}_\alpha$'s. Using transfinite induction it is easy to see that $\underline{X}_{\mathcal{W}}$ is a legitimate category and that it is equivalent to $[\![\underline{X}^{op}, \underline{\mathrm{Set}}]\!]_{\mathcal{W}}$.

The above transfinite construction also justifies our notation $[\![\underline{X}^{op}, \underline{\mathrm{Set}}]\!]_{\mathcal{W}}$ for the category of all $\mathcal{W}$-reachable functors; each $\mathcal{W}$-reachable functor is a $\underline{\mathrm{Set}}$-valued functor.

4. The category $\underline{X}_{\mathcal{W}}$ described above has the advantage that we have canonically *chosen* small colimits in $\underline{X}_{\mathcal{W}}$. This choice will be of great importance later.

The final remark is terminological: what we call reachable functors here is called *accessible functors* in [Ke82]. Since accessible functors have been defined as functors preserving filtered colimits (Definition 3.5.7), we prefer a different name — *reachable.* □

Before we establish the existence of a free $\mathcal{F}$-conservative $\mathcal{C}$-cocompletion for any category, let us introduce a useful notion of an *adjoint pair along a functor.*

Definition 5.1.10 Suppose $J : \underline{A} \longrightarrow \underline{B}$, $F : \underline{A} \longrightarrow \underline{X}$, $G : \underline{X} \longrightarrow \underline{B}$ are functors. We say that F *is a left adjoint of* G *along* J and we denote this fact by $F \dashv_J G$, provided there is a bijection of hom-sets

$$\underline{B}(Ja, Gx) \cong \underline{X}(Fa, x)$$

which is natural in x and a.

It is clear, that a left adjoint along a functor is determined uniquely up to a natural isomorphism. The following lemma is proved analogously to the case of an ordinary adjuction (i.e. to the case when $\underline{A} = \underline{B}$ and J is the identity functor):

Lemma 5.1.11 *Suppose that* $J : \underline{A} \longrightarrow \underline{B}$ *and* $G : \underline{X} \longrightarrow \underline{B}$ *are functors. Then the following hold:*

1. *If* $F : \underline{A} \longrightarrow \underline{X}$ *is a functor such that* $F \dashv_J G$*, then* F *preserves any colimit which is preserved by* J*.*

2. *If for each* a *in* $\underline{A}$ *the functor* $\underline{B}(Ja, G\_) : \underline{X} \longrightarrow \underline{\mathrm{Set}}$ *is representable by an* $\underline{X}$*-object* $F_0 a$*, then* F_0 *is an object function of a unique functor* $F : \underline{A} \longrightarrow \underline{X}$ *such that* $F \dashv_J G$*.*

We are now ready to prove our main result. Recall the notation for free $\mathcal{F}$-conservative $\mathcal{C}$-cocompletions (Notation 5.1.1) and recall that $\mathcal{F}^{op}$ denotes the class of all $\underline{D}^{op}$ with $\underline{D} \in \mathcal{F}$.

Theorem 5.1.12 *Let* $\mathcal{C}$ *and* $\mathcal{F}$ *be any classes of small categories and let* $\underline{X}$ *be any category. Then the following hold:*

I. *A free* $\mathcal{F}$*-conservative cocompletion* $\mathsf{Cocompl}(\underline{X})_{\mathcal{F}}$ *of* $\underline{X}$ *exists and it is equivalent to the category* $[\![\underline{X}^{op}, \underline{\mathrm{Set}}]\!]_{\mathcal{F}^{op}}$ *of all* $\mathcal{F}^{op}$*-reachable functors.*

II. A free $\mathcal{F}$-conservative $\mathcal{C}$-cocompletion $\mathcal{C}\text{-}\mathsf{Cocompl}(\underline{\mathrm{X}})_{\mathcal{F}}$ of $\underline{\mathrm{X}}$ exists and it is equivalent to the closure under $\mathcal{C}$-colimits of $\underline{\mathrm{X}}$ in the category $[\![\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]\!]_{\mathcal{F}^{op}}$ of all $\mathcal{F}^{op}$-reachable functors.

Moreover, the full and faithful functor from $\underline{\mathrm{X}}$ to $\mathcal{C}\text{-}\mathsf{Cocompl}(\underline{\mathrm{X}})_{\mathcal{F}}$ which satisfies the universal property can be taken as a codomain restriction of the Yoneda embedding.

Proof. I. Given $\underline{\mathrm{X}}$, let us fix a category $\underline{\mathrm{X}}_{\mathcal{F}^{op}}$, constructed as in 5.1.9, which is a full subcategory of $[\![\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]\!]_{\mathcal{F}^{op}}$ and which is equivalent to $[\![\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]\!]_{\mathcal{F}^{op}}$. It therefore suffices to show that $\underline{\mathrm{X}}_{\mathcal{F}^{op}}$ is a free $\mathcal{F}$-conservative cocompletion of $\underline{\mathrm{X}}$. Recall that $\underline{\mathrm{X}}_{\mathcal{F}^{op}}$ has canonically chosen small colimits (see Remark 5.1.9).

It is well-known that the Yoneda embedding of $\underline{\mathrm{X}}$ to $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{F}^{op}}$ preserves $\mathcal{F}$-colimits (see [Sch70], Satz 10.3.5). Therefore its codomain restriction $Y_{\underline{\mathrm{X}}} : \underline{\mathrm{X}} \longrightarrow \underline{\mathrm{X}}_{\mathcal{F}^{op}}$ preserves $\mathcal{F}$-colimits.

To prove the universal property of $Y_{\underline{\mathrm{X}}}$, it suffices to show that for any cocomplete category $\underline{\mathrm{Y}}$ and any $\mathcal{F}$-cocontinuous functor $H : \underline{\mathrm{X}} \longrightarrow \underline{\mathrm{Y}}$ there exists a unique (up to a natural isomorphism) functor $H^+ : \underline{\mathrm{X}}_{\mathcal{F}^{op}} \longrightarrow \underline{\mathrm{Y}}$ which preserves all small colimits and fulfills $H^+ \cdot Y_{\underline{\mathrm{X}}} = H$.

It is clear that there is at most one functor H^+ which meets the stated requirements. Therefore it suffices to prove the existence of such H^+.

Let us first define a functor $H_+ : \underline{\mathrm{Y}} \longrightarrow [\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]_{\mathcal{F}^{op}}$ as follows:

> $H_+(y) = \underline{\mathrm{Y}}(H\_, y)$ for any $\underline{\mathrm{Y}}$-object y. This definition is correct, since H is assumed to preserve $\mathcal{F}$-colimits. $H_+(y)$, though it is certainly a $\underline{\mathrm{Set}}$-valued functor, need not be an $\mathcal{F}^{op}$-reachable functor in general.
>
> H_+ is defined on $\underline{\mathrm{Y}}$-morphisms in an obvious way: for a $\underline{\mathrm{Y}}$-morphism $f : y \longrightarrow y'$ the natural transformation $H_+(f)$ is $\underline{\mathrm{Y}}(H\_, f) : \underline{\mathrm{Y}}(H\_, y) \Longrightarrow \underline{\mathrm{Y}}(H\_, y')$.

Let us denote by $J : \underline{\mathrm{X}}_{\mathcal{F}^{op}} \longrightarrow [\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]_{\mathcal{F}^{op}}$ the inclusion. Note that the class $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]_{\mathcal{F}^{op}}(J(F), F')$ is a set for any F and F'.

The desired functor $H^+ : \underline{\mathrm{X}}_{\mathcal{F}^{op}} \longrightarrow \underline{\mathrm{Y}}$ is going to be a left adjoint of H_+ along J. Since J preserves colimits by the definition of $\underline{\mathrm{X}}_{\mathcal{F}^{op}}$, the functor H^+ will preserve colimits by Lemma 5.1.11.

Recall from 5.1.9 that $\underline{\mathrm{X}}_{\mathcal{F}^{op}}$ is constructed as a colimit of a transfinite chain of full embeddings $E_{\alpha,\beta} : \underline{\mathrm{X}}_\alpha \longrightarrow \underline{\mathrm{X}}_\beta$. Denote the colimit cocone by $E_\alpha : \underline{\mathrm{X}}_\alpha \longrightarrow \underline{\mathrm{X}}_{\mathcal{F}^{op}}$.

To define H^+ it therefore suffices to define a family of functors

$$H_\alpha : \underline{\mathrm{X}}_\alpha \longrightarrow \underline{\mathrm{Y}}$$

such that $H_\alpha \dashv_{J \cdot E_\alpha} H_+$ for each α.

We proceed by transfinite induction on α.

- Let $\alpha = 0$. Let F be an arbitrary object of $\underline{\mathrm{X}}_0$. Then $F = \underline{\mathrm{X}}(\_, x)$ for some x. Put $H_0(F) = \underline{\mathrm{Y}}(\_, Hx)$. By the Yoneda lemma we have for each y a bijection of sets

$$\begin{aligned} [\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]_{\mathcal{F}^{op}}(\underline{\mathrm{X}}(\_, x), H_+(y)) &= [\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}](\underline{\mathrm{X}}(\_, x), H_+(y)) \cong (H_+(y))(x) \\ &= \underline{\mathrm{Y}}(Hx, y) \end{aligned} \tag{5.1}$$

natural in y by the Yoneda lemma.

Define $B_{F,y} : [\underline{X}^{op}, \underline{Set}](J \cdot E_0(\underline{X}(\_, x)), H_+(y)) \longrightarrow \underline{Y}(H_0(F), y)$ to be the bijection from (5.1). By Lemma 5.1.11 there is a unique functor $H_0 : \underline{X}_0 \longrightarrow \underline{Y}$ such that $H_0 \dashv_{J \cdot E_0} H_+$.

- Suppose that $\alpha = \beta + 1$ and suppose that a functor $H_\beta : \underline{X}_\beta \longrightarrow \underline{Y}$ such that $H_\beta \dashv_{J \cdot E_\beta} H_+$ has been defined.

  We therefore have bijections

$$B_{F',y'} : [\underline{X}^{op}, \underline{Set}](J \cdot E_\beta(F'), H_+(y')) \longrightarrow \underline{Y}(H_\beta(F'), y')$$

  natural in y' and F' for all functors F' from $\underline{X}_\beta$ and all $\underline{Y}$-objects y'.

  Let F be an object of $\underline{X}_\alpha$ which is not in $\underline{X}_\beta$. By assumption, F is a chosen colimit F_D for a small diagram $D : \underline{D} \longrightarrow \underline{X}_\beta$. Let $\kappa_d : Dd \Longrightarrow F$ denote the colimit cocone of D. Let $H_\alpha(F)$ be a colimit of $H_\beta \cdot D$ in $\underline{Y}$ and let $\gamma_d : H_\beta(Dd) \Longrightarrow H_\alpha(F)$ denote a colimit cocone of $H_\beta \cdot D$.

  Then

$$\underline{Y}(\gamma_d, y) : \underline{Y}(H_\alpha(F), y) \longrightarrow \underline{Y}(H_\beta(Dd), y)$$

  is a limit cone in $\underline{Set}$. Since by the construction of $\underline{X}_{\mathcal{F}^{op}}$, the functor $J : \underline{X}_{\mathcal{F}^{op}} \longrightarrow [\underline{X}^{op}, \underline{Set}]_{\mathcal{F}^{op}}$ preserves all small colimits, the cone of all morphisms

$$[\underline{X}^{op}, \underline{Set}]_{\mathcal{F}^{op}}(J(\kappa_d), H_+(y))$$

  from $[\underline{X}^{op}, \underline{Set}]_{\mathcal{F}^{op}}(J(F), H_+(y))$ to $[\underline{X}^{op}, \underline{Set}]_{\mathcal{F}^{op}}(J(Dd), H_+(y))$ is a limit cone in $\underline{Set}$ as well.

  Note that since y was arbitrary, we have that $\underline{Y}(H_\alpha(F), \_)$ is a limit of a diagram of functors $\underline{Y}(H_\beta(Dd), \_)$ in $[\underline{Y}, \underline{Set}]$ with natural transformations $\underline{Y}(\gamma_d, \_)$ as a limit cone.

  Analogously $[\underline{X}^{op}, \underline{Set}]_{\mathcal{F}^{op}}(J(F), \_)$ is a limit of $[\underline{X}^{op}, \underline{Set}]_{\mathcal{F}^{op}}(J(Dd), H_+(\_))$ with a limit cone formed by natural transformations $[\underline{X}^{op}, \underline{Set}]_{\mathcal{F}^{op}}(J(\kappa_d), H_+(\_))$.

  Since by assumption, each $B_{Dd,\_}$ is a natural transformation, define the natural transformation

$$B_{F,\_} : \underline{Y}(H_\alpha(F), \_) \Longrightarrow [\underline{X}^{op}, \underline{Set}]_{\mathcal{F}^{op}}(J(F), H_+(\_))$$

  as a unique natural transformation such that the equality

$$[\underline{X}^{op}, \underline{Set}]_{\mathcal{F}^{op}}(J(\kappa_d), H_+(\_)) \cdot B_{F,\_} = B_{Dd,\_} \cdot \underline{Y}(\gamma_d, \_)$$

  holds for all d.

  It is clear that $B_{F,\_}$ is a natural isomorphism. Thus by Lemma 5.1.11 there is a unique functor $H_\alpha : \underline{X}_\alpha \longrightarrow \underline{Y}$ such that H_α extends H_β and $H_\alpha \dashv_{J \cdot E_\alpha} H_+$ holds.

- The case of a limit ordinal α is trivial: define H_α to be the union of H_β's for all $\beta < \alpha$.

The functor H^+ is defined as a unique functor extending all H_α's. It now follows that $H^+ \dashv_J H_+$ and therefore H^+ preserves small colimits. It is clear that H^+ is defined uniquely up to an isomorphism and that the equation $H^+ \cdot Y_{\underline{X}} = H$ holds. This finishes the proof of the first part of the theorem.

II. Analogously to the first part of the proof choose a category $\underline{X}_{\mathcal{F},\mathcal{C}}$ which is a full subcategory of $\underline{X}_{\mathcal{F}^{op}}$ and which is equivalent to the closure of $\underline{X}$ in $[\![\underline{X}^{op}, \underline{Set}]\!]_{\mathcal{F}^{op}}$ under $\mathcal{C}$-colimits. By definition, $\underline{X}_{\mathcal{F},\mathcal{C}}$ has all $\mathcal{C}$-colimits and the codomain restriction of $Y_{\underline{X}} : \underline{X} \longrightarrow \underline{X}_{\mathcal{F}^{op}}$ to $\underline{X}_{\mathcal{F},\mathcal{C}}$ preserves $\mathcal{F}$-colimits (see the first part of this proof). Let us denote this codomain restriction by $Z_{\underline{X}}$.

To establish the universal property of $Z_{\underline{X}}$, it suffices to show that for any $\mathcal{C}$-cocomplete category $\underline{Y}$ and any $\mathcal{F}$-cocontinuous functor $H : \underline{X} \longrightarrow \underline{Y}$ there exists a unique (up to a natural isomorphism) functor $H^* : \underline{X}_{\mathcal{F},\mathcal{C}} \longrightarrow \underline{Y}$ which is $\mathcal{C}$-cocontinuous and fulfills $H^* \cdot Z_{\underline{X}} = H$.

Observe that the domain restriction to $\underline{X}_{\mathcal{F},\mathcal{C}}$ of the functor H^+ constructed in the first part of this proof is $\mathcal{C}$-cocontinuous. Put $H^* : \underline{X}_{\mathcal{F},\mathcal{C}} \longrightarrow \underline{Y}$ to be this domain restriction of H^+. The proof of the second assertion of the theorem is finished.

The last assertion of the theorem follows from the above proof. □

Remark 5.1.13 Theorem 5.1.12 allows us to speak about "the" free $\mathcal{F}$-conservative $\mathcal{C}$-cocompletion $\mathcal{C}\text{-}\mathsf{Cocompl}(\underline{X})_{\mathcal{F}}$ for each category $\underline{X}$. More precisely, we have a *canonical choice* of a free $\mathcal{F}$-conservative $\mathcal{C}$-cocompletion for each category $\underline{X}$, namely the closure of $\underline{X}$ under $\mathcal{C}$-colimits in $[\![\underline{X}^{op}, \underline{Set}]\!]_{\mathcal{F}^{op}}$. □

Examples 5.1.14 By a special choice of $\mathcal{C}$ and $\mathcal{F}$ we can recover familiar cocompletions:

1. Choose $\mathcal{C}$ to contain all small categories, $\mathcal{F} = \emptyset$. Then a free $\mathcal{C}$-cocompletion $\mathcal{C}\text{-}\mathsf{Cocompl}(\underline{X})$ of a category $\underline{X}$ is a *free cocompletion of* $\underline{X}$ *w.r.t. all small colimits.* In case $\underline{X}$ is small this cocompletion is equivalent to the category of presheaves $[\underline{X}^{op}, \underline{Set}]$ (see [Lam66]).

   The case of a free $\mathcal{C}$-cocompletion for an *arbitrary* choice of $\mathcal{C}$ is fully described (in the general enriched case) in [Ke82], Theorem 5.35 as the closure of $\underline{X}$ in $[\![\underline{X}^{op}, \underline{Set}]\!]$ under $\mathcal{C}$-colimits.

2. Choose $\mathcal{C}$ to consist of all λ-filtered small categories (λ is a fixed regular cardinal), let $\mathcal{F} = \emptyset$. Then the free $\mathcal{C}$-cocompletion of a category $\underline{X}$ is the *free cocompletion of* $\underline{X}$ *w.r.t. λ-filtered colimits.* For $\underline{X}$ small, $\mathcal{C}\text{-}\mathsf{Cocompl}(\underline{X})$ is (equivalent to) the λ-accessible category $\mathsf{Flat}_\lambda(\underline{X})$ of λ-flat functors from $\underline{X}^{op}$ to $\underline{Set}$ (see Definition 3.5.10). Many basic properties of this cocompletion for a special choice $\lambda = \aleph_0$ and a not neccessarily small category $\underline{X}$ have been proved in Exposé I of [SGA4], see also [JJ82], where a different description of this cocompletion is called *inductive*, see also Section 5.4.

3. Choose $\mathcal{C}$ to consist of all small categories, let $\mathcal{F}$ consist of all categories having less than λ morphisms (λ is a fixed regular cardinal). Then the free $\mathcal{F}$-conservative $\mathcal{C}$-cocompletion $\mathcal{C}\text{-}\mathsf{Cocompl}(\underline{X})_{\mathcal{F}}$ of a small category $\underline{X}$ is equivalent to the locally λ-presentable category $\mathsf{Cont}_\lambda(\underline{X})$ defined in Section 1.42 of [AR94]. By Theorem 1.46

of [AR94], each locally λ-presentable category is equivalent to a category of the form $\mathsf{Cont}_\lambda(\underline{X})$ for a small category $\underline{X}$. As far as we know free $\mathcal{F}$-conservative $\mathcal{C}$-cocompletions of *arbitrary* categories for this particular choice of $\mathcal{F}$ and $\mathcal{C}$ have not been studied.

4. An interesting example is the case when $\mathcal{C} = \mathcal{F}$. In this case, a free $\mathcal{C}$-conservative $\mathcal{C}$-cocompletion $\mathcal{C}\text{-}\mathsf{Cocompl}(\underline{X})_{\mathcal{C}}$ of a category $\underline{X}$ is called an *idempotent $\mathcal{C}$-cocompletion of* $\underline{X}$.

□

5.2 Discrete Op-Fibrations

In this section we give an equivalent description of a free conservative cocompletion of a category (Theorem 5.2.12).

Recall Definition 3.5.8 of the category $\mathsf{Elts}(F)$ of elements of a functor $F : \underline{X}^{op} \longrightarrow \underline{\mathrm{Set}}$ and the notation ∂_F for the projection functor from $\mathsf{Elts}(F)$ to $\underline{X}^{op}$.

We need the following notion of an "abstract" category of elements of a functor $F : \underline{X}^{op} \longrightarrow \underline{\mathrm{Class}}$ (cf. [Ke82], Section 4.7 where discrete op-fibrations are defined for $\underline{\mathrm{Set}}$-valued functors on a small category):

Definition 5.2.1 Suppose that $\underline{A}$ and $\underline{X}$ are categories. A functor $\partial : \underline{A} \longrightarrow \underline{X}^{op}$ is called a *discrete op-fibration* (*d.o.f.* for short) if there exists a functor $F : \underline{X}^{op} \longrightarrow \underline{\mathrm{Class}}$ and an isomorphism $T : \underline{A} \longrightarrow \mathsf{Elts}(F)$ such that $\partial = \partial_F \cdot T$.

A discrete op-fibration $\partial : \underline{A} \longrightarrow \underline{X}^{op}$ is called $\underline{\mathrm{Set}}$-*based* if the functor $F : \underline{X}^{op} \longrightarrow \underline{\mathrm{Class}}$ factors through the inclusion of $\underline{\mathrm{Set}}$ in $\underline{\mathrm{Class}}$.

Given two d.o.f.'s $\partial_1 : \underline{A}_1 \longrightarrow \underline{X}^{op}$ and $\partial_2 : \underline{A}_2 \longrightarrow \underline{X}^{op}$, then a *morphism from ∂_1 to ∂_2* is a functor $G : \underline{A}_1 \longrightarrow \underline{A}_2$ such that $\partial_1 = \partial_2 \cdot G$. The *quasicategory* of d.o.f.'s and all morphisms between them is denoted by $\mathsf{dof}(\underline{X}^{op})$.

Remark 5.2.2 Discrete op-fibrations play the rôle of generalized *ideals* of a poset in the process of cocompleting a category — see [Er86] for a description of various cocompletions of posets by means of ideals. □

Lemma 5.2.3 *Let $\partial : \underline{A} \longrightarrow \underline{X}^{op}$ be a functor. Then the following are equivalent:*

1. *The functor ∂ is a d.o.f.*

2. *Every morphism $f : \partial(a) \longrightarrow x$ in $\underline{X}^{op}$ is of the form $\partial(g)$ for a unique $\underline{A}$-morphism g with domain a.*

Proof. 1.⇒2.: This is trivial.

2.⇒1.: Define the functor $F : \underline{X}^{op} \longrightarrow \underline{\mathrm{Class}}$ as follows: $Fx = \{a \mid \partial(a) = x\}$ for each $\underline{X}^{op}$-object x, for a $\underline{X}^{op}$-morphism $f : x \longrightarrow x'$ define Ff as the mapping from Fx to Fx'

which sends $a \in Fx$ to $a' \in Fx'$, where a' is the codomain of a unique g with $\partial(g) = f$. □

Remark 5.2.4 The above lemma can be modified for $\underline{\text{Set}}$-based d.o.f.'s in the following manner: a functor $\partial : \underline{\text{A}} \longrightarrow \underline{\text{X}}^{op}$ is a $\underline{\text{Set}}$-based d.o.f. iff it verifies Condition 2. of the above lemma plus the requirement that the class $\{a \mid \partial(a) = x\}$ is a set for each $\underline{\text{X}}$-object x. □

Lemma 5.2.5 *Let $\underline{\text{X}}$ be any category. Then the quasicategories $[\underline{\text{X}}^{op}, \underline{\text{Class}}]$ and $\mathsf{dof}(\underline{\text{X}}^{op})$ are equivalent.*

Proof. One can use verbatim the proof of the corresponding result in Section 4.7 of [Ke82]. Since we refer to this equivalence later, we give a sketch of the proof here.

The desired equivalence functor $\Phi : [\underline{\text{X}}^{op}, \underline{\text{Class}}] \longrightarrow \mathsf{dof}(\underline{\text{X}}^{op})$ sends a functor $F : \underline{\text{X}}^{op} \longrightarrow \underline{\text{Class}}$ to its corresponding d.o.f. ∂_F and it sends a natural transformation $\tau : F_1 \Longrightarrow F_2$ to the functor $G : \mathsf{Elts}(F_1) \longrightarrow \mathsf{Elts}(F_2)$, where G is defined as follows: $G(\langle x, a\rangle) = \tau_a \cdot x$ and $G(f) = f$ for $f : \langle x, a\rangle \longrightarrow \langle y, b\rangle$. It is easy to verify that Φ is an equivalence. □

The following fact will be very useful (cf. [Ke82], Section 4.7).

Lemma 5.2.6 *Suppose that $F : \underline{\text{X}}^{op} \longrightarrow \underline{\text{Class}}$ is a functor. Let $\mathbf{1}$ denote a one element set. Then the following hold:*

1. *Each functor $D : \underline{\text{D}} \longrightarrow \mathsf{Elts}(F)$ induces a functor $\tilde{D} : \underline{\text{D}} \longrightarrow \underline{\text{X}}^{op}$ and a natural transformation $\delta : const_{\mathbf{1}} \Longrightarrow F \cdot \tilde{D}$, such that $\tilde{D} = \partial_F \cdot D$.*

2. *Each pair consisting of a functor $D : \underline{\text{D}} \longrightarrow \underline{\text{X}}^{op}$ and a natural transformation $\delta : const_{\mathbf{1}} \Longrightarrow F \cdot D$ induces a functor $\hat{D} : \underline{\text{D}} \longrightarrow \mathsf{Elts}(F)$ such that $D = \partial_F \cdot \hat{D}$.*

The correspondence $D \mapsto (\tilde{D}, \delta)$ is bijective.

Proof. 1. Denote $Dd = \langle x_d, a_d\rangle$ and $Du = f_u : \langle x_d, a_d\rangle \longrightarrow \langle x_{d'}, a_{d'}\rangle$ for $u : d \longrightarrow d'$ in $\underline{\text{D}}$. Define $\tilde{D}d = a_d$, $\tilde{D}u = f_u$ and $\delta_d = x_d : \mathbf{1} \longrightarrow Fa_d$.

2. Denote $\tilde{D}d = a_d$, $\tilde{D}u = f_u : a_d \longrightarrow a_{d'}$ for $ud \longrightarrow d'$ in $\underline{\text{D}}$. Denote $\delta_d = x_d : \mathbf{1} \longrightarrow Fa_d$ Put $\hat{D}d = \langle x_d, a_d\rangle$ and $\hat{D}u = f_u$.

The last assertion of the lemma is obvious. □

Definition 5.2.7 Suppose a class $\mathcal{W}$ of small categories is given. A functor $H : \underline{\text{X}} \longrightarrow \underline{\text{Y}}$ is said to *create $\mathcal{W}$-limits* provided that the following holds for any $\underline{\text{D}} \in \mathcal{W}$: for any diagram $D : \underline{\text{D}} \longrightarrow \underline{\text{X}}$ and any limit $\gamma : y \Longrightarrow H \cdot D$ of $H \cdot D$ there is a unique cone $\kappa : x \Longrightarrow D$ such that $Hx = y$, $H(\kappa) = \gamma$ and κ is a limit of D.

Lemma 5.2.8 *Let $\mathcal{W}$ be a class of small categories, let $\underline{\text{X}}$ be any category. A functor $F : \underline{\text{X}}^{op} \longrightarrow \underline{\text{Class}}$ preserves $\mathcal{W}$-limits iff the corresponding d.o.f. $\partial_F : \mathsf{Elts}(F) \longrightarrow \underline{\text{X}}^{op}$ creates them.*

Proof. 1. Suppose that F preserves $\mathcal{W}$-limits and let $D : \underline{\mathrm{D}} \longrightarrow \mathsf{Elts}(F)$ be a diagram with $\underline{\mathrm{D}} \in \mathcal{W}$. Suppose that $\kappa : a \Longrightarrow \partial_F \cdot D$ is a limit in $\underline{\mathrm{X}}^{op}$. Put $Dd = \langle x_d, a_d \rangle$ and $Du = f_u : \langle x_d, a_d \rangle \longrightarrow \langle x_{d'}, a_{d'} \rangle$ for $u : d \longrightarrow d'$ in $\underline{\mathrm{D}}$.

Recall from Lemma 5.2.6 that giving D is equivalent to giving $\tilde{D} : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{X}}^{op}$ and $\delta : \underline{\mathbf{1}} \Longrightarrow F \cdot \tilde{D}$ such that $\tilde{D} = \partial_F \cdot D$ and $\delta_d = x_d$.

By assumption, $F(\kappa) : Fa \Longrightarrow F \cdot \partial_F \cdot D$ is a limit cone. Since $\delta : \mathbf{1} \Longrightarrow F \cdot \tilde{D}$ is a cone on $F \cdot \tilde{D} = F \cdot \partial_F \cdot D$, there exists a unique $x : \mathbf{1} \longrightarrow Fa$ such that $F(\kappa) \cdot x = \delta$. The last equation states that $\langle x, a \rangle$ is a vertex of a cone on D in $\mathsf{Elts}(F)$. Denote this cone by $\gamma : \langle x, a \rangle \Longrightarrow D$. It is clear that $\partial_F(\gamma) = \kappa$ and that γ is a unique cone with this property.

To prove that γ is a limit, consider another cone $\gamma' : \langle x', a' \rangle \Longrightarrow D$.

Since $\partial_F(\gamma)$ is a limit, there is a unique $\underline{\mathrm{X}}^{op}$-morphism $f : a' \longrightarrow a$ such that $\partial_F(\gamma) \cdot f = \partial_F(\gamma')$. We have to prove that $Ff \cdot x' = x$.

By assumption, $(F \cdot \partial_F)(\gamma) = F(\kappa) : Fa \Longrightarrow F \cdot \tilde{D}$ is a limit cone of $F \cdot \partial_F \cdot D = F \cdot \tilde{D}$ in $\underline{\mathrm{Set}}$. Therefore the equality

$$(F \cdot \partial_F)(\gamma) \cdot Ff = (F \cdot \partial_F)(\gamma') \tag{5.2}$$

holds. Since γ' is a cone on D, we have the equality

$$(F \cdot \partial_F)(\gamma') \cdot x' = \delta \tag{5.3}$$

Therefore $(F \cdot \partial_F)(\gamma) \cdot Ff \cdot x' = \delta$. Since x is a unique mapping such that $(F \cdot \partial_F)(\gamma) \cdot x = \delta$, we proved that $Ff \cdot x' = x$.

2. Conversely, suppose that ∂_F creates $\mathcal{W}$-limits. Let $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{X}}^{op}$ be a diagram with $\underline{\mathrm{D}} \in \mathcal{W}$ and let $\kappa : a \Longrightarrow D$ be a limit of D.

To prove that $F(\kappa) : Fa \Longrightarrow F \cdot D$ is a limit, consider any cone $\gamma : Z \Longrightarrow F \cdot D$. To define a unique mediating mapping $f : Z \longrightarrow Fa$, take an element $z \in Z$ and identify it with a mapping $z : \mathbf{1} \longrightarrow Z$. Then $\gamma \cdot z : \mathbf{1} \Longrightarrow F \cdot D$ is a cone and together with $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{X}}^{op}$ it defines a functor $\hat{D} : \underline{\mathrm{D}} \longrightarrow \mathsf{Elts}(F)$ such that $\partial_F \cdot \hat{D} = D$.

Since ∂_F creates $\mathcal{W}$-limits, there is a unique cone $\tau : \langle x, a \rangle \Longrightarrow \hat{D}$ such that $\partial_F(\tau) = \kappa$ and τ is a limit cone. Put $f(z) = x$. It is easy to see that $F(\kappa) \cdot f = \gamma$ and we have proved that $F(\kappa)$ is a limit cone. □

Remark 5.2.9 It is clear that the above proof works for $\underline{\mathrm{Set}}$-valued functors as well. Explicitly: a functor $F : \underline{\mathrm{X}}^{op} \longrightarrow \underline{\mathrm{Set}}$ preserves $\mathcal{W}$-limits iff the corresponding $\underline{\mathrm{Set}}$-based d.o.f. creates them. □

Notation 5.2.10 Let $\mathcal{W}$ be a class of small categories, let $\underline{\mathrm{X}}$ be any category. Denote by $\mathcal{W}\text{-}\mathsf{dof}(\underline{\mathrm{X}}^{op})$ the full subquasicategory of $\mathsf{dof}(\underline{\mathrm{X}}^{op})$ determined by those d.o.f.'s which create $\mathcal{W}$-limits. □

The following lemma is easy to prove.

Lemma 5.2.11 *Let $\mathcal{W}$ be any class of small categories, let $\underline{\mathrm{X}}$ be any cateory. Then the quasicategories $[\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}}$ and $\mathcal{W}\text{-}\mathsf{dof}(\underline{\mathrm{X}}^{op})$ are equivalent.*

The correspondence of $\underline{\mathrm{Class}}$-valued functors and discrete op-fibrations from the above lemma provides us with a description of conservative cocompletions which resembles the approach used while cocompleting posets.

Given a class $\mathcal{W}$ of small categories, one can restrict the equivalence functor

$$E : [\underline{\mathrm{X}}^{op}, \underline{\mathrm{Class}}]_{\mathcal{W}} \longrightarrow \mathcal{W}\text{-}\mathsf{dof}(\underline{\mathrm{X}}^{op})$$

to the category $[\![\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]\!]_{\mathcal{W}}$ of $\mathcal{W}$-reachable functors. Call those d.o.f.'s, which lie in the image of the restriction of E, *$\mathcal{W}$-reachable discrete op-fibrations.*

Theorem 5.2.12 *Let $\mathcal{C}$ and $\mathcal{F}$ be classes of small categories, let $\underline{\mathrm{X}}$ be any category. Define $J : \underline{\mathrm{X}} \longrightarrow \mathcal{F}^{op}\text{-}\mathsf{dof}(\underline{\mathrm{X}}^{op})$ to be the codomain restriction of the full embedding of $\underline{\mathrm{X}}$ to $\mathsf{dof}(\underline{\mathrm{X}}^{op})$, which sends an $\underline{\mathrm{X}}$-object x to the discrete op-fibration $d_x : \underline{\mathrm{X}}^{op}/x \longrightarrow \underline{\mathrm{X}}^{op}$, where d_x is the canonical forgetful functor. Then $\mathcal{C}\text{-}\mathsf{Cocompl}(\underline{\mathrm{X}})_{\mathcal{F}}$ is equivalent to the closure of $J(\underline{\mathrm{X}})$ in $\mathcal{F}^{op}$-reachable discrete op-fibrations under $\mathcal{C}$-colimits. The codomain restriction of the functor J can be taken to be the full and faithful functor from $\underline{\mathrm{X}}$ to $\mathcal{C}\text{-}\mathsf{Cocompl}(\underline{\mathrm{X}})_{\mathcal{F}}$ which satisfies the universal property.*

5.3 Functorial Behaviour of Cocompletions

In this section we prove that assignment of "the" free $\mathcal{F}$-conservative $\mathcal{C}$-cocompletion (in the sense of Remark 5.1.13) is functorial.

For classes $\mathcal{F}$ and $\mathcal{C}$ of small categories we define the following 2-quasicategories as follows:

1. $\mathsf{CAT}_{\mathcal{F}}$ is the 2-quasicategory of all categories, all functors preserving $\mathcal{F}$-colimits and all natural transformations.

2. $\mathcal{C}\text{-}\mathsf{CAT}_{\mathcal{C}}$ is the 2-quasicategory of all $\mathcal{C}$-cocomplete categories, all functors preserving $\mathcal{C}$-colimits and all natural transformations.

Theorem 5.3.1 *Let $\mathcal{F}$ and $\mathcal{C}$ be classes of small categories. Let $\Phi : \underline{\mathrm{X}} \mapsto \Phi(\underline{\mathrm{X}})$ be a mapping which assigns to a category $\underline{\mathrm{X}}$ the closure of $\underline{\mathrm{X}}$ under $\mathcal{C}$-colimits in $[\![\underline{\mathrm{X}}^{op}, \underline{\mathrm{Set}}]\!]_{\mathcal{F}^{op}}$. Then Φ is an object function of a pseudofunctor $\Phi : \mathsf{CAT}_{\mathcal{F}} \longrightarrow \mathcal{C}\text{-}\mathsf{CAT}_{\mathcal{C}}$.*

Proof. For each category $\underline{\mathrm{X}}$ denote by $I_{\underline{\mathrm{X}}} : \underline{\mathrm{X}} \longrightarrow \Phi(\underline{\mathrm{X}})$ the embedding of $\underline{\mathrm{X}}$ to "the" free $\mathcal{F}$-conservative $\mathcal{C}$-cocompletion.

We now extend the assignment $\underline{\mathrm{X}} \mapsto \Phi(\underline{\mathrm{X}})$ to a pseudofunctor.

Let $\underline{\mathrm{X}}$ and $\underline{\mathrm{Y}}$ be arbitrary categories. The functor

$$\Phi_{\underline{\mathrm{X}},\underline{\mathrm{Y}}} : \mathsf{CAT}_{\mathcal{F}}(\underline{\mathrm{X}}, \underline{\mathrm{Y}}) \longrightarrow \mathcal{C}\text{-}\mathsf{CAT}_{\mathcal{C}}(\Phi(\underline{\mathrm{X}}), \Phi(\underline{\mathrm{Y}}))$$

is defined as follows:

- From the proof of Theorem 5.1.12 it follows that for each $\mathcal{F}$-cocontinuous functor $F : \underline{\mathrm{X}} \longrightarrow \underline{\mathrm{Y}}$ a left Kan extension of $I_{\underline{\mathrm{Y}}} \cdot F$ along $I_{\underline{\mathrm{X}}}$ exists (and preserves $\mathcal{C}$-colimits). Let us denote by $\Phi_{\underline{\mathrm{X}},\underline{\mathrm{Y}}}(F)$ a choice of $Lan_{I_{\underline{\mathrm{X}}}}(I_{\underline{\mathrm{Y}}} \cdot F)$ and denote by η^F a unit of left

Kan extension. We can make this choice of a left Kan extension in such a way that $\Phi(1_{\underline{X}})$ is equal to $1_{\Phi(\underline{X})}$ and $\eta^{1_{\underline{X}}}$ is the identity natural transformation. This follows from Theorem 5.1 of [Ke82]: the functor from $\Phi(\underline{X})$ to $[\underline{X}^{op}, \underline{Set}]$, which assigns $[\underline{X}^{op}, \underline{Set}](I_{\underline{X}-}, F)$ to a $\Phi(\underline{X})$-object F, is naturally isomorphic to the inclusion and therefore is full and faithful.

We have defined $\Phi_{\underline{X},\underline{Y}}$ on objects.

- For a morphism $\tau : F_1 \Longrightarrow F_2$ in $\mathsf{CAT}_{\mathcal{F}}(\underline{X}, \underline{Y})$, put

$$\Phi_{\underline{X},\underline{Y}}(\tau) : \Phi_{\underline{X},\underline{Y}}(F_1) \Longrightarrow \Phi_{\underline{X},\underline{Y}}(F_2)$$

to be the unique natural transformation such that the square

$$\begin{tikzcd}
I_{\underline{Y}} \cdot F_1 \arrow[r, "\eta^{F_1}"] \arrow[d, "I_{\underline{Y}}\tau"'] & \Phi_{\underline{X},\underline{Y}}(F_1) \cdot I_{\underline{X}} \arrow[d, "\Phi_{\underline{X},\underline{Y}}(\tau) I_{\underline{X}}"] \\
I_{\underline{Y}} \cdot F_2 \arrow[r, "\eta^{F_2}"'] & \Phi_{\underline{X},\underline{Y}}(F_2) \cdot I_{\underline{X}}
\end{tikzcd}$$

commutes (use the universal property of η^{F_1}). This defines $\Phi_{\underline{X},\underline{Y}}$ on morphisms and from the universal property of a unit of a left Kan extension it follows that $\Phi_{\underline{X},\underline{Y}}$ is indeed a functor.

To show that Φ bears a structure of a pseudofunctor we have to define the comparison natural isomorphisms

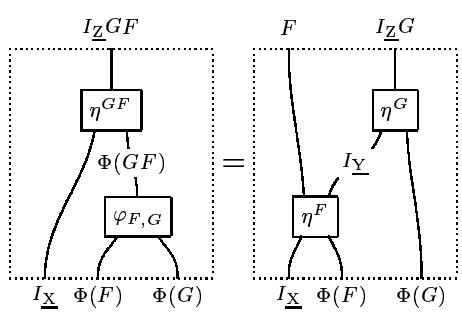
$$\psi_{\underline{X}} : \Phi_{\underline{X},\underline{X}}(1_{\underline{X}}) \Longrightarrow 1_{\Phi(\underline{X})}$$

and

$$\varphi_{F,G} : \Phi_{\underline{X},\underline{Z}}(G \cdot F) \Longrightarrow \Phi_{\underline{Y},\underline{Z}}(G) \cdot \Phi_{\underline{X},\underline{Y}}(Y)$$

and verify the appropriate coherence conditions for them (see Definition 3.3.11):

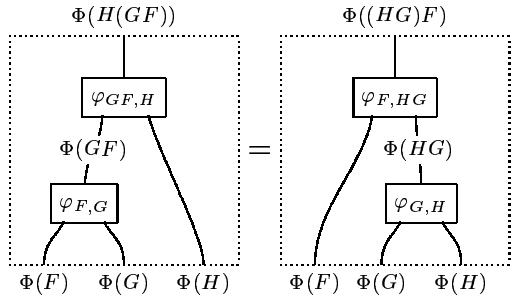
- $\psi_{\underline{X}}$ is the identity natural transformation.
- For $\mathcal{F}$-cocontinuous functors $F : \underline{X} \longrightarrow \underline{Y}$, $G : \underline{Y} \longrightarrow \underline{Z}$, $\varphi_{F,G}$ is defined as a unique natural transformation such that the equality



(5.4)

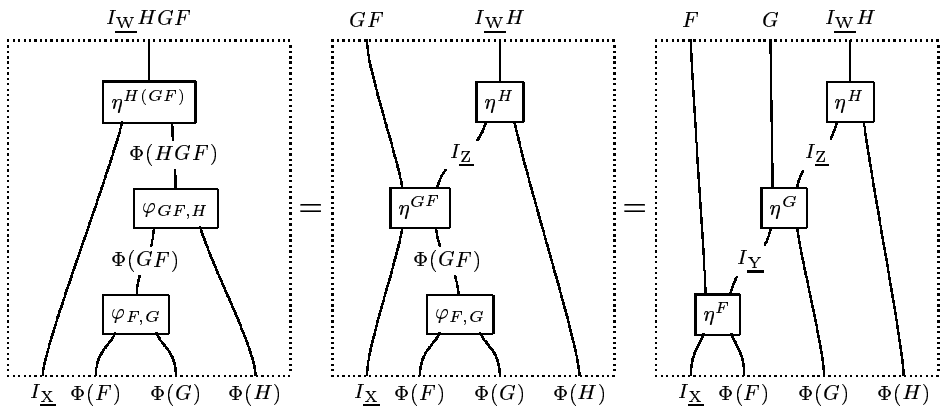
holds. To prove that $\psi_{F,G}$ is a natural isomorphism, recall from the proof of 5.1.12 that both $\Phi_{\underline{X},\underline{Z}}(G \cdot F)$ and $\Phi_{\underline{Y},\underline{Z}}(G) \cdot \Phi_{\underline{X},\underline{Y}}(Y)$ are defined on objects in an iterative way using $\mathcal{C}$-colimits starting from hom-functors. Since, by definition, $\psi_{F,G} I_{\underline{X}}$ is a natural isomorphism and both $\Phi_{\underline{X},\underline{Z}}(G \cdot F)$ and $\Phi_{\underline{Y},\underline{Z}}(G) \cdot \Phi_{\underline{X},\underline{Y}}(Y)$ are $\mathcal{C}$-cocontinuous, it follows that $\varphi_{F,G}$ is a natural isomorphism.

The associativity coherence (recall (3.18))



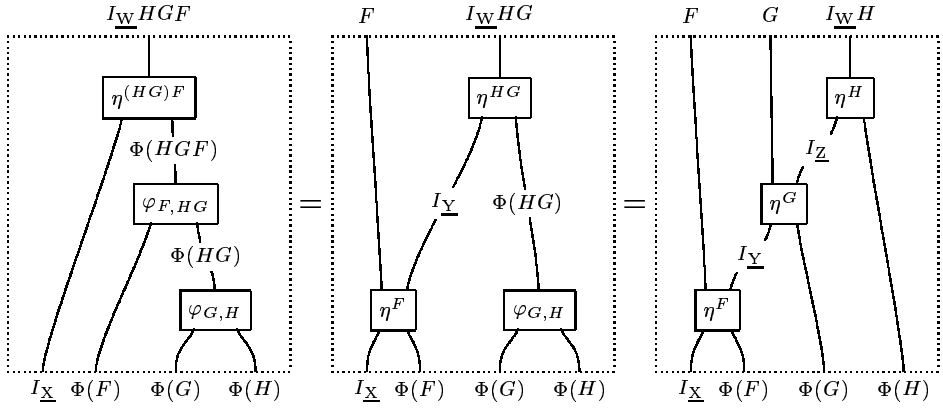
(5.5)

for functors $F : \underline{\mathrm{X}} \longrightarrow \underline{\mathrm{Y}}$, $G : \underline{\mathrm{Y}} \longrightarrow \underline{\mathrm{Z}}$ $H : \underline{\mathrm{Z}} \longrightarrow \underline{\mathrm{W}}$ follows from the universal property of η^{HGF}: the left hand side of (5.5) precomposed with η^{HGF} equals to:



(5.6)

where the equalities follow by definition of φ. Analogously, the right hand side of (5.5) precomposed with η^{HGF} equals to:

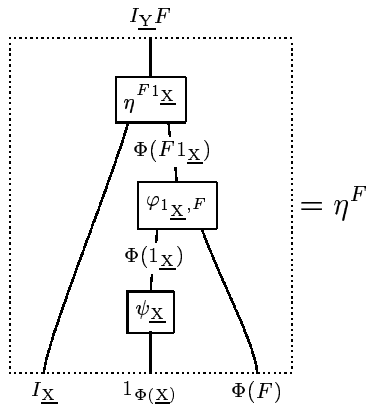


(5.7)

Therefore the equality (5.5) holds.

The identity coherence conditions (see (3.19)) for a functor $F : \underline{\mathrm{X}} \longrightarrow \underline{\mathrm{Y}}$ follow from the universal property of η^F and the fact that we have defined ψ as the identity natural transformation. We verify one of these conditions, the second is verified analogously.

Since the equation



$$= \eta^F \tag{5.8}$$

holds, it follows from the universal property of η^F that $(\Phi(F)\psi_{\underline{X}}) \cdot \varphi_{1_{\underline{X}},F}$ is the identity natural transformation on $\Phi(F)$. □

5.4 Free Cocompletions w.r.t. Filtered Colimits

We now treat the special case of a free $\mathcal{F}$-conservative $\mathcal{C}$-cocompletion when $\mathcal{F} = \emptyset$ and $\mathcal{C}$ consists of all small filtered categories.

An explicit description of a free cocompletion of a category w.r.t. small filtered colimits has been given in Expose I of [SGA4] and [JJ82]. We recall it in the following definition:

Definition 5.4.1 Let $\underline{X}$ be a category. An *inductive cocompletion of* $\underline{X}$ — denoted by $\mathrm{Ind}(\underline{X})$ — is the following category:

- $\mathrm{Ind}(\underline{X})$-objects are functors $D : \underline{D} \longrightarrow \underline{X}$ with $\underline{D}$ small and filtered,
- an $\mathrm{Ind}(\underline{X})$-morphism from D to D' is a family $\{f_d \mid d \text{ is a } \underline{D}\text{-object}\}$, where each f_d is an equivalence class of $\underline{X}$-morphisms from Dd to some $D'd'$ under the following equivalence:

  $g_1 : Dd \longrightarrow Dd'_1$ and $g_2 : Dd \longrightarrow Dd'_2$ are equivalent, if there exists a $\underline{D'}$-object d' and $\underline{D'}$-morphisms $\delta_1 : d'_1 \longrightarrow d'_2$ such that the square

$$\begin{tikzcd} Dd \arrow[r, "g_1"] \arrow[d, "g_2"'] & D'd'_1 \arrow[d, "D'\delta_1"] \\ D'd'_2 \arrow[r, "D'\delta_2"'] & D'd' \end{tikzcd} \tag{5.9}$$

  commutes.

  Each class f_d is required to satisfy the following compatibility condition:

  if $\delta : d \longrightarrow d_1$ is a $\underline{D}$-morphism and if $g : Dd_1 \longrightarrow D'd'$ represents f_{d_1}, then $g \cdot D\delta$ represents f_d.

- Composition in $\mathrm{Ind}(\underline{\mathrm{X}})$ is defined in the following way: suppose
$$f = \{f_d \mid d \text{ is a } \underline{\mathrm{D}}\text{-object}\} : D \longrightarrow D'$$
and
$$g = \{g_{d'} \mid d' \text{ is a } \underline{\mathrm{D}}'\text{-object}\} : D' \longrightarrow D''$$
are $\mathrm{Ind}(\underline{\mathrm{X}})$-morphisms, then their composite $g \cdot f : D \longrightarrow D''$ is a family
$$\{h_d \mid d \text{ is a } \underline{\mathrm{D}}\text{-object}\}$$
where h_d is an equivalence class represented by the $\underline{\mathrm{X}}$-morphism $\bar{g}_{d'} \cdot \bar{f}_d$ with $\bar{f}_d : Dd \longrightarrow D'd'$ representing f_d and $\bar{g}_{d'} : D'd' \longrightarrow D''d''$ representing $g_{d'}$.

It is proved in Expose I, [SGA4], Proposition 8.7.3 that $\mathrm{Ind}(\underline{\mathrm{X}})$ together with an obvious functor $I_{\underline{\mathrm{X}}} : \underline{\mathrm{X}} \longrightarrow \mathrm{Ind}(\underline{\mathrm{X}})$ (which sends an $\underline{\mathrm{X}}$-object x to $I_{\underline{\mathrm{X}}}(x) : \underline{\mathbf{1}} \longrightarrow \underline{\mathrm{X}}$) is a free cocompletion of $\underline{\mathrm{X}}$ w.r.t. small filtered colimits.

We have already given an alternative description of a free cocompletion w.r.t. filtered colimits in Theorem 5.1.12 but the description of $\mathrm{Ind}(\underline{\mathrm{X}})$ despite of being technical has the following advantage:

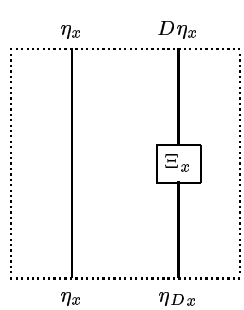
1. Objects of $\mathrm{Ind}(\underline{\mathrm{X}})$ are *canonical* filtered colimits of $\underline{\mathrm{X}}$-objects.
2. The assignment $\underline{\mathrm{X}} \mapsto \mathrm{Ind}(\underline{\mathrm{X}})$ is an object function of a *2-functor* $\mathrm{Ind}(\_) : \mathsf{CAT} \longrightarrow \mathsf{FILT}$ (compare with Theorem 5.3.1). This is proved in Section 8.6 of Expose I, [SGA4].
3. The 2-functor $\mathrm{Ind}(\_)$ induces a structure of a *KZ doctrine* on CAT — see Definition 5.4.2 below. The machinery of KZ doctrines will be essential in Chapter 6.

In his paper [Ko93] Anders Kock distinguishes certain structures on CAT called *KZ doctrines*, which are typical for free cocompletions of categories w.r.t. a class of small colimits. The following definitions come from [Ko93].

Definition 5.4.2 A *KZ doctrine* on a 2-(quasi)category K is a 4-tuple $\mathbb{D} = (D, \eta, \mu, \Xi)$, where $D : \mathsf{K} \longrightarrow \mathsf{K}$ is a 2-functor, $\eta : 1_{\mathsf{K}} \Longrightarrow D$ and $\mu : D \cdot D \Longrightarrow D$ are 2-natural transformations and $\Xi : D\eta \rightsquigarrow \eta D$ is a modification such that the following axioms are satisfied:

KZ-1 η is a two-sided identity for μ:
$$\mu \cdot \eta D = \mu \cdot D\eta = 1_D \tag{5.10}$$

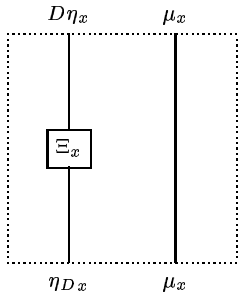
KZ-2 The 2-cell



(5.11)

is an identity 2-cell on $D\eta_x \cdot \eta_x = \eta_{Dx} \cdot \eta_x$ (η is 2-natural) for any 0-cell x.

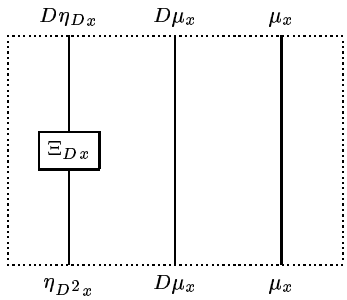
KZ-3 The 2-cell



(5.12)

is an identity 2-cell on $\mu_x \cdot D\eta_x = \mu_x \cdot \eta_{Dx}$ (equality is due to (5.10)) for any 0-cell x.

KZ-4 The 2-cell



(5.13)

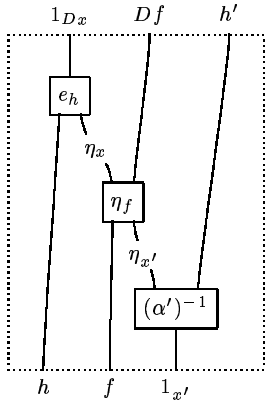
is an identity 2-cell on $\mu_x \cdot D\mu_x \cdot D\eta_{Dx} = \mu_x \cdot D\mu_x \cdot \eta_{D^2x}$ (equality is due to (5.10)) for any 0-cell x.

A KZ doctrine on K can be thought of as of a "lax idempotent" 2-monad on K.

Definition 5.4.3 An *algebra for a KZ doctrine* (D, η, μ, Ξ) is a triple (x, h, α), where $h : Dx \longrightarrow x$ is a 1-cell (called a *structure 1-cell of an algebra*) left adjoint to $\eta_x : x \longrightarrow Dx$ and $\alpha : 1_x \Longrightarrow h \cdot \eta_x$ is an isomorphism 2-cell, such that $\alpha^{-1} : h \cdot \eta_x \Longrightarrow 1_x$ is the counit of $h \dashv \eta_x$.

Definition 5.4.4 Suppose (x, h, α) and (x', h', α') are algebras for a KZ doctrine $\mathbb{D}$. Let $e_h : 1_{Dx} \Longrightarrow \eta_x \cdot h$ and $e_{h'} : 1_{Dx'} \Longrightarrow \eta_{x'} \cdot h'$ denote units of $h \dashv \eta_x$ and $h' \dashv \eta_{x'}$.

Let $f : x \longrightarrow x'$ be a 1-cell. A *canonical 2-cell for* f is a mate of the identity 2-cell $\eta_f : Df \cdot \eta_x \Longrightarrow \eta_{x'} \cdot f$ (η is 2-natural). Explicitly, it is the 2-cell



(5.14)

If the canonical 2-cell φ for f is an isomorphism, then f is called a *homomorphism from* (x, h, α) *to* (x', h', α').

Remark 5.4.5 Structure 1-cells are essentially unique: suppose that $A_1 = (x, h_1, \alpha_1)$ and $A_2 = (x, h_2, \alpha_2)$ are algebras with the same underlying 0-cell x. Then it is easy to see that the canonical 2-cell for $1_x : x \longrightarrow x$ is an isomorphism, therefore A_1 and A_2 are isomorphic. □

For a KZ doctrine $\mathbb{D}$ on K one can define the 2-quasicategory $\mathbb{D}$-Alg of algebras for $\mathbb{D}$ if we take algebras for 0-cells, homomorphisms for 1-cells and *all* 2-cells from K for 2-cells.

Kock explains in [Ko93], page 8, that free cocompletions w.r.t. filtered colimits yield a KZ doctrine $\mathbb{IND} = (D, \eta, \mu, \Xi)$ on CAT. Let us gather basic facts (proved in [Ko93]) how this KZ doctrine is formed:

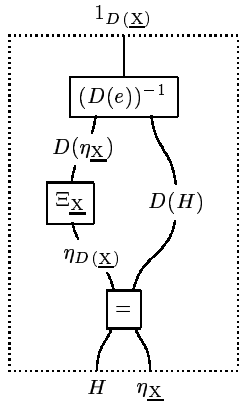
1. The inclusion 2-functor $E : \mathsf{FILT} \longrightarrow \mathsf{CAT}$ is a right adjoint to $\mathrm{Ind} : \mathsf{CAT} \longrightarrow \mathsf{FILT}$.
2. The 2-functor $D : \mathsf{CAT} \longrightarrow \mathsf{CAT}$ is the composition $E \cdot \mathrm{Ind}$.
3. The 2-natural transformation $\eta : 1_{\mathsf{CAT}} \Longrightarrow D$ is the unit of $\mathrm{Ind} \dashv E$.
4. The 2-natural transformation $\mu : DD \Longrightarrow D$ is the composition $E\varepsilon\mathrm{Ind}$, where $\varepsilon : \mathrm{Ind} \cdot E \Longrightarrow 1_{\mathsf{FILT}}$ denotes the counit of $\mathrm{Ind} \dashv E$.
5. Pointwise, the counit ε is a functor from $\mathrm{Ind}E(\underline{\mathrm{X}})$ to $\underline{\mathrm{X}}$ which assigns a colimit to a filtered diagram in $\underline{\mathrm{X}}$. Then $\mu_{\underline{\mathrm{X}}}$ is a left adjoint to $\eta_{D(\underline{\mathrm{X}})}$ with a natural isomorphism as a counit ($\eta_{D(\underline{\mathrm{X}})}$ is a full and faithful functor). Then the mate calculus provides us with a modification $\Xi : D\eta \rightsquigarrow \eta D$.
6. The quadruple (D, η, μ, Ξ) defined in the above way is a KZ doctrine.

The next result is contained in the proof of Theorem 3.5 of [Ko93].

Lemma 5.4.6 *Let $\mathbb{D} = (D, \eta, \mu, \Xi)$ be a KZ doctrine on CAT. Let $\underline{\mathrm{X}}$ be any category and suppose that $\eta_{\underline{\mathrm{X}}}$ is a full and faithful. Moreover, suppose that idempotents split in the quasicategory $[D(\underline{\mathrm{X}}), \underline{\mathrm{X}}]$. Then the following are equivalent:*

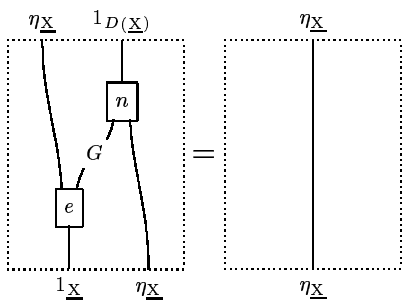
1. *There exists a functor $H : D(\underline{\mathrm{X}}) \longrightarrow \underline{\mathrm{X}}$ such that $H \cdot \eta_{\underline{\mathrm{X}}}$ is naturally isomorphic to $1_{\underline{\mathrm{X}}}$.*
2. *The functor $\eta_{\underline{\mathrm{X}}}$ has a left adjoint L and the counit of this adjunction is an isomorphism.*

Proof. 1.⇒2.: Let $e : H \cdot \eta_{\underline{X}} \Longrightarrow 1_{\underline{X}}$ denote a natural isomorphism. Define a natural transformation $n : 1_{D(\underline{X})} \Longrightarrow \eta_{\underline{X}} \cdot H$ by the equality

$n =$ 

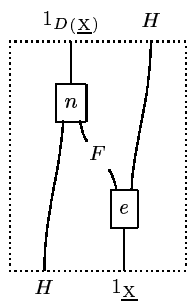
(5.15)

Since Ξ is a modification, we conclude that one of the triangle identities holds, namely



(5.16)

The second composition

$r =$ 

(5.17)

is in general not identity but it is an idempotent natural transformation $r : H \Longrightarrow H$, i.e. an idempotent in $[D(\underline{X}), \underline{X}]$. Due to our assumption r splits, say as $r = a \cdot b$ for some $a : L \Longrightarrow H$ and $b : H \Longrightarrow L$ with $b \cdot a = 1_F$.

Then due to Lemma 3.2.3 we have that L is a left adjoint to $\eta_{\underline{X}}$ with the counit $e \cdot a\eta_{\underline{X}}$. The last natural transformation is an isomorphism since $\eta_{\underline{X}}$ is full and faithful.

The implication 2.⇒1. is trivial. □

We now show that the 2-quasicategories FILT and $\mathbb{IND}$-Alg are essentially the same. First, Lemma 5.4.6 allows us to characterize categories which have filtered colimits.

Lemma 5.4.7 *Let $\underline{X}$ is a category which has filtered colimits. Then there is a structure of an $\mathbb{IND}$-algebra on $\underline{X}$. Conversely, if $(\underline{X}, H, \alpha)$ is an $\mathbb{IND}$-algebra, then $\underline{X}$ has filtered colimits.*

Proof. 1. By assumption, the functor quasicategory $[D(\underline{X}), \underline{X}]$ has filtered colimits and therefore idempotents split in $[D(\underline{X}), \underline{X}]$. Due to Lemma 5.4.6 $\underline{X}$ carries a $\mathbb{IND}$-algebra structure iff there is a functor $H : D(\underline{X}) \longrightarrow \underline{X}$ such that $H \cdot \eta_{\underline{X}}$ is naturally isomorphic to $1_{\underline{X}}$.

Since we assume that $\underline{X}$ has filtered colimits, let H be a functor which assigns a colimit to each $\mathrm{Ind}(\underline{X})$-object. Then clearly $H \cdot \eta_{\underline{X}} \cong 1_{\underline{X}}$.

2. The second assertion is trivial, since then $\underline{X}$ is equivalent to a full reflective subcategory of the category $D(\underline{X})$ which has filtered colimits. □

The following result appears in a more general setting in Remark 2.3 of [Ko93].

Lemma 5.4.8 *Suppose that $\underline{X}$, $\underline{X}'$ have filtered colimits and let $F : \underline{X} \longrightarrow \underline{X}'$ be a finitary functor. Then F is a homomorphism of $\mathbb{IND}$-algebras. Conversely, if F is a homomorphism of $\mathbb{IND}$-algebras, then F is finitary.*

Proof. Let $(\underline{X}, H, \alpha)$ and $(\underline{X}', H', \alpha')$ denote the structures of $\mathbb{IND}$-algebras which exist due to the previous lemma.

Let $\varphi : H' \cdot D(F) \Longrightarrow F \cdot H$ denote the canonical 2-cell for F. To say that φ is a natural isomorphism is the same as to say that F is finitary, since both H and H' are functors which assign a chosen colimit to filtered diagrams in $\underline{X}$, resp $\underline{Y}$. □

Theorem 5.4.9 *The 2-quasicategories* FILT *and* $\mathbb{IND}$-Alg *are biequivalent.*

Proof. Let us define a 2-functor $\Phi : \mathbb{IND}\text{-}\mathsf{Alg} \longrightarrow \mathsf{FILT}$ as follows:

Put $\Phi(\underline{X}, H, \alpha) = \underline{X}$ for any $\mathbb{IND}$-algebra $(\underline{X}, H, \alpha)$.

$\Phi(F) = F$ for each homomorphism of $\mathbb{IND}$-algebras.

$\Phi(\tau) = \tau$ for any 2-cell τ in $\mathbb{IND}$-Alg.

Lemmas 5.4.7 and 5.4.8 assure that the above definition is correct. Φ is clearly a biequivalence. □

Remark 5.4.10 Let us remark that the 2-functor Φ from the previous proof is *not* an isomorphism, since algebra structures are not determined uniquely but only up to an isomorphism (cf. 5.4.5). □

Chapter 6

Continuous Categories

In this chapter we are going to present a categorical generalization of another type of domains — namely *continuous domains*. The resulting concept of a *continuous category* goes back to [JJ82].

Apart from collecting known results from [JJ82] about continuous categories, the import of the current chapter lies in the fact that one can exploit the machinery of KZ doctrines from Section 5.4 and one can view continuous categories as certain coalgebras. These coalgebras have naturally defined morphisms among themselves — this allows us to define *way-below preserving functors* without mentioning the way-below relation explicitly, see Definition 6.2.3.

6.1 A Generalization of Continuous Domains

The notion of a *continuous domain* is another important concept of domain theory — see [AbJ96]. In this section we give the basic definition and properties of *continuous categories* which provide a categorical generalization of continuous domains. The definitions and results of this section come mostly from the paper of Peter Johnstone and André Joyal [JJ82]. They work with an explicit description of a free cocompletion of a category w.r.t. filtered colimits, namely with the category $\mathrm{Ind}(\underline{X})$ (see Definition 5.4.1).

We first give a definition of a continuous domain (see [AbJ96], Chapter 2).

Definition 6.1.1 Let $\langle X, \sqsubseteq \rangle$ be a DCPO.

We say that $x \in X$ *is way-below* $y \in X$ (notation $x \ll y$), provided for any directed set $A \subseteq X$ it holds: if $y \sqsubseteq \bigsqcup A$, then $x \sqsubseteq a$ for some $a \in A$.

A set $B \subseteq X$ is called a *basis of* $\langle X, \sqsubseteq \rangle$ if for each element $x \in X$ the set

$$B_x = \{y \in B \mid y \ll x\}$$

contains a directed subset with supremum x.

A DCPO $\langle X, \sqsubseteq \rangle$ is called a *continuous domain* (or a *continuous poset*), if it has a basis.

Lemma 6.1.2 *Let $\langle X, \sqsubseteq \rangle$ be a DCPO. If $x \ll y$, then $x \sqsubseteq y$.*

Proof. Suppose $x \ll y$. Use the fact that $y = \bigsqcup\{z \in X \mid z \sqsubseteq y\}$ and the definition of the way-below relation. □

Remark 6.1.3 The converse of the preceding lemma does not hold: for an element x of a DCPO $\langle X, \sqsubseteq\rangle$ we have that $x \ll x$ holds iff x is compact.

Therefore the concept of a continuous domain generalizes the concept of an (unpointed) algebraic DCPO (cf. Definition 2.1.3): each algebraic DCPO has a basis formed by the set of all its compact elements. □

Example 6.1.4 A classical example of a continuous poset which is not an algebraic DCPO is the unit interval $[0, 1]$ together with the usual order. Since $[0, 1]$ is a complete lattice, it is a DCPO. The following is easy to verify: $x \ll y$ iff either $x = y = 0$ or $x < y$. Thus 0 is the only compact element of $[0, 1]$ and hence $[0, 1]$ is not algebraic. □

Recall the notion of an *ideal completion* $\langle X, \sqsubseteq\rangle^*$ of a poset $\underline{\mathrm{X}} = \langle X, \sqsubseteq\rangle$ from 2.1.10, which is the free cocompletion of $\langle X, \sqsubseteq\rangle$ w.r.t. directed suprema. Thus, viewed as categories, $\langle X, \sqsubseteq\rangle^*$ and $\mathrm{Ind}(\underline{\mathrm{X}})$ are equivalent. Then each DCPO $\langle X, \sqsubseteq\rangle$ can be equipped with a continuous map $\sup_{\langle X,\sqsubseteq\rangle} : \langle X, \sqsubseteq\rangle^* \longrightarrow \langle X, \sqsubseteq\rangle$ which assigns $\bigsqcup I$ to each ideal I.

The following characterization of continuous domains will be useful for us (cf. [JJ82], Lemma 2.1):

Lemma 6.1.5 *A DCPO $\langle X, \sqsubseteq\rangle$ is a continuous domain iff the map $\sup_{\langle X,\sqsubseteq\rangle}$ has a left adjoint.*

Let us gather important facts about inductive cocompletions:

Theorem 6.1.6 *The following properties hold for any category $\underline{\mathrm{X}}$:*

1. *$I_{\underline{\mathrm{X}}}(x)$ is finitely presentable in $\mathrm{Ind}(\underline{\mathrm{X}})$ for any $\underline{\mathrm{X}}$-object x.*
2. *If idempotents split in $\underline{\mathrm{X}}$, then each finitely presentable object in $\mathrm{Ind}(\underline{\mathrm{X}})$ is isomorphic to $I_{\underline{\mathrm{X}}}(x)$ for some x.*
3. *$\underline{\mathrm{X}}$ has filtered colimits iff $I_{\underline{\mathrm{X}}}$ has a left adjoint.*

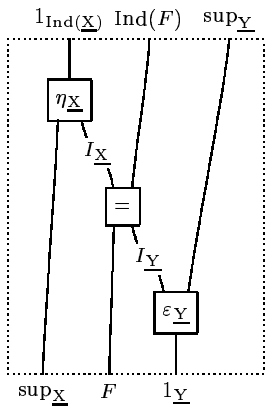
Proof. See [JJ82]. □

Recall that FILT denotes the 2-quasicategory of categories having all small filtered colimits as 0-cells, finitary functors as 1-cells and natural transformations as 2-cells. Also recall from Theorem 5.4.9 that $\mathbb{IND}$-Alg and FILT are biequivalent. Therefore we do not distinguish between algebras for $\mathbb{IND}$ and categories having filtered colimits.

Let us denote a choice of a left adjoint to $I_{\underline{\mathrm{X}}}$ by $\sup_{\underline{\mathrm{X}}} : \mathrm{Ind}(\underline{\mathrm{X}}) \longrightarrow \underline{\mathrm{X}}$ for any category $\underline{\mathrm{X}}$ with filtered colimits. Choosing a functor $\sup_{\underline{\mathrm{X}}}$ provides us with a choice of a colimit for each filtered diagram in $\underline{\mathrm{X}}$. It is clear that the collection of all $\sup_{\underline{\mathrm{X}}}$'s constitutes a pseudonatural transformation $\sup : \mathrm{Ind} \Longrightarrow 1_{\mathsf{FILT}}$ with the natural isomorphism

$$\sup_F : F \cdot \sup_{\underline{\mathrm{X}}} \Longrightarrow \sup_{\underline{\mathrm{Y}}} \cdot \mathrm{Ind}(F)$$

for each finitary functor $F : \underline{X} \longrightarrow \underline{Y}$ defined as follows: let s_F denote a mate of the identity natural transformation from $\mathrm{Ind}(F) \cdot I_{\underline{X}}$ to $I_{\underline{Y}} \cdot F$:

$s_F =$ 

(6.1)

where $\eta_{\underline{X}}$ denotes the unit of $\sup_{\underline{X}} \dashv I_{\underline{X}}$ and $\varepsilon_{\underline{Y}}$ denotes the counit of $\sup_{\underline{Y}} \dashv I_{\underline{Y}}$. s_F is an isomorphism, since F is finitary. Put

$$\sup_F = (s_F)^{-1}$$

The following definition comes from [JJ82]:

Definition 6.1.7 Suppose $\underline{X}$ has filtered colimits. Then $\underline{X}$ is called *continuous*, it the functor $\sup_{\underline{X}}$ has a left adjoint. For a continuous category $\underline{X}$, a choice of a left adjoint of $\sup_{\underline{X}}$ will be denoted by $w_{\underline{X}}$ and called a *way-below functor*.

Example 6.1.8 There are many examples of continuous categories:

1. A poset is a continuous domain iff it is a continuous category ([JJ82], Lemma 2.1).
2. For any category $\underline{X}$, $\mathrm{Ind}(\underline{X})$ is a continuous category (Proposition 2.4 of [JJ82]). Thus, in particular, each generalized domain is a continuous category.

□

It is clear from the definition that $w_{\underline{X}}$ is determined uniquely up to a natural isomorphism. Indeed, a continuous category is better to be seen as a *pair* $(\underline{X}, w_{\underline{X}})$ rather than $\underline{X}$ alone. The pair $(\underline{X}, w_{\underline{X}})$ is a kind of a *coalgebra structure* in the sense of [Ko93], Section 4.

Also, since the counit of $\sup_{\underline{X}} \dashv I_{\underline{X}}$ is always an isomorphism ($I_{\underline{X}}$ is a full embedding), the unit of $w_{\underline{X}} \dashv \sup_{\underline{X}}$ is always an isomorphism. Thus, each functor $w_{\underline{X}}$ is essentially an embedding of a full coreflective subcategory.

Denote by CONT the *2-quasicategory* having all continuous categories as 0-cells, all finitary functors as 1-cells and all natural transformations as 2-cells.

Definition 6.1.9 Let $E : \underline{Y} \longrightarrow \underline{X}$, $R : \underline{X} \longrightarrow \underline{Y}$ be a pair of finitary functors such that there is a natural isomorphism $\alpha : 1_{\underline{Y}} \Longrightarrow R \cdot E$. The triple (E, R, α) is called a *finitary retraction from* $\underline{Y}$ *to* $\underline{X}$ (and the category $\underline{Y}$ is called a *finitary retract of* $\underline{X}$).

Lemma 6.1.10 *Suppose $\mathcal{W}$ is a class of small categories. Let (E, R, α) be a finitary retraction from $\underline{\mathrm{Y}}$ to $\underline{\mathrm{X}}$. If the category $\underline{\mathrm{X}}$ has $\mathcal{W}$-colimits, then so does $\underline{\mathrm{Y}}$.*

Proof. Suppose $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{Y}}$ is a diagram with $\underline{\mathrm{D}}$ in $\mathcal{W}$. By assumption, $\operatorname{colim} E \cdot D$ exists in $\underline{\mathrm{X}}$. It is easy to verify that $R(\operatorname{colim} E \cdot D)$ is a colimit of D in $\underline{\mathrm{Y}}$. □

The following theorem generalizes the result of Achim Jung ([Ju88b], Proposition 1.16).

Theorem 6.1.11 CONT *is closed under finitary retracts, i.e.: suppose that $(\underline{\mathrm{X}}, w_{\underline{\mathrm{X}}})$ is a continuous category, $\underline{\mathrm{Y}}$ is any category and let (E, R, α) be a finitary retraction from $\underline{\mathrm{Y}}$ to $\underline{\mathrm{X}}$. Then $\underline{\mathrm{Y}}$ has a structure of a continuous category.*

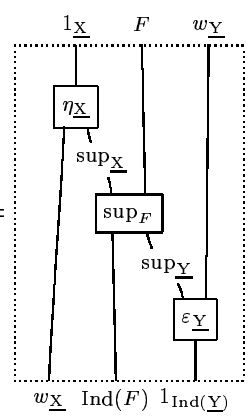
Proof. Since $\underline{\mathrm{Y}}$ has filtered colimits by Lemma 6.1.10, to prove the first statement of the lemma it suffices to prove that $\sup_{\underline{\mathrm{Y}}}$ has a left adjoint. This is Proposition 2.7 of [JJ82]. □

Remark 6.1.12 In fact, it is proved in Theorem 2.8 of [JJ82] that a category is continuous iff it is a finitary retract of a category having the form $\mathrm{Ind}(\underline{\mathrm{X}})$ for some $\underline{\mathrm{X}}$. □

6.2 The Way-Below Relation

In this section we are going to introduce another type of a morphism between continuous categories. The resulting concept of a *way-below preserving functor* is a generalization of way-below preserving monotone maps between DCPOs introduced by Jürgen Koslowski in [Kos96].

Definition 6.2.1 Suppose $(\underline{\mathrm{X}}, w_{\underline{\mathrm{X}}})$ and $(\underline{\mathrm{Y}}, w_{\underline{\mathrm{Y}}})$ are continuous categories. Let $F : \underline{\mathrm{X}} \longrightarrow \underline{\mathrm{Y}}$ be a finitary functor. The natural transformation $w_F : w_{\underline{\mathrm{Y}}} \cdot F \Longrightarrow \mathrm{Ind}(F) \cdot w_{\underline{\mathrm{X}}}$ defined as a mate of the invertible natural transformation $\sup_F : F \cdot \sup_{\underline{\mathrm{X}}} \Longrightarrow \sup_{\underline{\mathrm{Y}}} \cdot \mathrm{Ind}(F)$ under the adjunctions $w_{\underline{\mathrm{X}}} \dashv \sup_{\underline{\mathrm{X}}}$ and $w_{\underline{\mathrm{Y}}} \dashv \sup_{\underline{\mathrm{Y}}}$ is called a *way-below mate of F*. Explicitly,

$w_F =$ 

(6.2)

where $\varepsilon_{\underline{\mathrm{Y}}}$ denotes the counit of $w_{\underline{\mathrm{Y}}} \dashv \sup_{\underline{\mathrm{Y}}}$ and $\eta_{\underline{\mathrm{X}}}$ denotes the unit of $w_{\underline{\mathrm{X}}} \dashv \sup_{\underline{\mathrm{X}}}$.

Lemma 6.2.2 *Way-below mates of finitary functors form a lax natural transformation* $w : 1_{\mathsf{CONT}} \Longrightarrow \mathrm{Ind}$.

Proof. Using the fact that $\sup : \mathrm{Ind} \Longrightarrow 1_{\mathsf{FILT}}$ is a pseudonatural transformation, it is easy to verify all the appropriate equalities from Definition 3.3.17. □

Definition 6.2.3 Suppose $(\underline{X}, w_{\underline{X}})$ and $(\underline{Y}, w_{\underline{Y}})$ are continuous categories. Let $F : \underline{X} \longrightarrow \underline{Y}$ be a finitary functor. F is said to be *way-below preserving*, if the way-below mate w_F of F is a natural isomorphism.

It is clear that way-below preserving functors are closed under composition, thus one can form a *2-quasicategory* $\mathsf{CONT}_{\mathrm{wb}}$ having all continuous categories as 0-cells, all way-below preserving functors as 1-cells and all natural transformations as 2-cells.

It is straightforward that if $\underline{X}$ and $\underline{Y}$ are continuous domains, then a continuous function $f : \underline{X} \longrightarrow \underline{Y}$ is way-below preserving precisely when the following holds for any x, y in $\underline{X}$:

$$\text{if} \quad x \ll_{\underline{X}} y \quad \text{then} \quad f(x) \ll_{\underline{Y}} f(y)$$

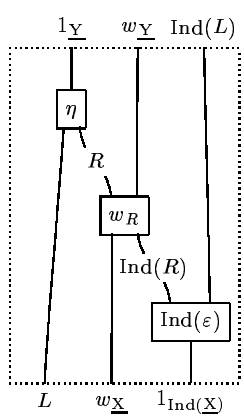
It follows that for a finitary retraction (E, R, α) from a continuous category $\underline{Y}$ to a continuous category $\underline{X}$, the functor E *need not* be way-below preserving as the following example shows:

Example 6.2.4 Let $\underline{Y}$ be a one-element poset, let $\underline{X}$ be the unit interval with the usual order. Then both $\underline{Y}$ and $\underline{X}$ are continuous DCPOs ($\underline{Y}$ is even algebraic). Define E as a constant with value 1 and let R be the unique (continuous) mapping from $\underline{X}$ to $\underline{Y}$. It is clear that $(E, R, 1_{\underline{Y}})$ is a finitary retraction, but E is not way-below preserving. □

The above example shows that $\mathsf{CONT}_{\mathrm{wb}}$ is not closed under finitary retracts in CAT. We however show in Corollary 6.2.7 that $\mathsf{CONT}_{\mathrm{wb}}$ is closed in CAT under (e,p)-adjunctions (recall 4.1.7). We prove a slightly more general result first.

Theorem 6.2.5 *Suppose* $(\underline{X}, w_{\underline{X}})$ *and* $(\underline{Y}, w_{\underline{Y}})$ *are continuous categories. Let* $L \dashv R$ *be a finitary adjunction with* $L : \underline{Y} \longrightarrow \underline{X}$. *Then* L *is a way-below preserving functor.*

Proof. Denote by η and ε the unit and the counit of $L \dashv R$. Since R is a finitary functor, we can compute the way-below mate w_R of R. We claim that the natural transformation

$\tau =$ 

(6.3)

is the inverse of w_L. This is straightforward: use the facts that $w : 1_{\mathsf{CONT}} \Longrightarrow \mathrm{Ind}$ is a lax natural transformation and triangle identities for $L \dashv R$. □

Remark 6.2.6 The above theorem is a generalization of Proposition 3.1.14 of [AbJ96] which states that a left adjoint with a finitary right adjoint between continuous domains preserves the way-below relation. □

Corollary 6.2.7 *Suppose* $(\underline{\mathrm{X}}, w_{\underline{\mathrm{X}}})$ *is a continuous category. Let* $\underline{\mathrm{Y}}$ *be an arbitrary category and let* $E \dashv P$ *be an (e,p)-adjunction with* $E : \underline{\mathrm{Y}} \longrightarrow \underline{\mathrm{X}}$. *Then* $\underline{\mathrm{Y}}$ *is a continuous category and* E *is way-below preserving.*

Proof. Since the counit of $E \dashv P$ is an isomorphism, $\underline{\mathrm{Y}}$ is a finitary retract of $\underline{\mathrm{X}}$ and therefore by Theorem 6.1.11 it has a structure of a continuous category $(\underline{\mathrm{Y}}, w_{\underline{\mathrm{Y}}})$. By the previous theorem E is way-below preserving. □

The following observation is easy to prove:

Lemma 6.2.8 *Let* $\langle X, \sqsubseteq \rangle$ *be a poset, let* x, y *be arbitrary elements of* X. *Then the following are equivalent:*

1. $x \ll y$.
2. $\{z \in X \mid z \sqsubseteq x\} \subseteq \{z \in X \mid z \ll y\}$.

The above lemma leads to the following concept of an *abstract way-below relation* in a continuous category (see [JJ82], p. 267):

Definition 6.2.9 Let $\underline{\mathrm{X}}$ be a continous category. A morphism $f : I_{\underline{\mathrm{X}}}(x) \longrightarrow w_{\underline{\mathrm{X}}}(y)$ in $\mathrm{Ind}(\underline{\mathrm{X}})$ is called a *wavy arrow from* x *to* y *in* $\underline{\mathrm{X}}$.

Recall the notion of a flat distributor from Definition 4.4.2. The following is proved in [JJ82]:

Theorem 6.2.10 *Let* $\underline{\mathrm{X}}$ *be a continuous category. If we denote by* $W(x, y)$ *the set of all wavy arrows from* x *to* y, *then* $W(x, y)$ *is an object function of a flat distributor* $W : \underline{\mathrm{X}} \longrightarrow \underline{\mathrm{X}}$. *Moreover, there is a natural transformations* ε *from* W *to the identity distributor* $i_{\underline{\mathrm{X}}}$ *on* $\underline{\mathrm{X}}$ *and a natural isomorphism* δ *from* W *to* $W \cdot W$ *which make the triple* (W, ε, δ) *a comonad of distributors.*

Remark 6.2.11 The definition of a distributor $\varphi : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ required categories $\underline{\mathrm{A}}$ and $\underline{\mathrm{B}}$ to be small. One can obviously drop this requirement and define a distributor $\varphi : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{B}}$ to be a functor $\varphi : \underline{\mathrm{B}}^{op} \times \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{Set}}$. There is a problem that such things cannot be composed in general, since this would involve a quotient of a class which may not be a set. In the special case of a distributor W described above one can, however, replace this class by a small set (see [JJ82], p. 269). Thus, $W \cdot W$ is a distributor in the extended sense, i.e. it is a functor from $\underline{\mathrm{X}}^{op} \times \underline{\mathrm{X}} \longrightarrow \underline{\mathrm{Set}}$.

Since distributors form a bicategory one also has to be careful about the notion of a comonad. By saying that (W, ε, δ) is a *comonad of distributors* we mean that the following diagrams of natural transformations commute (in the notation of the proof of Lemma 4.4.5):

$$\begin{tikzcd}
 & W \arrow[dl, "\rho_W^{-1}"'] \arrow[d, "\delta"] \arrow[dr, "\lambda_W^{-1}"] & \\
W \cdot i_{\underline{X}} & W \cdot W \arrow[l, "W\varepsilon"] \arrow[r, "\varepsilon W"'] & i_{\underline{X}} \cdot W
\end{tikzcd} \tag{6.4}$$

$$\begin{tikzcd}
W \cdot (W \cdot W) \arrow[r, "\alpha_{W,W,W}"] & (W \cdot W) \cdot W & W \cdot W \arrow[l, "\delta W"'] \\
W \cdot W \arrow[u, "W\delta"] \arrow[rr, "\delta"'] & & W \arrow[u, "\delta"']
\end{tikzcd} \tag{6.5}$$

□

Comonads of flat distributors on a category form, seemingly, a natural candidate for generalizing abstract bases in the sense of Abramsky and Jung ([AbJ96], Chapter 2) — see also Definition 6.1.1. This is a work in progress and we do not include any results in this spirit.

Chapter 7

Subobjects and Universal Objects

In this chapter we first define two notions which generalize the notion of a *subobject* of a domain. Since domains are categories, subobjects should be (full) subcategories and the embedding should somehow reflect the corresponding structures.

We have already encountered generalizations of classical notions of

- *finitary retracts* — Definition 6.1.9,
- *embedding-projection pairs* — Definition 4.1.7.

The contents of Section 7.1 is a more thorough study of the above notions in the context of various defininitions of a domain; we are interested in the following question:

> Suppose that K is a 2-quasicategory of domains and suppose we are given a notion of a subobject. Is K closed under this notion of a subobject in CAT?

Subdomains in classical domain theory are important because of a technique (developed by Dana Scott) of solving recursive domain equations of type $X = F(X)$ based on Theorem 2.2.3 and Scott's result on the existence of a universal domain.

The idea is that there is a special Scott domain U and a process which assigns a continuous mapping $f : U \longrightarrow U$ to the functor F. Solving the equation $X = F(X)$ then boils down to finding a fixed point of f. There is, however, a restriction: we have to deal with the so called *countably based Scott domains* — i.e. domains having a countable set of compact elements. This restriction is nothing grave: Scott domains in practical applications are countably based.

For the above process, the domain U must have the following property:

> any countably based domain is isomorphic to a subdomain of U.

Such objects U are called universal. Since it is completely legitimate to talk about CUSL embeddings instead of embedding-projection subdomains, the existence of a universal domain restricts to the *existence of a universal CUSL*. In [Sc82b] Dana Scott has constructed a universal countable CUSL K, i.e. he proved that for any countable CUSL A there is a CUSL embedding $f : A \longrightarrow K$.

The existence of universal domains in classical domain theory is of a great importance. If $\underline{\mathrm{D}}$ is any category of domains and embeddings (i.e. morphisms which correspond to

subobjects), then the existence of a universal domain in $\underline{\mathrm{D}}$ asserts the existence of a $\underline{\mathrm{D}}$-object U, such that for any $\underline{\mathrm{D}}$ object D there is an embedding of D to U. In categorical terminology, U is a *weakly terminal object* of $\underline{\mathrm{D}}$.

In later sections of this chapter, we prove the existence of weakly terminal objects in:

- $(\mathsf{PLOT}_{\mathrm{e}})_o$ — the *quasicategory* of all Plotkin categories (see Definition 7.3.1) and normal embeddings — Corollary 7.3.10,
- $(\mathsf{FCC}_{\mathrm{e}})_o$ — the *quasicategory* of all FCC categories and FCC embeddings — Corollary 7.4.7.

Since normal functors correspond to (e,p)-adjunctions by Theorem 4.3.12, results on weakly terminal objects in $(\mathsf{PLOT}_{\mathrm{e}})_o$ and $(\mathsf{FCC}_{\mathrm{e}})_o$ provide us with universal domains w.r.t. (e,p)-adjunctions.

7.1 Subobjects

Recall that we have defined the following 2-quasicategories of domains:

- FILT — the 2-quasicategory of all categories having small filtered colimits, all finitary functors and all natural transformations,
- IND — the 2-quasicategory of all inductive categories (=categories having small filtered colimits and initial objects), all finitary functors and all natural transformations,
- $\aleph_0$-ACC — the 2-quasicategory of all finitely accessible categories, all finitary functors and all natural transformations,
- GDOM — the 2-quasicategory of all generalized domains (=finitely accessible categories having initial objects), all finitary functors and all natural transformations,
- SC — the 2-quasicategory of all Scott complete categories, all finitary functors and all natural transformations,
- CONT — the 2-quasicategory of all continuous categories, all finitary functors and all natural transformations.
- $\mathsf{CONT}_{\mathrm{wb}}$ — the 2-quasicategory of all continuous categories, all way-below preserving functors and all natural transformations.

All the above 2-quasicategories are viewed as sub-2-quasicategories of FILT.

Finitary Retracts as Subobjects

Recall the definition of a finitary retract (Definition 6.1.9).

Definition 7.1.1 Let K be a 2-quasicategory of domains. We say that K *is closed under finitary retracts in* CAT provided the following holds: if $\underline{X}$ and $\underline{Y}$ are arbitrary categories with $\underline{X}$ in K and if (E, R, α) is a finitary retraction from $\underline{Y}$ to $\underline{X}$ then $\underline{Y}$ is in K.

The first result on finitary retracts follows immediately from Lemma 6.1.10 and Theorem 6.1.11:

Theorem 7.1.2 *The 2-quasicategories* FILT, IND *and* CONT *are closed under finitary retracts in* CAT.

Negative results are as follows:

Theorem 7.1.3 *Neither* $\mathsf{CONT}_{\mathrm{wb}}$, *nor* SC, *nor* GDOM *nor* $\aleph_0$-ACC *are closed under finitary retracts in* CAT.

Proof. Recall that we have shown in Example 6.2.4 that $\mathsf{CONT}_{\mathrm{wb}}$ is not closed under finitary retracts.

To prove the rest of the statement, let $\underline{X}$ denote the unit interval with the usual order. The ideal completion $\underline{X}^*$ of $\underline{X}$ is a locally finitely presentable category with an initial object. $\underline{X}$ is a finitary retract of $\underline{X}^*$, since it is a continuous lattice. $\underline{X}$ is, however, not a finitely accessible category — its only finitely presentable object is 0. □

(e,p)-Adjunctions as Subobjects

Recall the notion of an (e,p)-adjunction (Definition 4.1.7). An (e,p)-adjunction $L \dashv R$ with $L : \underline{Y} \longrightarrow \underline{X}$ exhibits $\underline{Y}$ essentially as a full coreflective subcategory of $\underline{X}$.

Definition 7.1.4 Let K be a 2-quasicategory of domains. We say that K *is closed under (e,p)-subobjects in* CAT provided the following holds: if $\underline{X}$ and $\underline{Y}$ are arbitrary categories with $\underline{X}$ in K and if $L \dashv R$ is an (e,p)-adjunction with $L : \underline{Y} \longrightarrow \underline{X}$ then $\underline{Y}$ is in K.

Theorem 7.1.5 *The 2-quasicategories* FILT, IND, CONT *and* $\mathsf{CONT}_{\mathrm{wb}}$ *are closed under (e,p)-subobjects in* CAT.

Proof. The 2-quasicategories FILT, IND and CONT are closed under (e,p)-subobjects in CAT since each (e,p)-adjunction is a special case of a finitary retraction. Corollary 6.2.7 shows that $\mathsf{CONT}_{\mathrm{wb}}$ is closed under (e,p)-subobjects. □

Negative results are as follows:

Theorem 7.1.6 *Neither* SC, *nor* GDOM *nor* $\aleph_0$-ACC *are closed under (e,p)-subobjects in* CAT.

Proof. Use the unit interval $\underline{X}$ from the proof of Theorem 7.1.3. There is an (e,p)-adjunction $w_{\underline{X}} \dashv \sup_{\underline{X}}$ from $\underline{X}$ to $\underline{X}^*$, since the unit of this adjunction is an isomorphism (see page 86). $\underline{X}^*$ is a locally finitely presentable category, but $\underline{X}$ is not finitely accessible. □

7.2 The General Embedding Theorem

The following proofs of the existence of weakly terminal objects use the Embedding Theorem of Věra Trnková from [Tr66b]. Let EMB denote the *quasicategory* of all categories and full embeddings. For a (quasi)category $\mathcal{C}$, denote by $\mathcal{C}^{\longrightarrow}$ the (quasi)category of all $\mathcal{C}$-morphisms. The Embedding Theorem reads as follows:

Theorem 7.2.1 *Suppose that $\mathcal{C}$ is a (not neccessarily full) subquasicategory of EMB, and that $\underline{\mathrm{C}}$ is a small category. Then $\mathcal{C}$ has a weakly terminal object, provided that the following conditions hold:*

1. *$\mathcal{C}^{\longrightarrow}$ has colimits of transfinite chains $D : (T, <) \longrightarrow \mathcal{C}^{\longrightarrow}$, where Dt have small domain and codomain for all $t \in T$,*
2. *each $\mathcal{C}$-object contains an isomorphic copy of $\underline{\mathrm{C}}$,*
3. *if $\underline{\mathrm{K}}$ is a $\mathcal{C}$-object, $\underline{\mathrm{L}}$ contains $\underline{\mathrm{C}}$ and $F : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{L}}$ is an isomorphism which is identity on $\underline{\mathrm{C}}$, then $\underline{\mathrm{L}}$ is a $\mathcal{C}$-object,*
4. *for any small cone $E_i : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{K}}_i$, $(i \in I)$, of small categories in $\mathcal{C}$ there exists a cocone $F_i : \underline{\mathrm{K}}_i \longrightarrow \underline{\mathrm{L}}$, $(i \in I)$, with $\underline{\mathrm{L}}$ a small category in $\mathcal{C}$ such that $F_i \cdot E_i = F_j \cdot E_j$ for all $i, j \in I$,*
5. *a category $\underline{\mathrm{K}}$ is a $\mathcal{C}$-object iff there is a transfinite chain $D : (T, <) \longrightarrow \mathcal{C}$ with Dt small for all $t \in T$, s.t. $\underline{\mathrm{K}} = \operatorname{colim} D$.*

Proof. See [Tr66b]. □

7.3 A Universal Plotkin Category

Recall from Definition 4.3.14 that an $\aleph_0$-category $\underline{\mathrm{K}}$ with an initial object is an $\aleph_0$-*Plotkin category* if each finite diagram $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{K}}$ factors through a normal embedding $F : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{K}}$ where $\underline{\mathrm{A}}$ is a finite category with an initial object. Our definition was motivated by the notion of a *Plotkin poset*. Plotkin posets are precisely posets arising as posets of compact elements of SFP domains (see e.g. [GS90]). We have proved a similar characterization of $\aleph_0$-Plotkin categories in Lemma 4.3.16. Let us now drop the requirement that the category $\underline{\mathrm{K}}$ should be a category with countably many objects and finite hom-sets. More precisely, we define:

Definition 7.3.1 Let $\underline{\mathrm{K}}$ be a category with an initial object. Then $\underline{\mathrm{K}}$ is called a *Plotkin category* if each small diagram $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{K}}$ factors through a normal embedding $F : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{K}}$ where $\underline{\mathrm{A}}$ is a small category with an initial object.

Remark 7.3.2 By definition, *every* small category with initial object is a Plotkin category, since the identity functor is a normal embedding. □

Let $\mathsf{PLOT}_{\mathrm{e}}$ denote the *2-quasicategory* of all Plotkin categories, all normal embeddings and all natural transformations between them. As usual, $(\mathsf{PLOT}_{\mathrm{e}})_o$ denotes the corresponding underlying quasicategory.

The following lemma provides an easy characterization of Plotkin categories:

Lemma 7.3.3 *$\underline{\mathrm{K}}$ is a Plotkin category iff there is a transfinite chain $D : (T, <) \longrightarrow (\mathsf{PLOT}_{\mathrm{e}})_o$ with all Dt small, s.t. $\underline{\mathrm{K}} = \operatorname{colim} D$.*

Proof. The proof is essentially the same as the proof of Lemma 4.3.16. □

The existence of a Plotkin category which is universal w.r.t. normal embeddings will follow after verifying all assumptions of Theorem 7.2.1 for the data $\mathcal{C} = (\mathsf{PLOT}_{\mathrm{e}})_o$ and $\underline{\mathrm{C}}$ the one-morphism category $\underline{\mathbf{1}}$.

Lemma 7.3.3 is precisely Condition 5. of Theorem 7.2.1. It remains to verify Conditions 1.–4. of the Embedding Theorem. We do this in a series of lemmas below.

Lemma 7.3.4 *$(\mathsf{PLOT}_{\mathrm{e}})_o^{\longrightarrow}$ has colimits of transfinite chains $D : (T, <) \longrightarrow (\mathsf{PLOT}_{\mathrm{e}})_o^{\longrightarrow}$, where all Dt have small domains and codomains.*

Proof. Suppose $D : (T, <) \longrightarrow (\mathsf{PLOT}_{\mathrm{e}})_o^{\longrightarrow}$ is a transfinite chain. Denote $Dt = F_t : \underline{\mathrm{K}}_t \longrightarrow \underline{\mathrm{L}}_t$ and $D(s \leq t) = (i_{st}, j_{st})$, i.e. for $s \leq t$ we have a commutative square

$$\begin{tikzcd} \underline{\mathrm{K}}_s \arrow[r, "i_{st}"] \arrow[d, "F_s"'] & \underline{\mathrm{K}}_t \arrow[d, "F_t"] \\ \underline{\mathrm{L}}_s \arrow[r, "j_{st}"'] & \underline{\mathrm{L}}_t \end{tikzcd} \tag{7.1}$$

of small Plotkin categories and normal embeddings.

First, it is clear that the colimit of D exists in $\mathsf{EMB}^{\longrightarrow}$: denote by $\underline{\mathrm{K}}_\infty$ the category on the class of objects $\bigcup\{Ob(\underline{\mathrm{K}}_t) \mid t \in T\}$ such that each category $\underline{\mathrm{K}}_t$ is a full subcategory of $\underline{\mathrm{K}}_\infty$. Denote the full embeddings by $i_{t\infty} : \underline{\mathrm{K}}_t \longrightarrow \underline{\mathrm{K}}_\infty$. The category $\underline{\mathrm{L}}_\infty$ and full embeddings $j_{t\infty} : \underline{\mathrm{L}}_t \longrightarrow \underline{\mathrm{L}}_\infty$ are defined in a similar way. There is a unique functor $F_\infty : \underline{\mathrm{K}}_\infty \longrightarrow \underline{\mathrm{L}}_\infty$ such that all the squares

$$\begin{tikzcd} \underline{\mathrm{K}}_s \arrow[r, "i_{st}"] \arrow[d, "F_s"'] & \underline{\mathrm{K}}_\infty \arrow[d, "F_\infty"] \\ \underline{\mathrm{L}}_s \arrow[r, "j_{st}"'] & \underline{\mathrm{L}}_\infty \end{tikzcd} \tag{7.2}$$

commute. Then $F_\infty : \underline{\mathrm{K}}_\infty \longrightarrow \underline{\mathrm{L}}_\infty$ is a colimit of D in $\mathsf{EMB}^{\longrightarrow}$. It remains to prove that $\underline{\mathrm{K}}_\infty$, $\underline{\mathrm{L}}_\infty$ are Plotkin categories and that the functors $i_{t\infty}$, $j_{t\infty}$ and F_∞ are normal embeddings.

Clearly, both the categories $\underline{\mathrm{K}}_\infty$ and $\underline{\mathrm{L}}_\infty$ are Plotkin categories (Lemma 7.3.3). The functors $i_{t\infty}$, $j_{t\infty}$ and F_∞ clearly preserve an initial object.

The functors $i_{t\infty}$, $j_{t\infty}$ and F_∞ are normal embeddings if $(T, <)$ has the greatest element. Suppose this is not the case.

To prove that $i_{t\infty}$ is a normal embedding, take any $\underline{\mathrm{K}}_\infty$-object b and any finite non-empty diagram $X : \underline{\mathrm{X}} \longrightarrow \underline{\mathrm{K}}_\infty/b$. Then there exists $s \geq t$ such that b is $\underline{\mathrm{K}}_s$-object and the whole diagram X lies in $\underline{\mathrm{K}}_s/b$. Then $1_b : b \longrightarrow b$ is the vertex of a compatible cocone on X.

The proofs that $j_{t\infty}$ and F_∞ are normal embeddings are similar. □

Lemma 7.3.5 *Each Plotkin category contains an isomorphic copy of* $\underline{\mathbf{1}}$. *If* $\underline{\mathrm{K}}$ *is a Plotkin category,* $\underline{\mathrm{L}}$ *contains* $\underline{\mathbf{1}}$ *and* $F : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{L}}$ *is an isomorphism which is the identity functor on* $\underline{\mathbf{1}}$, *then* $\underline{\mathrm{L}}$ *has an initial object.*

Proof. Trivial. □

We are now going to recall, for notational convenience, some of the standard concepts concerning the presentation of categories via graphs (in the categorical sense, i.e. large directed multigraphs) and commutatitvity conditions — see e.g. [Bo94] or [Sch70].

Definition 7.3.6 A *graph* $\mathbf{G}$ consists of a set $V(\mathbf{G})$, called a set of *vertices* of $\mathbf{G}$ and for each pair v, v' of vertices a set $\mathbf{G}(v, v')$, called the set of *arrows* from v to v'. For $f \in \mathbf{G}(v, v')$ the vertex v is called the *domain* of f, v' the *codomain* of f. A *path* from v to v' is a finite sequence $(v_{n+1}, f_n, v_n, \dots v_2, f_1, v_1)$ of alternating vertices and arrows such that $v_1 = v$, $v_{n+1} = v'$ and the domain of f_i is v_i, the codomain of f_i is v_{i+1} for $i \in \{1, \dots, n\}$. Any sequence $(v_{j+1}, f_j, v_j, \dots v_{i+1}, f_i, v_i)$ with $1 \leq i < j + 1 \leq n + 1$ will be called a *subpath* of $(v_{n+1}, f_n, v_n, \dots v_2, f_1, v_1)$.

Clearly, each small category $\underline{\mathrm{K}}$ gives rise to a graph $\mathbf{Graph}(\underline{\mathrm{K}})$. On the other hand, there are graphs which are not categories. Given a graph $\mathbf{G}$, we will describe how to construct a category from it. A *path category* $\underline{\mathrm{Path}}(\mathbf{G})$ is defined as follows:

- $\underline{\mathrm{Path}}(\mathbf{G})$-objects are vertices of $\mathbf{G}$,
- the set of $\underline{\mathrm{Path}}(\mathbf{G})$-morphisms from v to v' is the set of all paths from v to v',
- the composition is defined as a concatenation

  $$(v'_{m+1}, g_m, \dots, g_1, v'_1) * (v_{n+1}, f_n, \dots, f_1, v_1) = (v'_{m+1}, g_m, \dots g_1, v_{n+1}, f_n, \dots, f_1, v_1)$$

  provided v_{n+1} equals v'_1.

These data obviously define a small category. If we start with a small category $\underline{\mathrm{K}}$, then the categories $\underline{\mathrm{Path}}(\mathbf{Graph}(\underline{\mathrm{K}}))$ and $\underline{\mathrm{K}}$ are different. To regain the category $\underline{\mathrm{K}}$ some paths would have to be identified. For identifying some paths in the path category we introduce the notion of a set of commutativity conditions.

Definition 7.3.7 Let $\underline{\mathrm{Path}}(\mathbf{G})$ be a path category of a graph $\mathbf{G}$.

1. A *commutativity condition* is a pair $c = \langle \alpha_1, \alpha_2 \rangle$ of paths with the same domain and codomain.

2. Given a commutativity condition $c = \langle \alpha_1, \alpha_2 \rangle$, two paths ω_1, ω_2 with the same domain and codomain will be called *c-equivalent*, if

   either they are equal,

   or ω_1 contains α_1 as a subpath and ω_2 contains α_2 as a subpath,

   or ω_1 contains α_2 as a subpath and ω_2 contains α_1 as a subpath.

3. Given a set *Comm* of commutativity conditions, paths ω and ω' with the same domain and codomain will be called *equivalent*, if there is a finite sequence of paths $\omega_1, \ldots, \omega_n$ and commutativity conditions $c_1, \ldots, c_{n-1}$ in *Comm* such that $\omega = \omega_1$, $\omega' = \omega_n$ and for all $i \in \{1, \ldots, n-1\}$ the paths ω_i and ω_{i+1} are c_i-equivalent.

It is well-known (and easy to verify) that each set *Comm* of commutativity conditions defines a congruence on the category $\underline{\mathrm{Path}}(\mathbf{G})$ of paths — we will denote the factor category by $\underline{\mathrm{Path}}(\mathbf{G})/Comm$. We denote by $[v_{n+1}, f_n, \ldots, f_1, v_1]$ the equivalence class of a path $(v_{n+1}, f_n, \ldots, f_1, v_1)$. Without loss of generality we can always assume that if $\langle \alpha_1, \alpha_2 \rangle \in Comm$, then $\langle \alpha_2, \alpha_1 \rangle \in Comm$.

Lemma 7.3.8 *For any small cone $E_i : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{K}}_i$, $(i \in I)$, of small categories in $(\mathsf{PLOT}_e)_o$ there exists a cocone $F_i : \underline{\mathrm{K}}_i \longrightarrow \underline{\mathrm{L}}$, $(i \in I)$, of small categories in $(\mathsf{PLOT}_e)_o$ such that $F_i \cdot E_i = F_j \cdot E_j$ for all $i, j \in I$.*

Proof. Without loss of generality, suppose that $\underline{\mathrm{K}}$ is actually a full subcategory of each $\underline{\mathrm{K}}_i$ and that each E_i is the inclusion functor. Let $\mathbf{G}$ be the graph induced by the union of all categories $\underline{\mathrm{K}}_i$ for $i \in I$. Consider the path category $\underline{\mathrm{Path}}(\mathbf{G})$ and introduce the set *Comm* which contains commutativity conditions of the following forms:

(i) $\langle (c, g, b, f, a), (c, g \cdot f, a) \rangle$ where $f : a \longrightarrow b$ and $g : b \longrightarrow c$ are $\underline{\mathrm{K}}_i$-morphisms,

(ii) $\langle (a, 1_a, a), (a) \rangle$ for all $\underline{\mathrm{K}}_i$-objects a,

(iii) $\langle (b, g, y, m, x, f, a), (b, g \cdot m, x, f, a) \rangle$ for $\underline{\mathrm{K}}$-morphism $m : x \longrightarrow y$, $\underline{\mathrm{K}}_i$-morphism $f : a \longrightarrow x$, $\underline{\mathrm{K}}_j$-morphism $g : y \longrightarrow b$, $i \neq j$,

(iv) $\langle (b, g, y, m, x, f, a), (b, g, y, m \cdot f, a) \rangle$ for $\underline{\mathrm{K}}$-morphism $m : x \longrightarrow y$, $\underline{\mathrm{K}}_i$-morphism $f : a \longrightarrow x$, $\underline{\mathrm{K}}_j$-morphism $g : y \longrightarrow b$, $i \neq j$.

Then the following facts are clear:

- if both a and b are $\underline{\mathrm{K}}_i$-objects, then each path from a to b is equivalent to a path (b, f, a) for a suitable $\underline{\mathrm{K}}_i$-morphism $f : a \longrightarrow b$,

- if a is a $\underline{\mathrm{K}}_i$-object and b is a $\underline{\mathrm{K}}_j$-object for $i \neq j$, then each path from a to b is equivalent to a suitable path (b, g, x, f, a), where x is a $\underline{\mathrm{K}}$-object and $f : a \longrightarrow x$ is a $\underline{\mathrm{K}}_i$-morphism, $g : x \longrightarrow b$ is a $\underline{\mathrm{K}}_j$-morphism,

- two paths (b, g, x, f, a) and (b, k, y, h, a) for a $\underline{\mathrm{K}}_i$-object a and a $\underline{\mathrm{K}}_j$-object b and $\underline{\mathrm{K}}$-objects x, y, with $i \neq j$, are equivalent if and only if there are

  (i) a finite "zig-zag" of $\underline{\mathrm{K}}$-morphisms $m_1 : x_0 \longrightarrow x_1$, $m_2 : x_2 \longrightarrow x_1$, $\ldots$, $m_{2n} : x_{2n} \longrightarrow x_{2n-1}$ for $n \geq 0$, where $x_0 = x$, $x_{2n} = y$,

  and

  (ii) $\underline{\mathrm{K}}_i$-morphisms $f_s : a \longrightarrow x_s$ for $s \in \{0, \ldots, 2n\}$ and $\underline{\mathrm{K}}_j$-morphisms $g_s : x_s \longrightarrow b$ for $s \in \{0, \ldots, 2n\}$, where $f_0 = f$, $g_0 = g$, $f_{2n} = h$, $g_{2n} = k$

  such that all the corresponding triangles of m's and f's commute in $\underline{\mathrm{K}}_i$ and all the corresponding triangles of m's and g's commute in $\underline{\mathrm{K}}_j$.

Define $\underline{\mathrm{L}}$ to be the factor category $\underline{\mathrm{Path}}(\mathbf{G})/\mathit{Comm}$. Define the functors $F_i : \underline{\mathrm{K}}_i \longrightarrow \underline{\mathrm{L}}$ to be the identity on $\underline{\mathrm{K}}_i$-objects and $F_i(f) = [k', f, k]$ for $\underline{\mathrm{K}}_i$-morphisms $f : k \longrightarrow k'$. Clearly, all F_i are full embeddings. We will consider each $\underline{\mathrm{K}}_i$ as a full subcategory of $\underline{\mathrm{L}}$.

It is clear that the category $\underline{\mathrm{L}}$ has initial object and that the equality $F_i \cdot E_i = F_j \cdot E_j$ holds for all $i, j \in I$. Since $\underline{\mathrm{L}}$ is a small category, it is a Plotkin category.

Let us verify that each embedding F_i is normal.

Take any $\underline{\mathrm{L}}$-object z. We will prove that F_i/z is filtered. This is clear, if z is a $\underline{\mathrm{K}}_i$-object. Suppose this is not the case, i.e. suppose that z is a $\underline{\mathrm{K}}_j$-object for some $j \neq i$.

1. The category F_i/z is non-empty, since F_i clearly preserves the initial object and hence $F_i\bot \longrightarrow z$ is a F_i/z-object.

2. Let $[z, g_a, a', f_a, a]$ and $[z, g_b, b', f_b, b]$ be F_i/z-objects. Then $[z, g_a, a']$ and $[z, g_b, b']$ are E_j/z-objects, therefore there is an E_j/z-object $[z, g, x]$ with x a $\underline{\mathrm{K}}$-object, which is the vertex of a compatible cocone for $[z, g_a, a']$ and $[z, g_b, b']$. Denote the morphisms of the cocone by $[x, h, a']$ and $[x, k, b']$. Then $[x, h, a', f_a, a] = [x, h \cdot f_a, a]$ and $[x, k, b', f_b, b] = [x, k \cdot f_b, b]$ are E_i/x-objects. Since x is a $\underline{\mathrm{K}}_i$-object, $[x, 1_x, x]$ is the vertex of a compatible cocone for $[x, h \cdot f_a, a]$ and $[x, k \cdot f_b, b]$. Therefore $[z, g, x, 1_x, x] = [z, g, x]$ is the vertex of a compatible cocone for $[z, g_a, a', f_a, a]$ and $[z, g_b, b', f_b, b]$ in F_i/z with the cocone morphisms $[x, h \cdot f_a, a]$ and $[x, k \cdot f_b, b]$.

3. Let $[b, f, a]$ and $[b, g, a]$ be parallel paths in F_i/z. This means that there is a path $[z, k, b', h, b]$ such that $[z, k, b', h, b] * [b, f, a] = [z, k, b', h, b] * [b, g, a]$. Then by the definition of composition we obtain $[z, k, b', h, b, f, a] = [z, k, b', h, g, a]$. Since

$$[z, k, b', h, b, f, a] = [z, k \cdot h, b, f, a] = [z, k, b', h \cdot f, a]$$

and

$$[z, k, b', h, b, g, a] = [z, k \cdot h, b, g, a] = [z, k, b', h \cdot g, a]$$

there is a finite "zig-zag" of $\underline{\mathrm{K}}$-morphisms $m_1 : x_0 \longrightarrow x_1$, $m_2 : x_2 \longrightarrow x_1$, $\ldots$, $m_{2n} : x_{2n} \longrightarrow x_{2n-1}$ for $n \geq 0$, where $b' = x_0$, $b' = x_{2n}$, and there are $\underline{\mathrm{K}}_i$-morphisms $f_s : a \longrightarrow x_s$ for $s \in \{0, \ldots, 2n\}$ and $\underline{\mathrm{K}}_j$-morphisms $g_s : x_s \longrightarrow b$ for $s \in \{0, \ldots, 2n\}$, where $f_0 = h \cdot f$, $g_0 = k$, $f_{2n} = h \cdot g$, $g_{2n} = k$ such that all the corresponding triangles of m's and f's commute in $\underline{\mathrm{K}}_i$ and all the corresponding triangles of m's and g's commute in $\underline{\mathrm{K}}_j$. Denote the diagram of the "zig-zag" by Z, put $S = \{0, \ldots, 2n\}$.

Then Z yields a finite non-empty diagram D in F_j/z: objects are g_i's and morphisms are m_i's. Since E_j/z is filtered, there is a vertex $[z, \bar{m}, \bar{x}]$ of a cocone for D with the cocone morphisms $[\bar{x}, k_s, x_s]$ for $s \in S$. Especially we have that

$$\bar{m} \cdot k_0 = g_0 = k = g_{2n} = \bar{m} \cdot k_{2n}$$

holds in $\underline{\mathrm{K}}_j$, thus, by the same argument as above there exists a compatible cocone for the parallel pair of E_j/z-morphisms $[\bar{x}, k_0, x_0]$ and $[\bar{x}, k_{2n}, x_0]$. Denote its vertex by $[z, \hat{m}, \hat{x}]$ and the cocone morphism by $[\hat{x}, c, \bar{x}]$.

Then $[b, f, a]$ and $[b, g, a]$ form a parallel pair of morphisms in $E_i/\hat{x}$, since

$$(c \cdot k_0 \cdot h) \cdot f = c \cdot (k_0 \cdot h \cdot f) = c \cdot (k_{2n} \cdot h \cdot g) = (c \cdot k_{2n}) \cdot h \cdot g = (c \cdot k_0 \cdot h) \cdot g.$$

Thus for $[b, f, a]$ and $[b, g, a]$ there exists a compatible cocone with vertex $[\hat{x}, m, x]$ and cocone morphism $[x, c, b]$.

The F_i/z-object $[z, \hat{m}, \hat{x}, m, x]$ forms the vertex of a compatible cocone for $[b, f, a]$ and $[b, g, a]$ with the cocone morphism $[x, c, b]$.

□

Remark 7.3.9 One can show more: each F_i in the above proof satisfies the following:

Suppose that $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{K}}_i$ is a finite non-empty diagram such that $F_i D$ is $\underline{\mathrm{L}}$-consistent. Then D is $\underline{\mathrm{K}}_i$-consistent and F_i preserves the colimit of D.

It suffices to prove the statement for $\underline{\mathrm{L}}$-consistent

(i) discrete binary diagrams in $\underline{\mathrm{K}}_i$,

(ii) parallel arrows diagrams in $\underline{\mathrm{K}}_i$.

Ad (i): Let a, b be $\underline{\mathrm{K}}_i$-objects such that there is a compatible cocone on a, b in $\underline{\mathrm{L}}$. First, we will show that the diagram a and b is compatible in $\underline{\mathrm{K}}_i$. This is clear if the compatible cocone lies in $\underline{\mathrm{K}}_i$. Suppose the vertex z of the compatible cocone lies in $\underline{\mathrm{K}}_j$ for $j \neq i$, i.e. there are $\underline{\mathrm{L}}$-morphisms $[z, g_a, a', f_a, a]$, $[z, g_b, b', f_b, b]$. Then $[z, g_a, a', f_a, a]$ and $[z, g_b, b', f_b, b]$ form a discrete diagram in F_i/z. By Lemma 7.3.8, a and b are compatible in $\underline{\mathrm{K}}_i$. It is clear that if the sum of a and b exists in $\underline{\mathrm{K}}_i$, then F_i preserves it.

Ad (ii): Let a, b be $\underline{\mathrm{K}}_i$-objects, and let $[b, f, a]$ and $[b, g, a]$ be parallel paths having a compatible cocone in $\underline{\mathrm{L}}$. Using Lemma 7.3.8 one can show that $[b, f, a]$ and $[b, g, a]$ have a compatible cocone in $\underline{\mathrm{K}}_i$. It is clear that if the coequalizer of f and g exists in $\underline{\mathrm{K}}_i$, then F_i preserves it.

Having in mind that each F_i preserves initial objects, we have proved that F_i's "behave like FCC embeddings". □

Corollary 7.3.10 *The quasicategory* $(\mathsf{PLOT}_{\mathrm{e}})_o$ *has a weakly terminal object, i.e. there exists a Plotkin category* $\underline{\mathrm{U}}$ *such that for any Plotkin category* $\underline{\mathrm{K}}$ *there is a normal embedding* $F : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{U}}$.

7.4 A Universal FCC Category

Small FCC categories have been characterized as categories of finitely presentable objects of Scott complete categories (see Definition 4.3.8).

Recall that FCC_e denotes the 2-quasicategory of all FCC categories, all normal embeddings and all natural transformations between them. The existence of a universal category among not neccessarily small FCC categories will follow after verifying all assumptions of Theorem 7.2.1 for the data $\mathcal{C} = (\mathsf{FCC}_e)_o$ and $\underline{C} = \underline{\mathbf{1}}$.

Lemma 7.4.1 $(\mathsf{FCC}_e)_o^{\longrightarrow}$ *has colimits of transfinite chains* $D : (T, <) \longrightarrow (\mathsf{FCC}_e)_o^{\longrightarrow}$, *where all* Dt *have small domain and codomain.*

Proof. The proof is similar to that of Lemma 7.3.4 and we keep the notation of that proof.

It suffices to prove that the categories $\underline{K}_\infty$, $\underline{L}_\infty$ are FCC categories and that the functors $i_{t\infty}$, $j_{t\infty}$ and F_∞ are FCC embeddings.

This is clear if $(T, <)$ has the greatest element. Suppose this is not the case.

Let $X : \underline{X} \longrightarrow \underline{K}_\infty$ be a finite $\underline{K}_\infty$-consistent diagram. Then there exists $t \in T$ such that X factors through $i_{t\infty}$ and X is $\underline{K}_t$-compatible. The colimit of X in $\underline{K}_t$ is a colimit of X in $\underline{K}_\infty$ by the definition of $\underline{K}_\infty$. Thus $\underline{K}_\infty$ is FCC category. The proof that $\underline{L}_\infty$ is FCC category is similar.

Let $X : \underline{X} \longrightarrow \underline{K}_\infty$ be a finite diagram such that $i_{t\infty} \cdot X$ is $\underline{K}_\infty$-consistent. Then there is $s \geq t$ such that the compatible cocone of $i_{t\infty} \cdot X$ lies in $\underline{K}_s$. It follows that X is $\underline{K}_t$-consistent, since i_{ts} is an FCC functor. Clearly $i_{t\infty}(\operatorname{colim} X) = \operatorname{colim}(i_{t\infty} \cdot X)$. Analogously, $j_{t\infty}$ is an FCC functor. The proof that F_∞ is an FCC functor is similar. □

Lemma 7.4.2 *Each FCC category has an initial object* $\perp$ *and the corresponding subcategory is an isomorphic copy of* $\underline{\mathbf{1}}$. *If* $\underline{K}$ *is an FCC category,* $\underline{L}$ *contains* $\underline{\mathbf{1}}$ *and* $F : \underline{K} \longrightarrow \underline{L}$ *is an isomorphism which is the identity on* $\underline{\mathbf{1}}$, *then* $\underline{L}$ *is an FCC category.*

Proof. Trivial. □

Lemma 7.4.3 *For any small cone* $E_i : \underline{K} \longrightarrow \underline{K}_i$, $(i \in I)$, *of small categories in* $(\mathsf{FCC}_e)_o$ *there exists a cocone* $G_i : \underline{K}_i \longrightarrow \underline{M}$, $(i \in I)$, *of small categories in* $(\mathsf{FCC}_e)_o$ *such that* $G_i \cdot E_i = G_j \cdot E_j$ *for all* $i, j \in I$.

By Remark 7.3.9 we have almost what we need. The functors F_i defined in 7.3.8 behave like FCC functors. The category $\underline{L}$, however, need not be FCC — it may lack the desired colimits of finite $\underline{L}$-consistent diagrams. We will "improve" this by an iterative process of adding colimits to $\underline{L}$ below.

We need the following auxiliary lemma before we will be able to prove Lemma 7.4.3.

Lemma 7.4.4 *Let* $\underline{A}$ *be a small category, let* $\mathcal{F}$ *denote a set of diagrams which have a compatible cone in* $\underline{A}$. *Then there is a small category* $\underline{A}^{(\mathcal{F})}$ *and a full embedding* $E : \underline{A} \longrightarrow \underline{A}^{(\mathcal{F})}$ *such that:*

1. *For each diagram* D *in* $\mathcal{F}$, *the functor* ED *has a limit.*

2. *E preserves all limits.*

3. *If $X : \underline{X} \longrightarrow \underline{A}$ is any diagram such that EX has a compatible cone in $\underline{A}^{(\mathcal{F})}$, then X has a compatible cone in $\underline{A}$.*

Proof. Denote by $Y : \underline{A} \longrightarrow [\underline{A}^{op}, \underline{Set}]$ the covariant Yoneda embedding. It is well-known that Y is a full embedding which preserves limits and that the category $[\underline{A}^{op}, \underline{Set}]$ is complete. Define $\underline{A}_{(\mathcal{F})}$ to be the full subcategory of $[\underline{A}, \underline{Set}]$ on objects of the form Ya for $\underline{A}$-objects a and a chosen limit $\lim YD$ for each $D \in \mathcal{F}$. Denote the limit cone for $\lim YD$ by $\lambda^{(D)} : \lim YD \longrightarrow YD$. Now, define $E : \underline{A} \longrightarrow \underline{A}^{(\mathcal{F})}$ to be the codomain corestriction of Y. Clearly, E is a full embedding which preserves limits. Let $X : \underline{X} \longrightarrow \underline{A}$ be any diagram such that $E \cdot X$ has a compatible cone $\gamma : const_x \Longrightarrow EX$ in $\underline{A}^{(\mathcal{F})}$. We will prove that X has a compatible cone in $\underline{A}$.

- If $const_x = E const_a$ for some $\underline{A}$-object a, then $\gamma = E\mu$ for $\mu : const_a \Longrightarrow X$, since E is full. Thus, X has a compatible cone in $\underline{A}$.

- If x is of the form $\lim YD$ for some $D \in \mathcal{F}$, consider the compatible cone $\delta : const_a \Longrightarrow D$ (D is supposed to have a compatible cone, say with domain a, in $\underline{A}$). Then there is a unique $\underline{A}^{(\mathcal{F})}$-morphism $f : Ea \longrightarrow x$ such that $E\delta = \lambda^{(D)} \cdot f$. Then $\gamma \cdot f : const_{Ea} \Longrightarrow E \cdot X$ is a compatible cone on $E \cdot X$. Since E is full, $\gamma \cdot f$ is of the form $E\mu$ for $\mu : const_a \Longrightarrow X$.

□

Remark 7.4.5 The dual construction is denoted by $\underline{A}_{(\mathcal{F})}$. □

Proof of Lemma 7.4.3. Recall that we are given a small cone $E_i : \underline{K} \longrightarrow \underline{K}_i$, $(i \in I)$, of small categories in $(\mathsf{FCC_e})_o$. We want to show the existence of a small FCC category $\underline{M}$ and of a cocone $G_i : \underline{K}_i \longrightarrow \underline{M}$, $(i \in I)$, in $(\mathsf{FCC_e})_o$ such that $G_i \cdot E_i = G_j \cdot E_j$ for all $i, j \in I$. Let $F_i : \underline{K}_i \longrightarrow \underline{L}$ be the cocone from Lemma 7.3.8. We know by 7.3.9 that each F_i satisfies the following:

> Suppose that $D : \underline{D} \longrightarrow \underline{K}_i$ is a finite non-empty diagram such that $F_i D$ is $\underline{L}$-consistent. Then D is $\underline{K}_i$-consistent and F_i preserves the colimit of D.

We will construct a countable chain of categories $(F^{(n,n+1)} : \underline{M}^{(n)} \longrightarrow \underline{M}^{(n+1)} \mid n \in \omega)$ as follows:

- Define $\underline{M}^{(0)} = \underline{L}$.

- Suppose we have constructed $\underline{M}^{(n)}$ for $n \in \omega$. Denote by $\mathcal{F}^{(n)}$ a (representative) set of all finite diagrams $D : \underline{D} \longrightarrow \underline{M}^{(n)}$ which are $\underline{M}^{(n)}$-consistent and which do not factor through any F_i. Define the category $\underline{M}^{(n+1)}$ to be $\underline{M}^{(n)}_{\mathcal{F}^{(n)}}$, and the functor $F^{(n,n+1)} : \underline{M}^{(n)} \longrightarrow \underline{M}^{(n+1)}$ to be the full embedding.

Then $(F^{(n,n+1)} : \underline{M}^{(n)} \longrightarrow \underline{M}^{(n+1)} \mid n \in \omega)$ is a chain of small categories and full embeddings. Define $\underline{M}$ to be a colimit of it in EMB, and define the functors $F^{(n,\omega)} : \underline{M}^{(n)} \longrightarrow \underline{M}$ to be the colimit embeddings.

(I) The category $\underline{\mathrm{M}}$ is an FCC category, because:

(i) It has the initial object $\perp$ inherited from $\underline{\mathrm{K}}$, since $\perp$ is preserved by each F_i and each $F^{(n,n+1)}$.

(ii) Let $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{L}}$ be a finite $\underline{\mathrm{M}}$-consistent diagram. Then there is $n \in \omega$ such that the given cocone of D lies in $\underline{\mathrm{M}}^{(n)}$, thus the colimit of D exists in $\underline{\mathrm{M}}^{(n+1)}$ and is preserved by $F^{(n+1,\omega)}$.

(II) Define the functors $G_i : \underline{\mathrm{K}}_i \longrightarrow \underline{\mathrm{M}}$ to be the composition $F^{(0,\omega)} \cdot F_i$. Then clearly $G_i \cdot E_i = G_j \cdot E_j$ for $i, j \in I$. Moreover:

(i) The functor G_i preserves the initial object.

(ii) Suppose $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{K}}_i$ is a finite $\underline{\mathrm{L}}$-consistent diagram. Then there is $n \in \omega$ such that the given cocone of D lies in $\underline{\mathrm{M}}^{(n)}$, therefore it is $\underline{\mathrm{M}}^{(n)}$-consistent (for simplicity we regard the embeddings as actual inclusions). Then applying the (dual of) Lemma 7.4.4, D is $\underline{\mathrm{M}}^{(0)}$-consistent and by Lemma 7.3.8 it is $\underline{\mathrm{K}}_i$-consistent. The functor G_i clearly preserves the colimit of D.

□

Lemma 7.4.6 *$\underline{\mathrm{K}}$ is an FCC category iff there is a transfinite chain $D : (T, <) \longrightarrow (\mathsf{FCC}_{\mathrm{e}})_o$ with all Dt small, s.t. $\underline{\mathrm{K}} = \operatorname{colim} D$.*

Proof. $\Rightarrow$: This is clear if the category $\underline{\mathrm{K}}$ is small. Suppose $\underline{\mathrm{K}}$ is large. Enumerate the class $Ob(\underline{\mathrm{K}})$ of all $\underline{\mathrm{K}}$-objects as $(a_\alpha \mid \alpha \in On)$ by the class On of all ordinals with $\perp = a_0$. The well-ordered class $(T, <)$ will be the class of all ordinals, the small categories Dt will be defined by transfinite induction as follows:

- $\underline{\mathrm{K}}_0$ is the one-morphism category $\underline{\mathbf{1}}$.

- Suppose t is an ordinal such that the small FCC categories Ds for $s < t$ and FCC functors $D(s \leq s')$ for $s \leq s' < t$ have been defined.

  If t is a limit ordinal, define Dt to be the colimit of the chain $(D(s \leq s') : Ds \longrightarrow Ds' \mid s \leq s' < t)$ in EMB, the functors $D(s < t) : Ds \longrightarrow Dt$ are the colimit embeddings. It is routine to verify that Dt is a small FCC category and that the functors $D(s < t)$ are FCC functors.

  If $t = s + 1$ is a successor ordinal and the FCC category Ds has been defined, the FCC category Dt is defined as follows:

  First, put $\underline{\mathrm{L}}^{(0)} = Ds$.

  Suppose the category $\underline{\mathrm{L}}^{(n)}$ has been defined. Pick α to be the least ordinal such that the $\underline{\mathrm{K}}$-object a_α is not in $\underline{\mathrm{L}}^{(n)}$. Define $\underline{\mathrm{L}}$ to be the full subcategory of $\underline{\mathrm{K}}$ on all $\underline{\mathrm{L}}^{(n)}$-objects and on object a_α, where α is the least ordinal such that the $\underline{\mathrm{K}}$-object a_α is not in $\underline{\mathrm{L}}^{(n)}$. Denote by $\mathcal{F}^{(n)}$ the set of all finite $\underline{\mathrm{L}}$-consistent diagrams which do not have a colimit in $\underline{\mathrm{L}}^{(n)}$. The category $\underline{\mathrm{L}}^{(n+1)}$ is the full subcategory of $\underline{\mathrm{K}}$ on $\underline{\mathrm{L}}^{(n)}$-objects and all objects of the form $\operatorname{colim} D$ for $D \in \mathcal{F}^{(0)}$. The functor $F^{(n,n+1)} : \underline{\mathrm{L}}^{(n)} \longrightarrow \underline{\mathrm{L}}^{(n+1)}$ is defined as the obvious full embedding.

The category $D(s+1)$ is defined as a colimit of the countable chain $F^{(n,n+1)}$: $\underline{\mathrm{L}}^{(n)} \longrightarrow \underline{\mathrm{L}}^{(n+1)}$. The functor $D(s < s+1)$ is defined as the full embedding of Ds in $D(s+1)$. The proof that $D(s+1)$ is an FCC category and $D(s < s+1)$ is an FCC functor is analogous to the proof of Lemma 7.4.3.

We have defined a transfinite chain of small FCC categories Dt, such that clearly $\underline{\mathrm{K}} = \operatorname{colim} D$.

$\Leftarrow$: Follows from Lemma 7.4.1. □

Corollary 7.4.7 *The quasicategory* $(\mathsf{FCC}_{\mathrm{e}})_o$ *has a weakly terminal object, i.e. there exists an FCC category* $\underline{\mathrm{U}}$ *such that for any FCC category* $\underline{\mathrm{K}}$ *there is an FCC functor* $F : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{U}}$.

7.5 Size Considerations

The categories $\underline{\mathrm{U}}$ from Corollaries 7.3.10 and 7.4.7 are not small, therefore their free completions w.r.t. small filtered colimits would not be generalized domains. This problem can be avoided if we restrict the notion of universality to "universal w.r.t. κ-categories", where κ is an inaccessible cardinal — recall Definition 3.1.1.

Let κ be an inaccessible cardinal (see page 14). Since the κ-th level $\mathbf{V}_\kappa$ of the cumulative hierarchy is a model of ZFC (cf. [Je78], Lemma 10.2), the Embedding Theorem 7.2.1 holds in $\mathbf{V}_\kappa$, if we replace "small" by "of cardinality $< \kappa$" and "large" by "of cardinality κ" in the definitions of all notions appearing in the hypotheses of the Embedding Theorem.

Also, modify Definition 7.3.1 in the obvious way to obtain the notion of a κ-Plotkin category:

Definition 7.5.1 Let $\underline{\mathrm{K}}$ be a κ-category with an initial object. Then $\underline{\mathrm{K}}$ is called a *κ-Plotkin category* if each κ-small diagram $D : \underline{\mathrm{D}} \longrightarrow \underline{\mathrm{K}}$ factors through a normal embedding $F : \underline{\mathrm{A}} \longrightarrow \underline{\mathrm{K}}$ where $\underline{\mathrm{A}}$ is a κ-small category with an initial object.

The above notion is clearly a generalization of the notion of an $\aleph_0$-Plotkin category (see 4.3.14).

Denote by $\underline{\kappa\mathrm{EMB}}$ the category having κ-categories as objects and full embeddings as morphisms, and by

- $\underline{\kappa\mathrm{PLOT}}_{\mathrm{e}}$ the subcategory of $\underline{\kappa\mathrm{EMB}}$ having κ-Plotkin categories as objects and normal embeddings as morphisms,
- $\underline{\kappa\mathrm{FCC}}_{\mathrm{e}}$ the subcategory of $\underline{\kappa\mathrm{EMB}}$ having FCC κ-categories as objects and FCC embeddings as morphisms.

Let $\underline{\mathrm{C}}$ be the one morphism category $\underline{\mathbf{1}}$. It can be verified in the same way as in sections 7.3 and 7.4 that the hypotheses the "κ-small version" of the Embedding Theorem are fulfilled. Therefore we obtain the following:

Corollary 7.5.2 *Let κ be an inaccessible cardinal. Then $\underline{\kappa\mathrm{PLOT}}_{\mathrm{e}}$ has a weakly terminal object, i.e. there exists a κ-Plotkin category $\underline{\mathrm{U}}$ such that for any κ-Plotkin category $\underline{\mathrm{K}}$ there is a normal embedding $F : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{U}}$.*

Corollary 7.5.3 *Let κ be an inaccessible cardinal. Then $\underline{\kappa\mathrm{FCC}}_{\mathrm{e}}$ has a weakly terminal object, i.e. there exists FCC κ-category $\underline{\mathrm{U}}$ such that for any FCC κ-category $\underline{\mathrm{K}}$ there is an FCC embedding $F : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{U}}$.*

7.6 Finitary Version of the General Embedding Theorem

The construction can also be restricted to $\kappa = \aleph_0$ for:

1. The category $\underline{\aleph_0\mathrm{PLOT}}_{\mathrm{e}}$ of $\aleph_0$-Plotkin categories and normal embeddings.

2. The category $\underline{\mathrm{PlotPos}}$ of Plotkin posets and normal embeddings. Our construction thus gives an independent proof that a universal SFP domain exists — Corollary 7.6.2.

3. The category $\underline{\aleph_0\mathrm{CUSL}}_{\mathrm{e}}$ of countable CUSLs and CUSL embeddings. In fact, in Corollary 7.6.3 we present the construction of a universal Scott domain, independent of the construction in [Sc82b].

We show, however, that the construction *cannot* be carried out for the category $\underline{\aleph_0\mathrm{FCC}}$ of FCC $\aleph_0$-categories and FCC embeddings. Namely, in Theorem 7.6.9 we show that $\underline{\aleph_0\mathrm{FCC}}$ does not have a weakly terminal object. This result comes from the joint paper of Věra Trnková and the author [TrV97].

In this section we work in the finite set theory. That is: "small" means "finite", "large" means "countable infinite".

Let us indicate how to verify the hypotheses of the finitary version of Theorem 7.2.1 for the category $\underline{\aleph_0\mathrm{PLOT}}_{\mathrm{e}}$. Recall that $\underline{\aleph_0\mathrm{PLOT}}$ is a subcategory of $\underline{\aleph_0\mathrm{EMB}}$ — the category of all $\aleph_0$-categories and full embeddings.

1. The category $\underline{\aleph_0\mathrm{PLOT}}_{\mathrm{e}}^{\longrightarrow}$ has colimits of countable chains — mimic the proof of Lemma 7.3.4.

2., 3. These hypotheses are trivially fulfilled by any $\aleph_0$-Plotkin category.

4. Given the diagram $E_i : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{K}}_i$, $(i \in I)$, I a finite set, then the proof of Lemma 7.3.8 provides us with a finite category $\underline{\mathrm{L}}$ and normal embeddings $F_i : \underline{\mathrm{K}}_i \longrightarrow \underline{\mathrm{L}}$

5. Any colimit of a countable chain of finite $\aleph_0$-Plotkin and normal embeddings is an $\aleph_0$-Plotkin category (see 4.3.16).

Thus we obtain the following corollary:

Corollary 7.6.1 *The category* $\underline{\aleph_0\mathrm{PLOT}}_e$ *has a weakly terminal object, i.e. there exists an* $\aleph_0$*-Plotkin category* $\underline{\mathrm{U}}$ *such that for any* $\aleph_0$*-Plotkin category* $\underline{\mathrm{K}}$ *there is a normal embedding* $F : \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{U}}$.

The assumptions of the finitary version of Theorem 7.2.1 are also satisfied as it is readily seen if in the previous considerations we replace $\underline{\aleph_0\mathrm{PLOT}}_e$ by the category $\underline{\mathrm{PlotPos}}$ of Plotkin posets and normal embeddings. Thus we obtain a proof of the existence of a universal SFP domain:

Corollary 7.6.2 (Universal SFP Domain) *The category* $\underline{\mathrm{PlotPos}}$ *has a weakly terminal object, i.e. there exists a Plotkin poset* U *such that for any Plotkin poset* K *there is a normal embedding* $F : K \longrightarrow U$.

The hypotheses of the finitary version of the Embedding Theorem are also fulfilled if we consider the category $\underline{\aleph_0\mathrm{CUSL}}$ of countable CUSLs and CUSL embeddings:

1. The category $\underline{\aleph_0\mathrm{CUSL}}^{\longrightarrow}$ has colimits of countable chains — mimic the proof of Lemma 7.4.1.

2., 3. These hypotheses are trivially fulfilled by any countable CUSL.

4. Given the diagram $E_i : K \longrightarrow K_i$, $(i \in I)$, I a finite set, then the proof of Lemma 7.3.8 provides us with a finite poset L and full embeddings $F_i : K_i \longrightarrow L$ which "behave like CUSL embeddings" by Remark 7.3.9. The crucial point here is that one can add all possibly missing colimits in one step — use the dual of Lemma 7.4.4 for the (finite) set $\mathcal{F}$ of finite consistent subsets of L to obtain the desired finite CUSL M and CUSL embeddings $G_i : K_i \longrightarrow M$.

5. Any colimit of countable chain of finite CUSLs and CUSL embeddings is clearly a countable CUSL. Any countable CUSL K can be expressed as a colimit of countable chain of CUSL embeddings $D : (T, <) \longrightarrow \underline{\aleph_0\mathrm{CUSL}}$ between finite CUSLs. This is clear if K is finite. If K is infinite, enumerate K-objects as $(a_n \mid n \in \omega)$ in such a way that $a_0 = \perp$. The chain will be indexed by natural numbers.

   Define $D0$ as $\{\perp\}$.

   Suppose that a finite CUSL Dn has been defined. Define the poset L to be the full subposet of K on Dn-objects and a_n, where n is the least index such that $a_n \notin Dn$. Let $\mathcal{F}$ be the finite set of all finite L-consistent subsets of L which do not have a colimit in L. The poset $D(n+1)$ is the full subposet of K on all L-objects and objects of the form $\sup D$ for $D \in \mathcal{F}$, $D(n < n+1) : Dn \longrightarrow Dn+1$ is the full embedding.

   It is clear that all Dn are CUSLs, $D(n < n+1)$ are CUSL embeddings and that $K = \operatorname{colim} D$.

Corollary 7.6.3 (Universal Scott Domain) *The category* $\underline{\aleph_0\mathrm{CUSL}}$ *has a weakly terminal object, i.e. there exists a countable CUSL* U *such that for any countable CUSL* K *there is a CUSL embedding* $f : K \longrightarrow U$.

Remark 7.6.4 The point that we work with countable CUSLs instead of $\aleph_0$-small FCC categories is really substantial:

- The existence of colimits in a partially ordered set (regarded as a category) reduces to the existence of sups. Thus in verifying the hypothesis 4. in the finitary version of the Embedding Theorem we do not need the countable iterative process of adding missing colimits from Lemma 7.4.3 — one can add the missing sups in one step.

- The finitary version of the Embedding Theorem cannot be used for the category $\underline{\aleph_0\mathrm{FCC}}$ of FCC $\aleph_0$-categories as objects and FCC functors as morphisms as the following counterexample shows:

  The $\aleph_0$-category $\underline{\mathrm{K}}$ of finite sets and set functions is FCC category, but it cannot be expressed as a colimit of a countable chain of $\aleph_0$-small FCC categories. In fact, the only full subcategories of $\underline{\mathrm{K}}$ which are FCC are $\{\perp\}$ and $\underline{\mathrm{K}}$. Thus Condition 5. of the finitary version of the Embedding Theorem is not fulfilled.

□

The *nonexistence* of universal $\aleph_0$-small universal FCC category (Theorem 7.6.9) has been proved in [TrV97]. It uses results from (categorical) universal algebra. For details on the following notions we refer to [AR94].

Let $\mathcal{V}$ be a variety of finitary one-sorted algebras with the signature Σ and the set of equations E, denote by $\underline{\mathrm{Alg}(\Sigma,\mathrm{E})}$ the category of $\mathcal{V}$-algebras and homomorphisms. Let $U : \underline{\mathrm{Alg}(\Sigma,\mathrm{E})} \longrightarrow \underline{\mathrm{Set}}$ be the forgetful functor and let $F : \underline{\mathrm{Set}} \longrightarrow \underline{\mathrm{Alg}(\Sigma,\mathrm{E})}$ be a left ajoint to U. Assume that there is a standard countable set V of variables. Due to the adjunction $F \dashv U$ each free algebra over a finite set of k generators is the k-th copower of a for $k \geq 0$, where a is the free algebra on a one-element set. We denote the free algebra on k generators by $k \otimes a$.

A *finitary algebraic theory* $\mathbf{T}$ *generated by* $\mathcal{V}$ is defined as the opposite of the category of all $\mathcal{V}$-free algebras FX, where X is a finite (including empty) subset of V, regarded as a full subcategory of $\underline{\mathrm{Alg}(\Sigma,\mathrm{E})}^{op}$. A finitary algebraic theory $\mathbf{T}$ will be called $\aleph_0$*-small*, if it is an $\aleph_0$-category.

Lemma 7.6.5 *Let $s = \{s_k\}_{k=0}^{\infty}$ be a sequence of positive natural numbers, $s_0 = 1$. Then there exists an $\aleph_0$-small finitary algebraic theory $\mathbf{T}$, such that for all $k \geq 0$ it holds that* $\operatorname{card} \mathbf{T}(k \otimes a, a) \geq s_k$.

Proof. We will desribe a variety $\mathcal{V}$ of finitary algebras which will determine $\mathbf{T}$.

We will denote the set of finitary operations for $\mathcal{V}$ as Σ. We will define the set of k-ary operations $\Sigma^{(k)}$ for each $k \geq 0$ and then let $\Sigma = \bigcup_{k=0}^{\infty} \Sigma^{(k)}$:

$$\Sigma^{(0)} = \{0\}, \qquad \Sigma^{(k)} = \{\sigma_1^{(k)}, \ldots, \sigma_{s_k}^{(k)}\}.$$

The set of equations E for $\mathcal{V}$ will be expressed as the union of the following three sets of equations:

1. $E_1 = \bigcup_{k=1}^{\infty} E_1^{(k)}$, where $E_1^{(k)}$ is the set of equations

$$\sigma_i^{(k)}(x_1, \ldots, x_k) = 0, \qquad \sigma_i^{(k)} \in \Sigma^{(k)},\ i \in \{1, \ldots, s_k\},$$

   whenever $x_\alpha = x_\beta$ for distinct α, β in $\{1, \ldots, k\}$.

2. $E_2 = \bigcup_{k=1}^{\infty} E_2^{(k)}$, where $E_2^{(k)}$ is the set of equations

$$\sigma_i^{(k)}(x_1, \ldots, x_k) = 0, \qquad \sigma_i^{(k)} \in \Sigma^{(k)},\ i \in \{1, \ldots, s_k\},$$

whenever $x_\alpha = 0$ for some $\alpha \in \{1, \ldots, k\}$.

3. $E_3 = \bigcup_{k=1}^{\infty} E_3^{(k)}$, where $E_3^{(k)}$ is the set of equations

$$\sigma_i^{(k)}(x_1, \ldots, x_\alpha, \tau_j^{(l)}(y_1, \ldots, y_l), x_{\alpha+2}, \ldots, x_k) = 0, \qquad \begin{array}{l} \sigma_i^{(k)} \in \Sigma^{(k)},\ \tau_j^{(l)} \in \Sigma^{(l)}, \\ l \geq 1,\ i \in \{1, \ldots, s_k\}, \\ j \in \{1, \ldots, s_l\}, \\ \alpha \in \{0, \ldots, k-1\}. \end{array}$$

Let us remark that above notations are indeed equations — one can easily "unwrap" the conditional statements and describe the full list of equations.

We show that for any $k \geq 0$ the $\mathcal{V}$-free algebra on k generators has a finite number of elements greater than s_k. This is clear for $k = 0$. Let $k > 0$ and denote the set of generators as $X = \{g_1, \ldots, g_k\}$. Distinct elements of FX are distinct (Σ, E)-terms over X. These are 0, $g_1, \ldots, g_k$ and all terms of the form $\sigma_i^{(l)}(g_{\pi(1)}, \ldots, g_{\pi(l)})$ for $l \leq k$, $i \in \{1, \ldots, s_l\}$, $\pi : \{1, \ldots, l\} \longrightarrow \{1, \ldots, k\}$ injective. All other terms are equal to 0 due to the set of equations E. It is clear that FX has finitely many elements and that $\operatorname{card} FX > s_k$.

From the adjunction $F \dashv U : \underline{\mathrm{Alg}(\Sigma, \mathrm{E})} \longrightarrow \underline{\mathrm{Set}}$ it follows that the set of homomorphisms from FX to FY is finite for any finite sets X, Y.

Define now a as the $\mathcal{V}$-free algebra on one generator and define $\mathbf{T}$ as the finitary theory of $\mathcal{V}$. Then $\mathbf{T}$ has the desired properties. □

The following lemma is probably a folklore in universal algebra. Nevertheless, we provide a simple proof here.

Lemma 7.6.6 *There exist uncountably many mutually non-isomorphic $\aleph_0$-small finitary algebraic theories.*

Proof. Suppose that, to the contrary, $\{\mathbf{T}^{(i)} \mid i \geq 0\}$ is the countable list of all mutually non-isomorphic $\aleph_0$-small finitary algebraic theories. By diagonal argument we will produce our $\aleph_0$-small finitary algebraic theory $\mathbf{T}$ which is not isomorphic to any $\mathbf{T}^{(i)}$.

For any $i \geq 0$ denote by $a_{(i)}$ the free algebra on one generator in $\mathbf{T}^{(i)}$ and define a sequence $s^{(i)} = \{s_k^{(i)}\}_{k=0}^{\infty}$ as follows: $s_0^{(i)} = 1$ and $s_k^{(i)} = \operatorname{card} \mathbf{T}^{(i)}(k \otimes a_{(i)}, a_{(i)})$ for $k \geq 1$.

Define a sequence $s = \{s_k\}_{k=0}^{\infty}$ as follows: $s_0 = 1$ and $s_k = 1 + \max\{s_k^{(i)} \mid i \in \{1, \ldots, k\}\}$ for $k \geq 1$. By Lemma 7.6.5 there exists a $\aleph_0$-small finitary algebraic theory $\mathbf{T}$, such that $\operatorname{card} \mathbf{T}(k \otimes a, a) \geq s_k$ for all $k \geq 0$, where a denotes the free algebra on one generator in $\mathbf{T}$. Then $\mathbf{T}$ clearly cannot be isomorphic to any $\mathbf{T}^{(i)}$ — a contradiction. □

Lemma 7.6.7 *Let $\mathbf{T}$ be an $\aleph_0$-small finitary algebraic theory. Then there exists a full embedding $E : \mathbf{T}^{op} \longrightarrow \underline{\mathrm{A}}$ such that $\underline{\mathrm{A}}$ is a FCC $\aleph_0$-category and E preserves finite colimits.*

Proof. We will define a countable chain $(F^{(i,i+1)} : \underline{A}_i \longrightarrow \underline{A}_{i+1} \mid i \in \omega)$ of $\aleph_0$-small categories and full embeddings.

Define $\underline{A}_0$ as $\mathbf{T}^{op}$.

Suppose that functors $F^{j,j+1}$ have been defined for all $j < i$. Let $\mathcal{F}$ be the set of finite non-empty diagrams in $\underline{A}_i$ which have a compatible cocone in $\underline{A}_0$ and which do not factor through any $F^{(j,j+1)}$ for $j < i$. Apply the dual of 7.4.4 to obtain $\underline{A}_{i+1}$ and the full embedding $F^{i,i+1} : \underline{A}_i \longrightarrow \underline{A}_{i+1}$.

Define $\underline{A}$ as a colimit of the chain $(F^{(i,i+1)} : \underline{A}_i \longrightarrow \underline{A}_{i+1} \mid i \in \omega)$. $\underline{A}$ is clearly an FCC $\aleph_0$-category. □

Remark 7.6.8 Using Lemma 7.6.6 we obtain the uncountable collection $E_i : (\mathbf{T}^{(i)})^{op} \longrightarrow \underline{A}_i$ $(i \in I)$ such that E_i preserves finite colimits and $\underline{A}_i$ is an FCC $\aleph_0$-category for any $i \in I$. □

Theorem 7.6.9 *The category $\underline{\aleph_0\mathrm{FCC}}$ does not have a weakly terminal object, i.e. there does not exist an FCC $\aleph_0$-category $\underline{U}$ such that for any FCC $\aleph_0$-category $\underline{K}$ there is an FCC embedding $F : \underline{K} \longrightarrow \underline{U}$.*

Proof. Suppose that a weakly terminal FCC $\aleph_0$-category $\underline{U}$ exists. Take the uncountable collection $E_i : (\mathbf{T}^{(i)})^{op} \longrightarrow \underline{A}_i$ from Remark 7.6.8. Regard the functors E_i as actual inclusions. Since $\underline{U}$ is weakly terminal, there exists an uncountable collection of FCC embeddings $F_i : \underline{A}_i \longrightarrow \underline{U}$. Then there exist i, j in I, $i \neq j$ such that $F_i a_{(i)} = F_j a_{(j)}$. Therefore $F_i(k \otimes a_{(i)}) = F_j(k \otimes a_{(j)})$ for any $k > 0$, hence the theories $\mathbf{T}^{(i)}$ and $\mathbf{T}^{(j)}$ are isomorphic, which is a contradiction. □

Chapter 8

The Fixed Point Calculus

The well-known limit-colimit coincidence for filtered colimits of domains and embedding-projection pairs (see e.g. [SP82]) has been generalized by Paul Taylor in [Tay87] to *general adjunctions* between domains (domains there are posets having directed sups). Although Taylor explicitly mentions the possibility of generalizing his results to categories having filtered colimits as domains, he does not explicitly state the result. The main result of the current chapter is that, with the "right" choice of a limit concept, Taylor's result has indeed a generalization to categories as domains.

In generalizing Taylor's original proof one has to be more careful about the proper notion of a limit. Since the dual equivalence of the category of domains and left adjoints and the category of domains and right adjoints must be replaced by a dual *biequivalence* of the respective 2-quasicategories, the proper notion here is that of a *bilimit of a pseudofunctor* — cf. 3.3.22. The reason is that a biequivalence need not preserve limits but it always preserves bilimits (of pseudofunctors).

The limit-colimit coincidence is a basic ingredient for solving recursive domain equations, i.e. equations of the form $X = F(X)$, where F is an *endofunctor* of a category of domains. The existence of a solution (determined up to an isomorphism of posets) of a recursive equation above is guaranteed — in the classical case, when X is a poset — by a property of the functor F which is analogous to continuity. We generalize this condition on F in Section 8.2 and we show that a variety of recursive equations can be solved (up to an equivalence of categories), e.g. in the 2-quasicategory FILT.

8.1 The Limit-Colimit Coincidence

Recall that:

1. FILT denotes the 2-quasicategory of all categories having small filtered colimits as 0-cells, all finitary functors as 1-cells and all natural transformations as 2-cells.

2. FILT^l denotes the 2-quasicategory of all categories having small filtered colimits as 0-cells, all left adjoint functors which have finitary right adjoints as 1-cells and all natural transformations as 2-cells.

3. FILT^r denotes the 2-quasicategory of all categories having small filtered colimits as 0-cells, all finitary right adjoint functors as 1-cells and all natural transformations

as 2-cells.

The following is rather straightforward:

Lemma 8.1.1 *The 2-quasicategory* FILT *is closed in the 2-quasicategory* CAT *under bilimits.*

Notation 8.1.2 Due to Remark 3.3.15 there is a contravariant pseudofunctor

$$(\Phi, \varphi, \psi) : \mathsf{FILT}^r \longrightarrow \mathsf{FILT}^l$$

which is a biequivalence (it is a domain-codomain restriction of the contravariant biequivalence from the proof of Theorem 3.3.14 — thus the same notation). We will use this pseudofunctor for the rest of this section. □

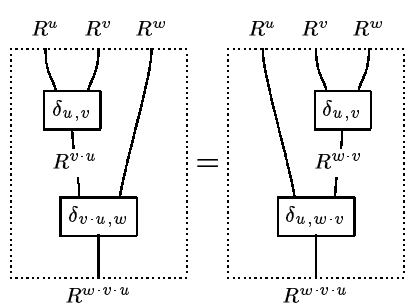
The following result appears in [Tay87] for categories of posets having directed sups. We believe, however, that an explicit formulation for categories having filtered colimits is new.

Theorem 8.1.3 *The quasicategory* FILT^r *has cofiltered bilimits.*

Proof. Let us fix first the following notation:

- $\underline{\mathrm{I}}$ is a small cofiltered category regarded as a discrete 2-category, $D : \underline{\mathrm{I}} \longrightarrow \mathsf{FILT}^r$ is a pseudofunctor, where $\underline{\mathrm{I}}$ is a small cofiltered category.

  For an $\underline{\mathrm{I}}$-morphism $u : i \longrightarrow j$, let $R^u : D(i) \longrightarrow D(j)$ denote the finitary right adjoint functor $D(u)$. Suppose that $D(1_i) = R^{1_i} = 1_{D(i)}$ for each i. Let $\delta_{u,v} : R^v \cdot R^u \Longrightarrow R^{v \cdot u}$ for $u : i \longrightarrow j$, $v : j \longrightarrow k$ denote the comparison natural isomorphism for D such that the collection of δ's satisfies the coherence conditions



(8.1)

  Since we suppose that $D(1_i) = 1_{D(i)}$, the identity coherence conditions are trivial.

  Let $L^u : D(j) \longrightarrow D(i)$ denote the left adjoint $\Phi(R^u)$ for any $\underline{\mathrm{I}}$-morphism u and let η^u and ε^u denote the unit and the counit of $L^u \dashv R^u$.

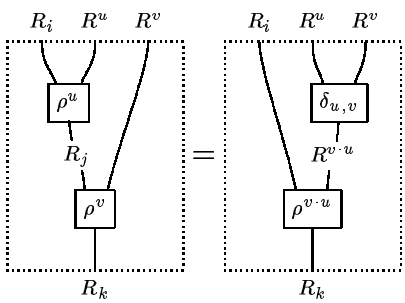
- Denote by $E : \mathsf{FILT}^r \longrightarrow \mathsf{FILT}$ the embedding 2-functor. Since E is an inclusion, we (whenever convenient) omit E for the sake of readability.

Form a bilimit of ED in FILT and denote by $\rho : const_{\underline{\mathrm{L}}} \Longrightarrow D$ the pseudonatural transformation which forms a bilimit cone. Explicitly, we have finitary functors R_j :

$\underline{\mathrm{L}} \longrightarrow D(j)$ for each $\underline{\mathrm{I}}$-object j and natural isomorphisms $\rho^w : R^w \cdot R_{j_1} \Longrightarrow R_{j_2}$ for each $\underline{\mathrm{I}}$-morphism $u : j_1 \longrightarrow j_2$. The collection of ρ^w's is subject to the following coherence conditions:

$$\rho^{1_i} : R_i \Longrightarrow R_i \text{ is the identity natural transformation for any } \underline{\mathrm{I}}\text{-object } i \tag{8.2}$$

and



for any $\underline{\mathrm{I}}$-morphisms $u : i \longrightarrow j, v : j \longrightarrow k$. (8.3)

Note that each functor R_i is finitary since the bilimit is formed in FILT, and note that (due to Lemma 8.1.1) a typical $\underline{\mathrm{L}}$-object is a *compatible thread*, i.e. a collection $\langle y_j, a_w \rangle$, where y_j is an object in $D(j)$ for any $\underline{\mathrm{I}}$-object j and $a_w : R^w(y_{j_1}) \longrightarrow y_{j_2}$ is an isomorphism in $D(j_2)$ for each $\underline{\mathrm{I}}$-morphism $w : j_1 \longrightarrow j_2$, such that the following equations hold:

$$a_{1_j} = 1_{y_j} \quad \text{for any } j \tag{8.4}$$

$$a_{w_2 \cdot w_1} = a_{w_2} \cdot R^{w_2}(a_{w_1}) \quad \text{for any } w_1 : j_1 \longrightarrow j_2, w_2 : j_2 \longrightarrow j_3 \tag{8.5}$$

Due to the definition of a bilimit we have that for any category $\underline{\mathrm{X}}$ having filtered colimits the quasicategory $\mathsf{FILT}(\underline{\mathrm{X}}, \underline{\mathrm{L}})$ having finitary functors from $\underline{\mathrm{X}}$ to $\underline{\mathrm{L}}$ as objects and natural transformations between such functors as morphisms is equivalent to the quasicategory $\mathrm{PseudoCone}(\underline{\mathrm{X}}, D)$ having psudonatural transformations from $const_{\underline{\mathrm{X}}}$ to D as objects and modifications between such pseudonatural transformations as morphisms. One part of the equivalence is a functor

$$\hat{\rho}_{\underline{\mathrm{X}}} : \mathsf{FILT}(\underline{\mathrm{X}}, \underline{\mathrm{L}}) \longrightarrow \mathrm{PseudoCone}(\underline{\mathrm{X}}, D) \tag{8.6}$$

sending a finitary functor $F : \underline{\mathrm{X}} \longrightarrow \underline{\mathrm{L}}$ to a pseudonatural transformation $\hat{\rho}_{\underline{\mathrm{X}}}(F) = (R_j F, \rho^w F)$ and sending a natural transformation $\tau : F_1 \Longrightarrow F_2$ to a modification $\hat{\rho}_{\underline{\mathrm{X}}}(\tau)$ with the j-th component $R_j\tau : R_j F_1 \Longrightarrow R_j F_2$. The functor $\hat{\rho}_{\underline{\mathrm{X}}}$ will be extensively used in the sequel.

Claim A. *Each functor R_i has a (finitary) left adjoint.*

Proof of A. Let us fix an arbitrary $\underline{\mathrm{I}}$-object i and let us show that R_i has a (finitary) left adjoint. The idea of the proof is as follows: we define a finitary functor $F_i : D(i) \longrightarrow \underline{\mathrm{L}}$ such that there is an idempotent natural transformation $F_i \Longrightarrow F_i$ whose image is the desired left adjoint of R_i.

Since we have an equivalence of categories $\mathsf{FILT}(D(i), \underline{\mathrm{L}})$ and $\mathrm{PseudoCone}(D(i), D)$ to define the functor F_i it suffices to define a pseudocone $\alpha = (A_j, \alpha^w)$ for ED with vertex $D(i)$.

Let j be an arbitrary object of $\underline{\mathrm{I}}$. The finitary functor $A_j : D(i) \longrightarrow D(j)$ is going to be a colimit of a filtered diagram $D_j : \underline{\mathrm{D}}_j \longrightarrow \mathsf{FILT}(D(i), D(j))$, where the auxiliary category $\underline{\mathrm{D}}_j$ is defined as follows:

Objects are triples (k, u, v) forming a "span"

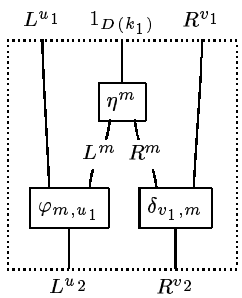
$$\begin{tikzcd} & i \\ k \arrow[ur, "u"] \arrow[dr, "v"'] & \\ & j \end{tikzcd}$$

in $\underline{\mathrm{I}}$.

Morphisms from (k_1, u_1, v_1) to (k_2, u_2, v_2) are those $\underline{\mathrm{I}}$-morphisms $m : k_2 \longrightarrow k_1$, such that $u_2 = u_1 \cdot m$ and $v_2 = v_1 \cdot m$.

Using the fact that $\underline{\mathrm{I}}$ is cofiltered, it is easy to show that $\underline{\mathrm{D}}_j$ is filtered.
The functor $D_j : \underline{\mathrm{D}}_j \longrightarrow \mathsf{FILT}(D(i), D(j))$ is defined as follows:

D_j sends an object (k, u, v) to $R^v \cdot L^u$ and it sends a morphism $m : (k_1, u_1, v_1) \longrightarrow (k_2, u_2, v_2)$ in $\underline{\mathrm{D}}_j$ to

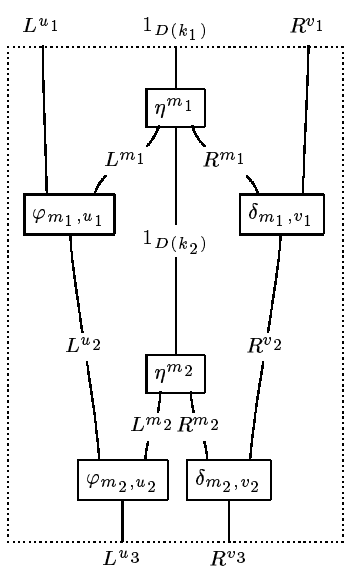


(8.7)

where φ_{m,u_1} is an abbreviated notation of the coherence isomorphism $\varphi_{R^m,R^{u_1}} : L^m \cdot L^{u_1} \Longrightarrow L^{u_1 \cdot m}$. (Use the facts that $u_1 \cdot m = u_2$ and $v_1 \cdot m = v_2$.)

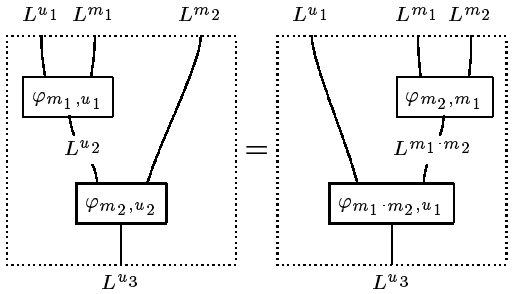
D_j is indeed a functor:
Let $m_1 : (k_1, u_1, v_1) \longrightarrow (k_2, u_2, v_2)$ and $m_2 : (k_2, u_2, v_2) \longrightarrow (k_3, u_3, v_3)$ be $\underline{\mathrm{D}}_j$-morphisms. Then due to the definition of D_j we have the equality

$D_j(m_2) \cdot D_j(m_1) =$



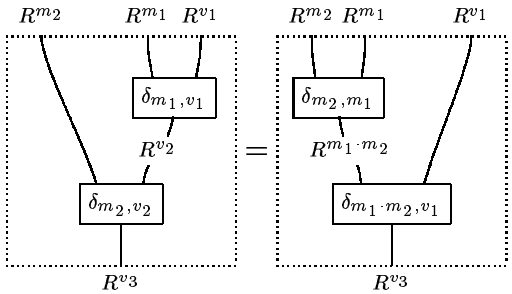
(8.8)

Since by the associativity coherence for φ (cf. (3.18)) the equality



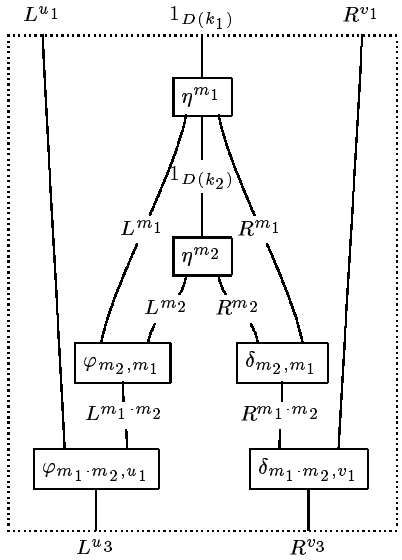
(8.9)

holds, and analogously by the associativity coherence for δ it holds that



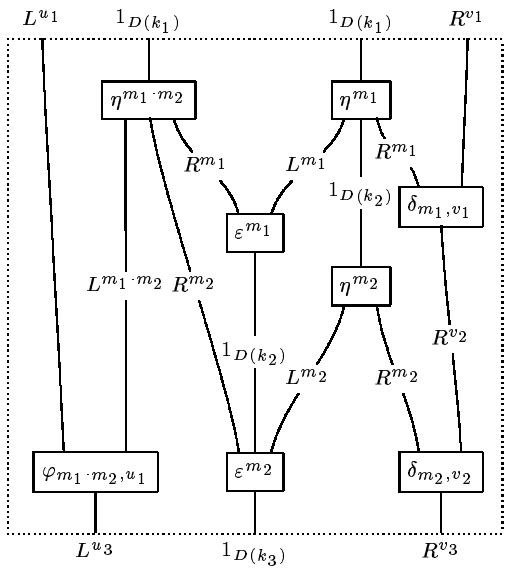
(8.10)

Therefore the right hand side of (8.8) is equal to:



(8.11)

Due to the definition of φ_{m_2,m_1} (see (3.21)), (8.11) equals to



$$= D_j(m_2 \cdot m_1) \tag{8.12}$$

where the last equality holds due to triangle equalities for $L^{m_1} \dashv R^{m_1}$ and $L^{m_2} \dashv R^{m_2}$, due to the associativity coherence for δ's and the fact that $\delta^{-1}_{m_2,m_1}$ and δ_{m_2,m_1} are mutually inverse.

It is trivial to show that D_j preserves identitites (recall that $\Phi(1_{\underline{A}}) = 1_{\underline{A}}$).

Define A_j as a colimit of D_j. (This colimit exists, since D_j is filtered and colimits in $\mathsf{FILT}(D(i), D(j))$ are computed pointwise.) Denote by $\kappa_j(k,u,v) : R^v \cdot L^u \Longrightarrow A_j$ the colimit morphisms.

It remains to define a natural isomorphism $\alpha^w : R^w \cdot A_{j_1} \Longrightarrow A_{j_2}$ for each $\underline{\mathrm{I}}$-morphism $w : j_1 \longrightarrow j_2$. Since R^w is finitary, the collection of natural transformations $R^w \kappa_{j_1}(k,u,v)$ forms a colimit cocone with $R^w \cdot A_{j_1}$ as a colimit.

Let α^w be the unique natural transformation such that the square

$$\begin{tikzcd}
R^w \cdot R^v \cdot L^u \arrow[r, "R^w \kappa_{j_1}(k,u,v)"] \arrow[d, "\delta_{v,w} L^u"'] & R^w \cdot A_{j_1} \arrow[d, "\alpha^w"] \\
R^{w \cdot v} \cdot L^u \arrow[r, "\kappa_{j_2}(k,u,w \cdot v)"'] & A_{j_2}
\end{tikzcd} \tag{8.13}$$

commutes for any $\underline{\mathrm{D}}_j$-object (k,u,v). (The coherence conditions for δ's assure that morphisms

$$\kappa_{j_2}(k,u,w \cdot v) \cdot \delta_{v,w} L^u : R^w \cdot R^v \cdot L^u \longrightarrow A_{j_2}$$

form a compatible cocone for $\mathsf{FILT}(D(i), R^w) \cdot D_{j_1}$.) To prove that α^w is an isomorphism, consider the following functor $H^w : \underline{\mathrm{D}}_{j_1} \longrightarrow \underline{\mathrm{D}}_{j_2}$:

H^w assings a $\underline{\mathrm{D}}_{j_2}$-object $(k,u,w \cdot v)$ to each $\underline{\mathrm{D}}_{j_1}$-object (k,u,v),

H^w is identity on morphisms.

Since $\underline{\mathrm{I}}$ is cofiltered, the functor H^w is cofinal. It is also clear that the collection of all morphisms $\delta_{v,w}L^u$ form a natural isomorphism

$$\theta : \mathsf{FILT}(D(i), R^w) \cdot D_{j_1} \Longrightarrow D_{j_2} \cdot H^w$$

Therefore a colimit of $\mathsf{FILT}(D(i), R^w) \cdot D_{j_1}$ is isomorphic to a colimit of $D_{j_2} \cdot H^w$ and this in turn is isomorphic to a colimit of D_{j_2} via α^w.

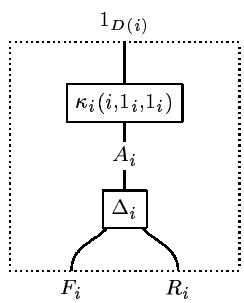
The collection (A_j, α^w) constitutes a pseudonatural transformation from $const_{D(i)}$ to ED: α^{1_j} is the identity and $\alpha^{w_2 \cdot w_1} = \alpha^{w_2} \cdot R^{w_2}\alpha^{w_1}$ by our definition for any pair of $\underline{\mathrm{I}}$-morphisms $w_1 : j_1 \longrightarrow j_2$, $w_2 : j_2 \longrightarrow j_3$.

Let $F_i : D(i) \longrightarrow \underline{\mathrm{L}}$ denote the finitary functor which corresponds to (A_j, α^w) under the equivalence $\hat{\rho}_{D(i)} : \mathsf{FILT}(D(i), \underline{\mathrm{L}}) \longrightarrow \mathrm{Pseudocone}(D(i), ED)$.

Since $\hat{\rho}_{D(i)}$ is an equivalence, we have an isomorphism modification $\Delta : (A_j, \alpha^w) \rightsquigarrow (R_jF_i, \rho^w F_i)$.

We are going to define a natural transformation $\eta_i : 1_{D(i)} \Longrightarrow R_i \cdot F_i$ and a natural transformation $\varepsilon_i : F_i \cdot R_i \Longrightarrow 1_{\underline{\mathrm{L}}}$.

Put

$\eta_i =$ 

(8.14)

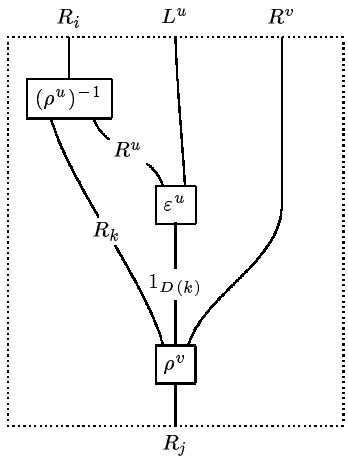
To define $\varepsilon_i : F_i \cdot R_i \Longrightarrow 1_{\underline{\mathrm{L}}}$, we use the equivalence

$$\hat{\rho}_{\underline{\mathrm{L}}} : \mathsf{FILT}(\underline{\mathrm{L}}, \underline{\mathrm{L}}) \longrightarrow \mathrm{Pseudocone}(\underline{\mathrm{L}}, ED)$$

It suffices to define a modification $(R_jF_iR_i, \rho^w F_iR_i) \rightsquigarrow (R_j, \rho^w)$ and define ε_i as its preimage under $\hat{\rho}_{\underline{\mathrm{L}}}$. Since R_i is a finitary functor, the cocone formed by $\kappa_j(k,u,v)R_i : R^vL^uR_i \Longrightarrow A_jR_i$ is a colimit. First we define τ_j as the unique natural transformation such that the triangle

$$\begin{tikzcd} R^vL^uR_i \arrow[rr, "\kappa_j(k,u,v)R_i"] \arrow[dr, "m_j(k,u,v)"'] & & A_jR_i \arrow[dl, "\tau_j"] \\ & R_j & \end{tikzcd} \tag{8.15}$$

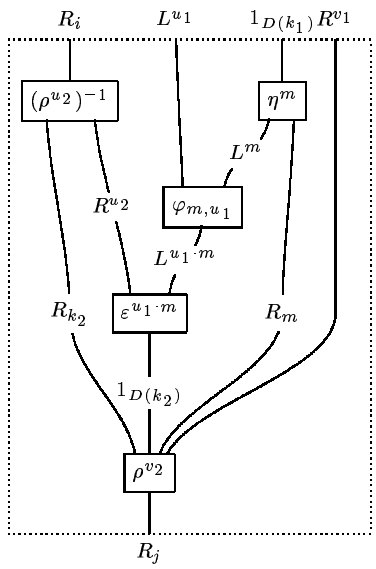
commutes for any $\underline{\mathrm{D}}_j$-object (k, u, v), where the natural transformation $m_j(k, u, v) : R^v L^u R_i \Longrightarrow R_j$ is defined as follows:

$m_j(k, u, v) =$ 

(8.16)

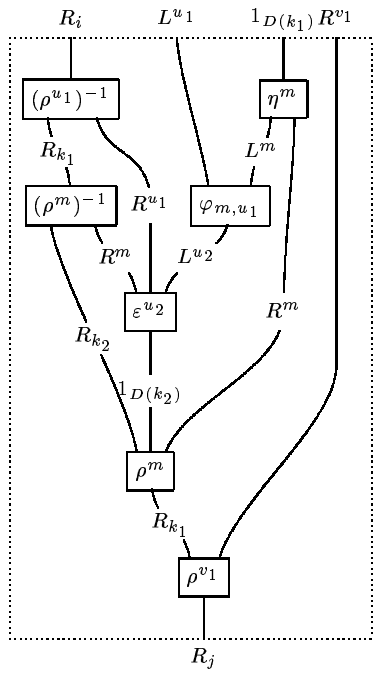
We have to verify that m_j's form a compatible cocone for the functor $\mathsf{FILT}(R_i, D(j)) \cdot D_j$. Take any $\underline{\mathrm{D}}_j$-morphism $m : (k_1, u_1, v_1) \longrightarrow (k_2, u_2, v_2)$ (i.e. we have that $m : k_2 \longrightarrow k_1$, $u_1 : k_1 \longrightarrow i$, $u_2 : k_2 \longrightarrow i$, $v_1 : k_1 \longrightarrow j$, $v_2 : k_2 \longrightarrow j$, $u_2 = u_1 \cdot m$ and $v_2 = v_1 \cdot m$ in $\underline{\mathrm{I}}$).

By the definition of D_j it holds that:

$m_j(k_2, u_2, v_2) \cdot (\mathsf{FILT}(R_i, D(j)) \cdot D_j)(m) =$ 

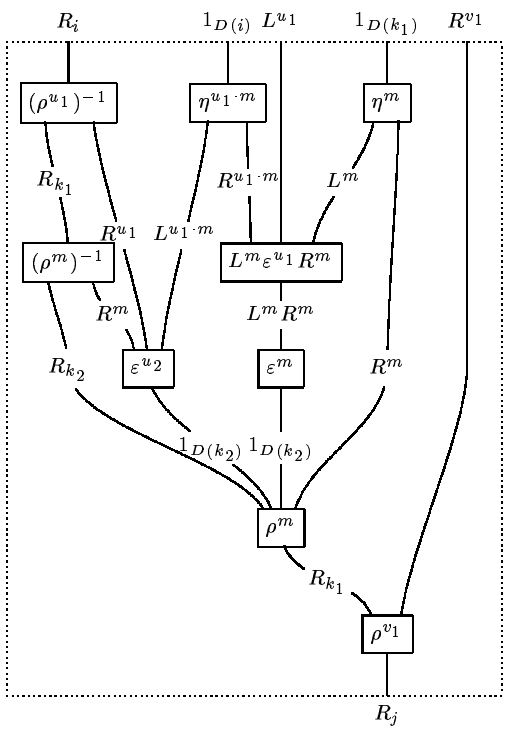
(8.17)

Due to coherence equalities for ρ and since $u_1 \cdot m = u_2$ and $v_1 \cdot m = v_2$, (8.17) equals to:



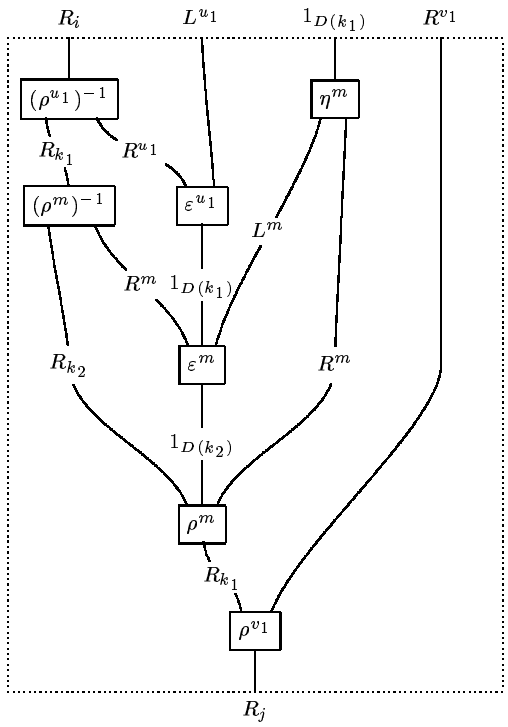
(8.18)

Due to the definition of φ_{m,u_1}, (8.18) equals to:



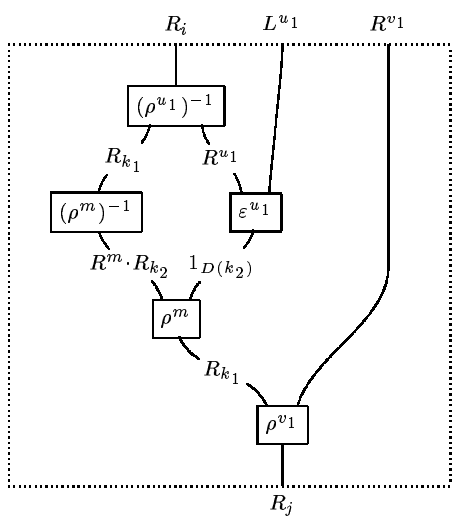
(8.19)

Due to a triangle equality for $L^{u_1 \cdot m} \dashv R^{u_1 \cdot m}$, (8.19) equals to:



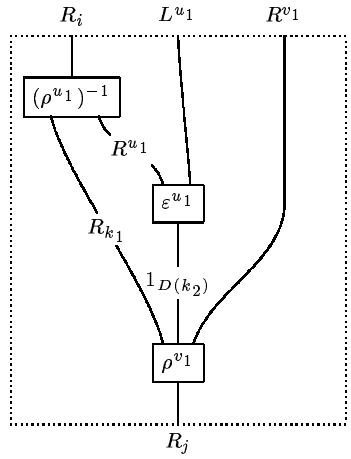
(8.20)

Due to a triangle equality for $L^m \dashv R^m$, (8.20) equals to:



(8.21)

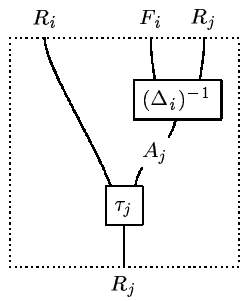
But (8.21) equals to



$$= m_j(k_1, u_1, v_1) \tag{8.22}$$

where the last equality is the definition of $m_j(k_1, u_1, v_1)$. Thus, τ_j is correctly defined. Put

$$\varepsilon_i^{(j)} =$$



(8.23)

We are going to verify that the collection of natural transformations $\varepsilon_i^{(j)}$ constitutes a modification from $(R_j F_i R_i, \rho^w F_i R_i)$ to (R_j, ρ^w) (cf. (3.29) and (3.30)). Since the only 2-cells in $\underline{\mathrm{I}}$ are identitites, the equation (3.30) is trivial and only the equation (3.29) has to be verified. Thus we have to verify the equality

$$\rho^w \cdot R^w \varepsilon_i^{(j_1)} = \varepsilon_i^{(j_2)} \cdot \rho^w F_i R_i \tag{8.24}$$

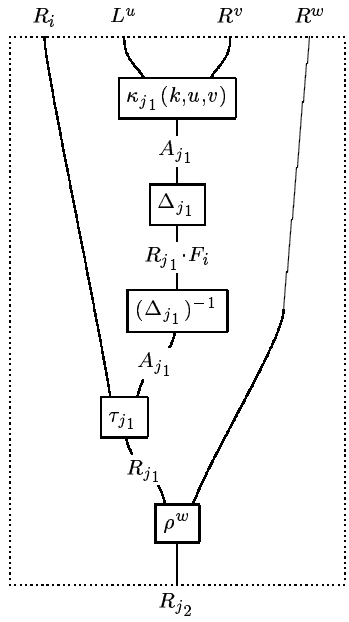
for any $\underline{\mathrm{I}}$-morphism $w : j_1 \longrightarrow j_2$.

Let us use the fact that natural transformations of the form

$$R^w \Delta_{j_1} R_i \cdot R^w \kappa_{j_1}(k, u, v) R_i$$

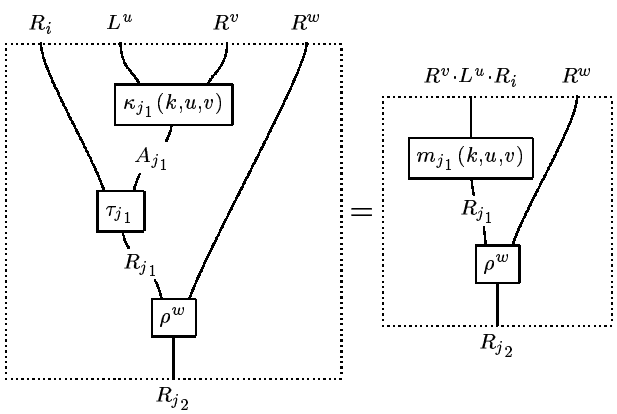
constitute a colimit cocone.

$$\rho^w \cdot R^w \varepsilon_i^{(j_1)} \cdot R^w \kappa_{j_1}(k,u,v) R_i =$$



(8.25)

Since Δ_{j_1} and $(\Delta_{j_1})^{-1}$ are mutually inverse, (8.25) is equal to:

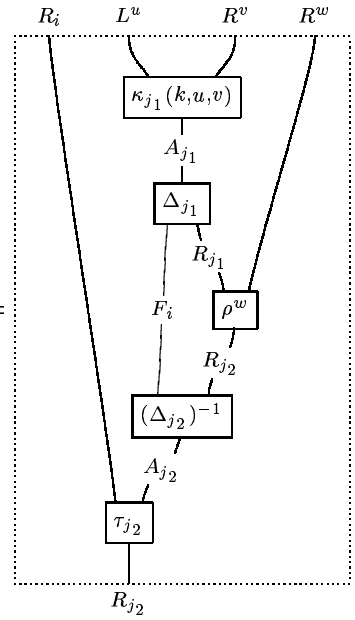


$$= m_{j_2}(k, u, w \cdot v) \qquad (8.26)$$

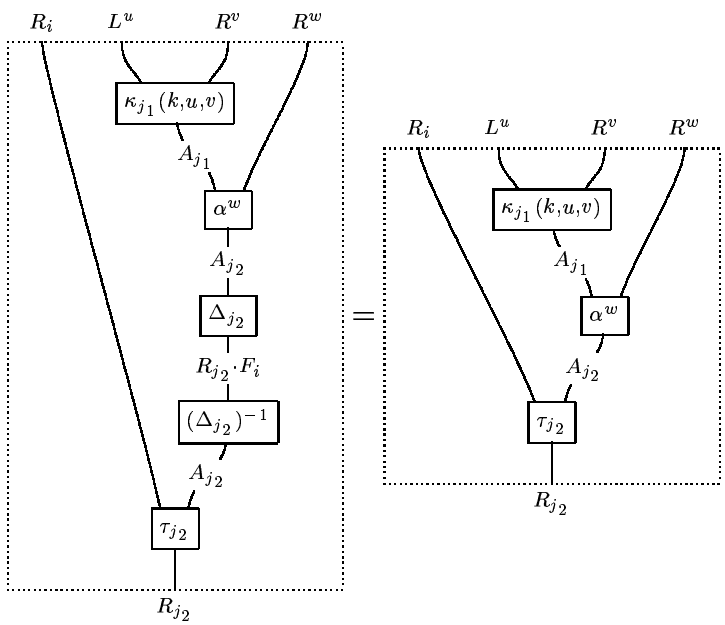
where the first equality follows from the definition of τ_{j_1} and the latter follows from the definition of $m_{j_2}(k, u, w \cdot v)$.

Let us investigate the right hand side of the equality (8.24).

$$\varepsilon_i^{(j_2)} \cdot \rho^w F_i R_i \cdot R^w \Delta_{j_1} R_i \cdot R^w \kappa_{j_1}(k,u,v) R_i = \tag{8.27}$$

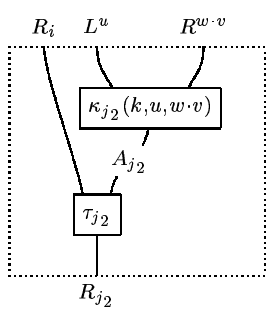


Since Δ is a modification, (8.27) is equal to:



(8.28)

By the definition of α^w, the right hand side of (8.28) equals to:



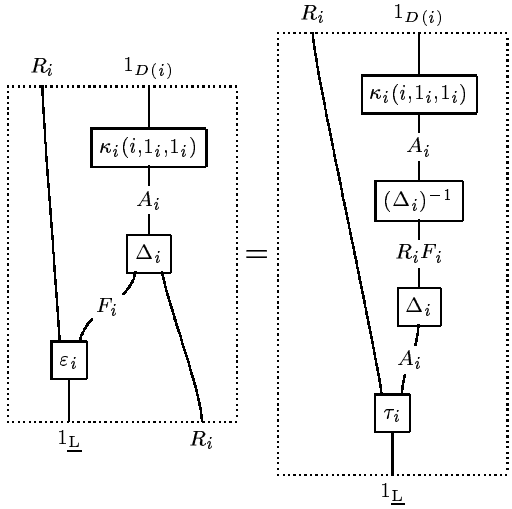
$$= m_{j_2}(k, u, w \cdot v) \tag{8.29}$$

We have proved that the equality (8.24) holds.

Due to the equivalence of $\mathsf{FILT}(\underline{\mathrm{L}}, \underline{\mathrm{L}})$ and Pseudocone$(\underline{\mathrm{L}}, ED)$ we have defined a natural transformation $\varepsilon_i : F_i R_i \Longrightarrow 1_{\underline{\mathrm{L}}}$.

Let us verify the equality $R_i\varepsilon_i \cdot \eta_i R_i = 1_{R_i}$.

By definitions we have

$R_i\varepsilon_i \cdot \eta_i R_i =$ 

$= m_i(i, 1_i, 1_i)$ (8.30)

where the last equation follows by the definition of m_i. Due to coherence conditions for ρ the transformation $m_i(i, 1_i, 1_i)$ equals to 1_{R_i}.

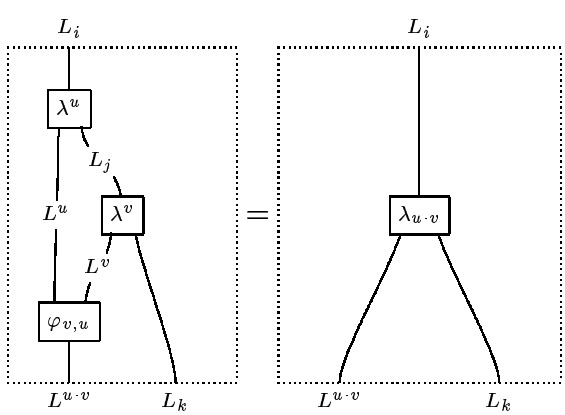
From (8.30) it follows that the natural transformation $\varepsilon_i F_i \cdot F_i\eta_i$ is an idempotent natural transformation from F_i to F_i. Since idempotents split in $\underline{\mathrm{L}}$, by Lemma 3.2.3 we conclude that R_i has a left adjoint which we denote by L_i. This finishes the proof of **A**.

Without loss of generality we can suppose that $L_i = \Phi(R_i)$ for any i. Denote by $\lambda^u : L_j \Longrightarrow L_i \cdot L^u$ the isomorphism comparison natural transformation $\Phi(\rho^u)$ for each $u : i \longrightarrow j$. Thus, (L_i, λ^u) is the image of (R_i, ρ^u) under the contravariant biequivalence Φ and therefore it is a pseudococone in FILT^l.

Let us spell out the coherence conditions for λ's:

$$\lambda^{1_i} = 1_{L_i} \quad \text{for each } \underline{\mathrm{I}}\text{-object } i \tag{8.31}$$

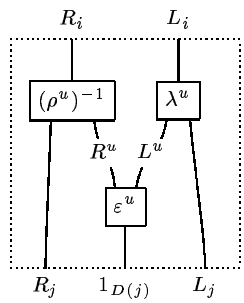
and the equation



(8.32)

holds for each pair of $\underline{\mathrm{I}}$-morphisms $u : j \longrightarrow i$, $v : k \longrightarrow j$.

Claim B. *There is a filtered diagram $W : \underline{\mathrm{I}}^{op} \longrightarrow \mathsf{FILT}(\underline{\mathrm{L}}, \underline{\mathrm{L}})$, which assigns a finitary functor $W(i) = L_i \cdot R_i$ to each $\underline{\mathrm{I}}$-object i and a natural transformation $W(u) : W(i) \Longrightarrow W(j)$ defined as*

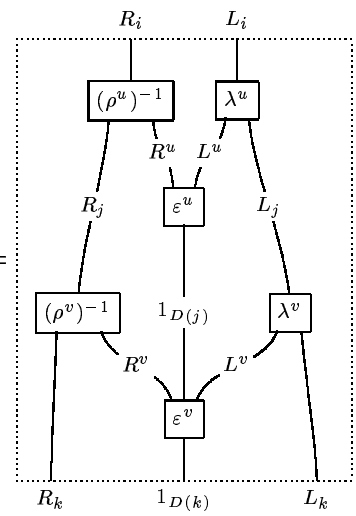


(8.33)

to each $\underline{\mathrm{I}}^{op}$-morphism $u : i \longrightarrow j$. The collection of counits $\varepsilon_i : L_i \cdot R_i \Longrightarrow 1_{\underline{\mathrm{L}}}$ forms a filtered colimit cocone for W in $\mathsf{FILT}(\underline{\mathrm{L}}, \underline{\mathrm{L}})$.

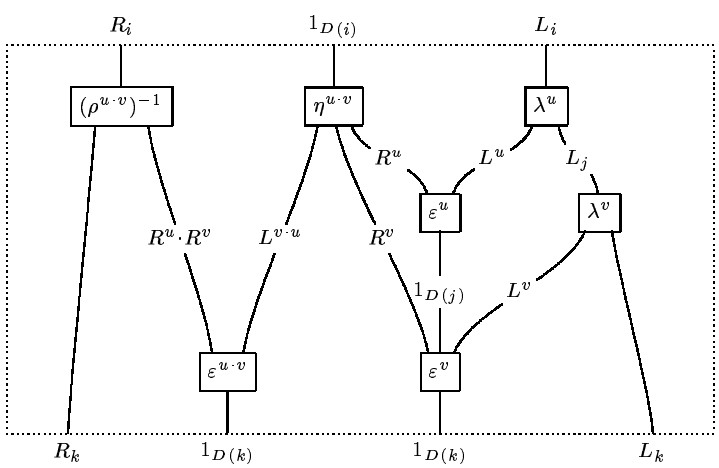
Proof of B.: To show that W is indeed a functor, consider $\underline{\mathrm{I}}^{op}$-morphisms $u : i \longrightarrow j$, $v : j \longrightarrow k$. Then, by definition,

$W(v) \cdot W(u) =$



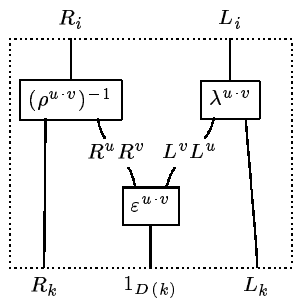
(8.34)

Using the triangle equality for $L^{u \cdot v} \dashv R^v \cdot R^u$, the right hand side of (8.34) is equal to



(8.35)

Due to the definition of $\varphi_{v,u}$ and due to coherence conditions for λ (cf. (8.32)), we have that (8.34) equals to



$$= W(v \cdot u) \tag{8.36}$$

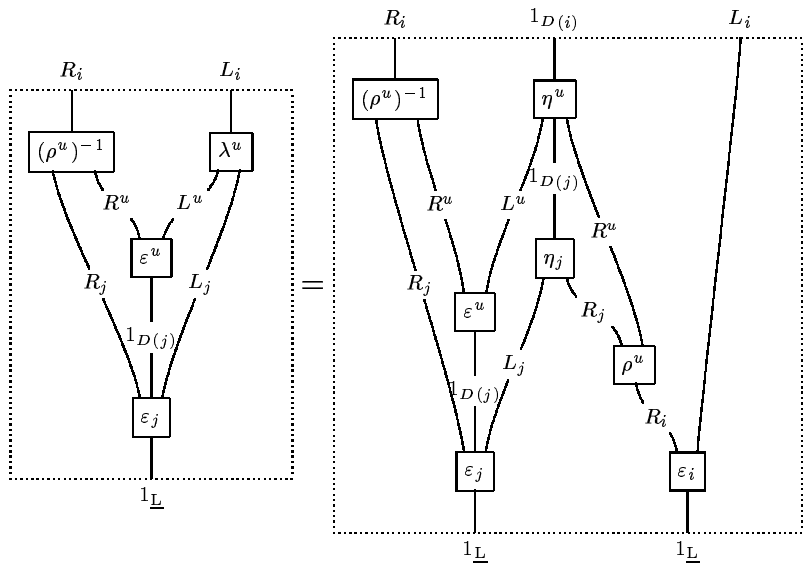
To prove that $W(1_i) = 1_{W(i)}$ use coherence conditions for ρ and λ and our choice of ε^{1_i}. This finishes the proof that W is a functor.

To show that the collection of counits ε_i forms a compatible cocone for W, we are to prove that for any $u : i \longrightarrow j$ in $\underline{\mathbf{I}}^{op}$ we have the equality:

$$\varepsilon_j \cdot W(u) = \varepsilon_i \tag{8.37}$$

The following equalities hold:

$\varepsilon_j \cdot W(u) =$



$= \varepsilon_i$ (8.38)

where the first equality holds due to the definition of $W(u)$, the second holds due to the definition of λ^u and the third holds due to triangle equalities for $L^u \dashv R^u$, $L_j \dashv R_j$ and the fact that ρ^u and $(\rho^u)^{-1}$ are mutually inverse.

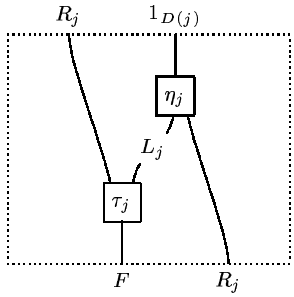
We have proved that ε_i's form a compatible cocone on W.

To prove that ε_i's form a colimit cocone, consider another W-compatible cocone formed by natural transformations $\tau_i : L_i R_i \Longrightarrow F$. We are going to define a unique natural transformation $\theta : 1_{\underline{\mathbf{L}}} \Longrightarrow F$, such that the equation

$$\theta \cdot \varepsilon_i = \tau_i \tag{8.39}$$

holds for all $\underline{\mathbf{I}}$-objects i.

First, for any $\underline{\mathrm{I}}$-object j define a natural transformation $\theta^{(j)} : R_j \Longrightarrow R_j \cdot F$ as follows:

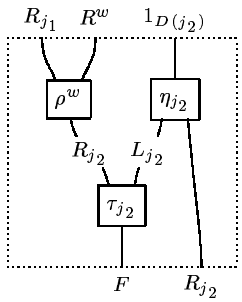
$\theta^{(j)} =$ 

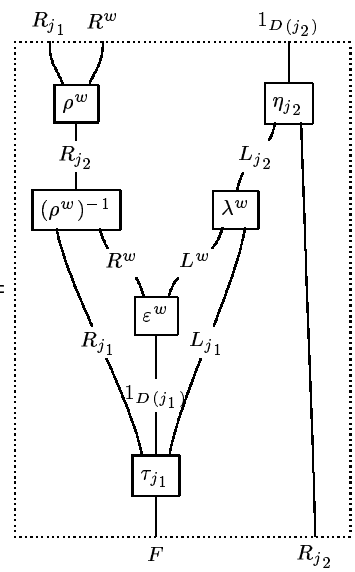
(8.40)

To prove that for any $\underline{\mathrm{L}}$-object x the collection $(\theta^{(j)} : R_j \Longrightarrow R_j F)$ constitutes a modification from (R_j, ρ^w) to $(R_j F, \rho^w F)$, it suffices to show that for any $\underline{\mathrm{I}}$-morphism $w : j_1 \longrightarrow j_2$ the equality

$$\theta^{(j_2)} \cdot \rho^w = \rho^w F \cdot R^w \theta^{(j_1)} \tag{8.41}$$

holds (cf. (3.29) and (3.30)). The following equalities hold:

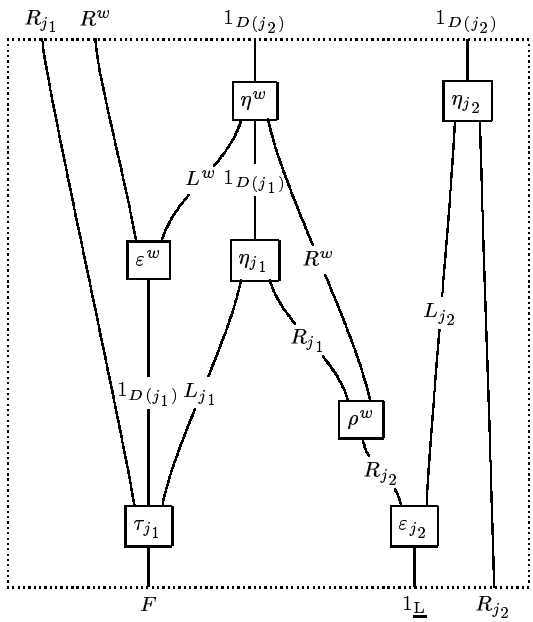
$\theta^{(j_2)} \cdot \rho^w =$ 

$=$ 

(8.42)

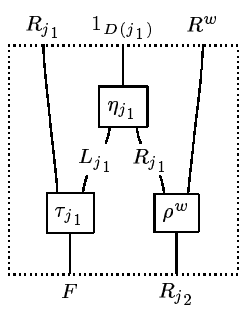
where the first equality holds due to the definition of $\theta^{(j_2)}$ and the second equality holds due to the assumption that $\tau_{j_2} = \tau_{j_1} \cdot W(u)$ and the definition of $W(u)$. The right hand

side of (8.42) equals to (use the definition of λ^w):



(8.43)

Using triangle equalities for $L^w \dashv R^w$ and $L_{j_2} \dashv R_{j_2}$, (8.43) equals to:



(8.44)

which equals to $\rho^w F \cdot R^w \theta^{(j_1)}$, therefore the equality (8.41) holds and the collection $(\theta^{(j)})$ constitutes a modification from (R_j, ρ^w) to $(R_j F, \rho^w F)$.

Define $\theta : 1_{\underline{L}} \Longrightarrow F$ to be the natural transformation corresponding to the modification $(\theta^{(j)})$.

Let us prove that for any $\underline{I}$-object i we have the equality

$$\tau_i = \theta \cdot \varepsilon_i \tag{8.45}$$

It suffices to show that for any $\underline{I}$-object j it holds that $R_j \tau_i = R_j \theta \cdot R_j \varepsilon_i$.

Since $R_j \varepsilon_i$ is natural, we have

$$R_j \theta \cdot R_j \varepsilon_i = R_j \varepsilon_i F \cdot R_j L_i R_i \theta \tag{8.46}$$

Due to the definition of $R_i \theta$, the equality

$$R_j L_i R_i \theta = R_j L_i R_i \tau_i \cdot R_j L_i \eta_i R_i \tag{8.47}$$

holds. Finally, due to naturality of $R_j \varepsilon_i$, the equality

$$R_j \varepsilon_i F \cdot R_j L_i R_i \tau_i = R_j \tau_i \cdot R_j \varepsilon_i L_i R_i \tag{8.48}$$

holds. Putting (8.46), (8.47) and (8.48) together, we obtain that the equality (8.45) holds.

To prove uniqueness of θ, suppose that there is another natural transformation $\sigma : 1_{\underline{\mathrm{L}}} \Longrightarrow F$ fulfilling $\sigma \cdot \varepsilon_i = \tau_i$ for all i. To show that then neccessarily $\theta = \sigma$, it suffices to prove that the image of $R_j\sigma$ under the adjunction $L_j \dashv R_j$ equals to τ_j. This is straightforward. This finishes the proof of **B**.

Due to Claim **A** we already know that we have a pseudocone (R_j, ρ^w) with vertex $\underline{\mathrm{L}}$ in FILT^r. Our next aim is to prove that it is a bilimit of D.

Claim C. (R_j, ρ^w) *is a bilimit of* D.

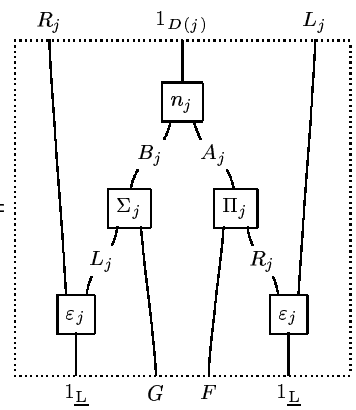
Proof of C. Let $\underline{\mathrm{X}}$ be an arbitrary category having filtered colimits. We are going to show that the categories $\mathsf{FILT}^r(\underline{\mathrm{X}}, \underline{\mathrm{L}})$ and $\mathrm{PseudoCone}(\underline{\mathrm{X}}, D)$ are equivalent.

Suppose that (A_j, α^w) is an arbitrary pseudocone for D in FILT^r with vertex $\underline{\mathrm{X}}$, that is, we have finitary functors $A_j : \underline{\mathrm{X}} \longrightarrow D(j)$, finitary adjunctions $B_j \dashv A_j$ with units n_j and counits e_j. Morover, we have natural isomorphisms $\alpha^w : R^w \cdot A_{j_1} \Longrightarrow A_{j_2}$ and $\beta^w : B_{j_2} \Longrightarrow L^w \cdot B_{j_1}$ for each $\underline{\mathrm{I}}$-morphism $w : j_1 \longrightarrow j_2$ and these isomorphisms satisfy the appropriate coherence conditions.

Since the categories $\mathsf{FILT}(\underline{\mathrm{X}}, \underline{\mathrm{L}})$ and $\mathrm{PseudoCone}(\underline{\mathrm{X}}, D)$ are equivalent, there is finitary functor $F : \underline{\mathrm{X}} \longrightarrow \underline{\mathrm{L}}$ and an isomorphism modification $\Pi : (A_j, \alpha^w) \rightsquigarrow (R_j \cdot F, \rho^w F)$.

Analogously, since the categories $\mathsf{FILT}(\underline{\mathrm{L}}, \underline{\mathrm{X}})$ and $\mathrm{PseudoCocone}(\Phi \cdot D, \underline{\mathrm{X}})$ are equivalent, there is finitary functor $F : \underline{\mathrm{L}} \longrightarrow \underline{\mathrm{X}}$ and an isomorphism modification $\Sigma : (B_j, \beta^w) \rightsquigarrow (G \cdot L_j, G\lambda^w)$.

We are going to prove that $G \dashv F$. To define the unit $\eta : 1_{\underline{\mathrm{L}}} \Longrightarrow F \cdot G$, use Claim **B**: $\varepsilon_j : L_j R_j \Longrightarrow 1_{\underline{\mathrm{L}}}$ is a colimit cocone. Let η be the unique natural transformation such that

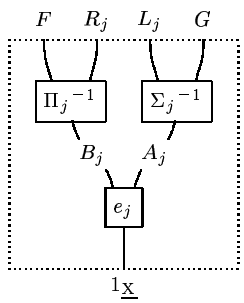
$\eta \cdot \varepsilon_j =$ 

(8.49)

for each j.

To define the counit $\varepsilon : G \cdot F \Longrightarrow 1_{\underline{\mathrm{X}}}$, use the fact that GF is a filtered colimit with a colimit cocone formed by $G\varepsilon_j F : GL_jR_jF \Longrightarrow GF$. Let ε be the unique natural

transformation such that the equality

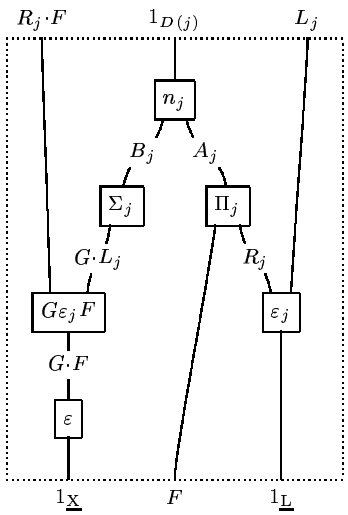
$\varepsilon \cdot G\varepsilon_j F =$ 

(8.50)

holds for each j.

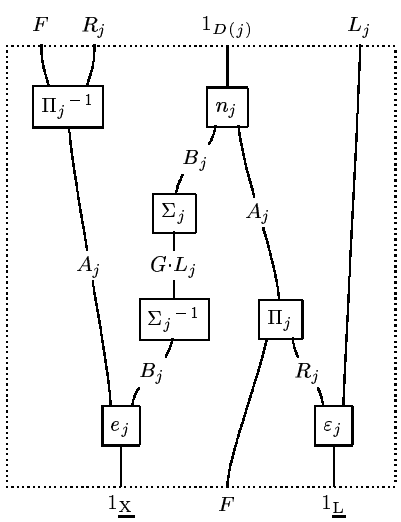
Let us verify that $F\varepsilon \cdot \eta F = 1_F$. Due to definition of η it suffices to show that $F\varepsilon \cdot \eta F \cdot \varepsilon_j F = \varepsilon_j F$ for all j.

Due to (8.49) we have

$F\varepsilon \cdot \eta F \cdot \varepsilon_j F =$ 

(8.51)

and due to (8.50) this equals to:

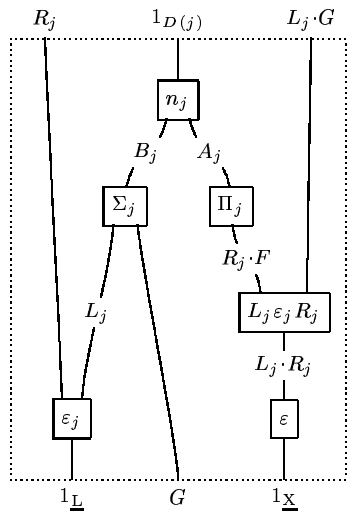


$= \varepsilon_j F$ (8.52)

where the last equation holds due to the fact that Σ^{-1} and Σ are mutually inverse, due to triangle equality for $B_j \dashv A_j$, and due to the fact that Π^{-1} and Π are mutually inverse.

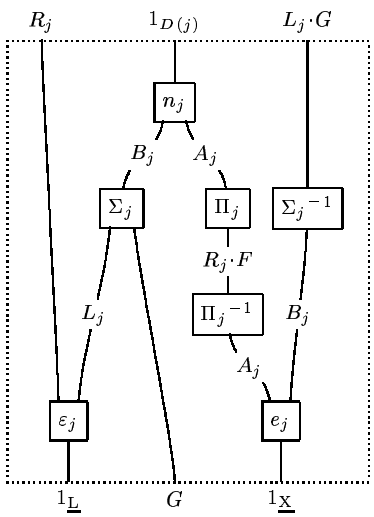
Analogously, to verify the triangle equality $\varepsilon G \cdot G\eta = 1_G$, it suffices to show that $\varepsilon G \cdot G\eta \cdot G\varepsilon_j = G\varepsilon_j$ for all j.

Due to (8.49) we have:

$\varepsilon G \cdot G\eta \cdot G\varepsilon_j =$ 

(8.53)

which due to (8.50) equals to:



$= G\varepsilon_j$ (8.54)

where the last equality holds due to the fact that Π^{-1} and Π are mutually inverse, due to triangle equality for $B_j \dashv A_j$ and the fact that Σ^{-1} and Σ are mutually inverse. The proof of Claim **C** is finished. □

As a corollary of 8.1.3 we obtain the following result (cf. [Ad97], Theorem 3):

Corollary 8.1.4 *The 2-quasicategories* $\aleph_0$-ACC^r, GDOM^r *and* SC^r *are closed under cofiltered bilimits in* FILT^r.

Proof. We keep the notation of the proof of Theorem 8.1.3.

1. $\aleph_0$-ACC^r: Suppose all categories $D(i)$ are finitely accessible. We have to show that the category $\underline{\mathrm{L}}$ is finitely accessible. Let $\underline{\mathrm{K}}$ denote the full subcategory of $\underline{\mathrm{L}}$ defined by objects

of the form $L_i(p)$ with p a $D(i)_{fin}$-object for all i. The category $\underline{\mathrm{K}}$ is small since $\underline{\mathrm{I}}$ is a small category and each $D(i)_{fin}$ is a small category. Moreover, each $\underline{\mathrm{K}}$-object is finitely presentable, since each L_i has a finitary right adjoint.

To finish the proof it suffices to show that each $\underline{\mathrm{L}}$-object is a filtered colimit of $\underline{\mathrm{K}}$-objects. Let y be an $\underline{\mathrm{L}}$-object.

Let us prove that the comma category $\underline{\mathrm{K}}/y$ is filtered.

(i) $\underline{\mathrm{K}}$ is non-empty: since $\underline{\mathrm{I}}$ is non-empty there is i such that $D(i)$ is finitely accessible. Therefore the comma category $D(i)_{fin}/R_i(y)$ contains e.g. $f : p \longrightarrow R_i(y)$. The adjoint of f under $L_i \dashv R_i$ is an object of $\underline{\mathrm{K}}/y$.

(ii) Let $f_1 : L_{j_1}(a) \longrightarrow y$ and $f_2 : L_{j_2}(a) \longrightarrow y$ be a pair of $\underline{\mathrm{K}}/y$-objects. Then by the cofilteredness of $\underline{\mathrm{I}}$ there exists an i in $\underline{\mathrm{I}}$ and a pair of morphisms $u : i \longrightarrow j_1$, $v : i \longrightarrow j_2$. Replace $f_1 : L_{j_1}(a) \longrightarrow y$ by the isomorphism $\lambda^u(a) : L_i \cdot L^u(a) \longrightarrow L_{j_1}(a)$ followed by f_1. Analogously, replace f_2 by the isomorphism $\lambda^v(b) : L_i \cdot L^v(b) \longrightarrow L_{j_2}(b)$ followed by f_2. Both $L^u(a)$ and $L^v(b)$ are finitely presentable, since both L^u and L^v have finitary right adjoints. We can thus assume without loss of generality that f_1 and f_2 are of the form $f_1 : L_i(a) \longrightarrow y$, $f_2 : L_i(b) \longrightarrow y$, respectively. Denote by $g_1 : a \longrightarrow R_i(y)$ and by $g_2 : b \longrightarrow R_i(y)$ the adjoints of f_1 and f_2 under $L_i \dashv R_i$. Then, since $R_i(y)$ is a filtered colimit of its canonical diagram in $D(i)$, there is a finitely presentable object p and a morphism $g : p \longrightarrow R_i(y)$ such that g_1 factors as $g \cdot g_1'$ and g_2 factors as $g \cdot g_2'$. The image of g under $L_i \dashv R_i$ provides us with a morphism $f : L_i(p) \longrightarrow y$ which is the desired vertex of a cocone on f_1 and f_2.

(iii) Let $f_1 : L_{j_1}(a) \longrightarrow y$ and $f_2 : L_{j_2}(a) \longrightarrow y$ be a pair of $\underline{\mathrm{K}}/y$-objects and suppose that $g : f_1 \longrightarrow f_2$ and $h : f_1 \longrightarrow f_2$ is a parallel pair of $\underline{\mathrm{K}}/y$-morphisms, i.e. that $g \cdot f_2 = f_1$ and $h \cdot f_2 = f_1$. Analogously as above we can suppose that $j_1 = j_2 = i$. Denote by $g_1 : a \longrightarrow R_i(a)$ and by $g_2 : b \longrightarrow R_i(b)$ the adjoints of f_1 and f_2 under $L_i \dashv R_i$. Then, since $R_i(y)$ is a filtered colimit of its canonical diagram in $D(i)$, there is a finitely presentable object p and a morphism $g : p \longrightarrow R_i(y)$ such that g_1 factors as $g \cdot g_1'$ and g_2 factors as $g \cdot g_2'$. The image of g under $L_i \dashv R_i$ provides us with a morphism $f : L_i(p) \longrightarrow y$ which is the desired vertex of a cocone on f_1 and f_2.

To prove that y is a colimit of the canonical forgetful functor from $\underline{\mathrm{K}}/y$ to $\underline{\mathrm{L}}$, use the fact that counits of $L_i \dashv R_i$ form a filtered colimit cocone in $\mathsf{FILT}(\underline{\mathrm{L}}, \underline{\mathrm{L}})$. We therefore have a filtered colimit cocone $\varepsilon_i(y) : L_i \cdot R_i(y) \longrightarrow y$ in $\underline{\mathrm{L}}$. Each $f : L_i(p) \longrightarrow y$ factors through some ε_j, since $L_i(p)$ is finitely presentable in $\underline{\mathrm{L}}$ and ε_j's form a filtered colimit cocone. Now use the fact that each $L_i \cdot R_i(y)$ is a filtered colimit of $L_i \cdot C_{R_i(y)}$, where $C_{R_i(y)}$ denotes the canonical filtered diagram for $R_i(y)$ in $D(i)$.

2. GDOM^r: Suppose that all categories $D(i)$ are generalized domains, i.e. they all are finitely accessible with initial objects. By 1. we know that $\underline{\mathrm{L}}$ is finitely accessible, it remains to verify that $\underline{\mathrm{L}}$ has an initial object. This is trivial: take an initial object $\perp$ in (any) $D(i)$. Then $L_i(\perp)$ is initial in $\underline{\mathrm{L}}$.

3. SC^r: Suppose all categories $D(i)$ are Scott complete categories. We only have to show that the category $\underline{\mathrm{L}}$ is a Scott complete category. Since by 2. we know that $\underline{\mathrm{L}}$ is a generalized domain, it remains to verify that $\underline{\mathrm{L}}$ is boundedly cocomplete.

To prove that $\underline{\mathrm{L}}$ is boundedly cocomplete, consider a non-empty diagram $C : \underline{\mathrm{C}} \longrightarrow \underline{\mathrm{L}}$ having a compatible cocone. Recall that by Claim B. from the proof of Theorem 8.1.3, each Cc can be expressed as a filtered colimit, with $\varepsilon_i(Cc) : L_i \cdot R_i(Cc) \longrightarrow Cc$ as the colimit morphisms. Since C has a compatible cocone, so does each functor $R_i \cdot C$ and hence $\mathrm{colim}_c\, R_i(Cc)$ exists. L_i, being a left adjoint, preserves this colimit, thus $L_i(\mathrm{colim}_c\, R_i(Cc)) = \mathrm{colim}_c\, L_i \cdot R_i(Cc)$ exists for each i and this collection forms a filtered diagram in $\underline{\mathrm{L}}$. Since colimits commute with colimits, we have that $\mathrm{colim}_i\, \mathrm{colim}_c\, L_i \cdot R_i(Cc) = \mathrm{colim}_c\, \mathrm{colim}_i\, L_i \cdot R_i(Cc) = \mathrm{colim}_c\, Cc$. □

8.2 Solving Recursive Domain Equations

In this section we give sufficient conditions for the existence of a solution of the recursive equation

$$\underline{\mathrm{X}} = F(\underline{\mathrm{X}}) \tag{8.55}$$

where F is an endo-2-functor on a 2-quasicategory K of domains. We say that a category $\underline{\mathrm{K}}$ is a *solution* of the equation (8.55), or that it is a fixed point of F, if it satisfies that equation up to an equivalence of categories, i.e. if $\underline{\mathrm{K}} \simeq F(\underline{\mathrm{K}})$ holds.

The following theorem generalizes Theorem 2.2.3. Let ω denote the first countable ordinal number and consider ω as a category w.r.t. ususal order.

Theorem 8.2.1 *Suppose that* K *is a sub-2-quasicategory of* FILT *such that* K^r *is closed under cofiltered bilimits in* FILT^r. *Let* $F : \mathsf{K}^r \longrightarrow \mathsf{K}^r$ *be a 2-functor which preserves cofiltered bilimits. Suppose that* $R : F(\underline{\mathrm{A}}) \longrightarrow \underline{\mathrm{A}}$ *is a 1-cell in* K^r. *Define the cofiltered diagram* $D : \omega^{op} \longrightarrow \mathsf{K}^r$ *as follows:*

$D(0) = \underline{\mathrm{A}}$, $D(n+1) = F(D(n))$ *for each* $n \in \omega$.

$D(1 \geq 0) = R : D(1) \longrightarrow D(0)$, $D(n+2 \geq n+1) = F(D(n+1 \geq n))$ *for each* $n \in \omega$.

Then a bilimit of D *is a fixed point of* F.

Proof. Denote by β a bilimit pseudocone of D with vertex $\underline{\mathrm{A}}_\infty$.

Since F preserves cofiltered bilimits, $F(\beta)$ is a bilimit pseudocone on $F \cdot D$ with vertex $F(\underline{\mathrm{A}}_\infty)$. Denote by E the inclusion functor of $\langle \{1, 2, \ldots\}, \leq \rangle$ in $\langle \omega, \leq \rangle$. Then E is a cofinal functor and therefore bilimits of D and $D \cdot E^{op}$ are the same. Since $F(\beta)$ is a bilimit pseudococone on $D \cdot E^{op}$, we conclude that the categories $\underline{\mathrm{A}}_\infty$ and $F(\underline{\mathrm{A}}_\infty)$ are equivalent. □

The above theorem indicates that the preservation of cofiltered bilimits is an important property.

Here is a "recipe" how to solve a recursive equation: when working in a 2-quasicategory K of domains, to solve a recursive equation $\underline{\mathrm{X}} = F(\underline{\mathrm{X}})$ with a 2-functor $F : \mathsf{K} \longrightarrow \mathsf{K}$, one has to first switch to a 2-quasicategory K^r. Since F is a 2-functor, it can be restricted to a 2-functor $F^r : \mathsf{K}^r \longrightarrow \mathsf{K}^r$ (see 8.2.2). If K^r has cofiltered bilimits and if F^r preserves them, then one applies Theorem 8.2.1 to obtain a fixed point of F.

Notation 8.2.2 Let K, L be sub-2-quasicategories of FILT, let $F : \mathsf{K} \longrightarrow \mathsf{L}$ be a 2-functor.

Recall that K^r denotes the sub-2-quasicategory having all finitary right adjoints as 1-cells and that K^l denotes the sub-2-quasicategory having left adjoints as 1-cells. Since F preserves adjunctions, we can define the following 2-functors:

1. $F^r : \mathsf{K}^r \longrightarrow \mathsf{L}^r$ as the domain-codomain restriction of F.
2. $F^l : \mathsf{K}^l \longrightarrow \mathsf{L}^l$ as the domain-codomain restriction of F.

□

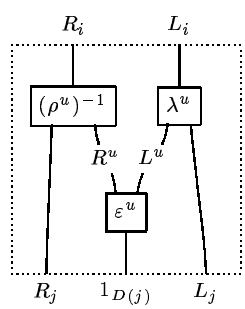
Notation 8.2.3 Let K be a sub-2-quasicategory of FILT. By 3.3.15 we have a contravariant pseudofunctor

$$(\Phi, \varphi, \psi) : \mathsf{K}^r \longrightarrow \mathsf{K}^l$$

which is a biequivalence (it is a domain-codomain restriction of the contravariant biequivalence from the proof of Theorem 3.3.14).

Let $D : \underline{\mathsf{I}} \longrightarrow \mathsf{K}^r$ be a pseudofunctor with $\underline{\mathsf{I}}$ a small cofiltered category. Denote $R^u = D(u)$ for each $\underline{\mathsf{I}}$-morphism u and let L^u be a left adjoint of R^u with ε^u the counit of $L^u \dashv R^u$. Let (R_i, ρ^w) be a pseudocone on D with vertex $\underline{\mathsf{L}}$. Denote by (L_i, λ^w) the pseudocone which is the image of (R_i, ρ^w) under the contravariant biequivalence (Φ, φ, ψ). Let ε_i denote the counit of $L_i \dashv R_i$.

Define a filtered diagram $W_D : \underline{\mathsf{I}}^{op} \longrightarrow \mathsf{K}(\underline{\mathsf{L}}, \underline{\mathsf{L}})$ as follows: W_D assigns a finitary functor $W_D(i) = L_i \cdot R_i$ to each $\underline{\mathsf{I}}$-object i and it assigns a natural transformation $W_D(u) : W_D(i) \Longrightarrow W_D(j)$ defined as



(8.56)

to each $\underline{\mathsf{I}}^{op}$-morphism $u : i \longrightarrow j$. It can be verified in the same way as in the proof of Theorem 8.1.3 that W_D is a functor. □

Definition 8.2.4 Let K be a sub-2-quasicategory of FILT. We say that K has *locally determined cofiltered bilimits*, provided that (in the notation of 8.2.3) the pseudocone (R_i, ρ^w) is a bilimit pseudocone if and only if $\varepsilon_i : L_i \cdot R_i \Longrightarrow 1_{\underline{\mathsf{L}}}$ is a colimit cocone for W_D.

The above definition is a generalization of the (dual) notion of *locally determined colimits* from [Gun92], p. 325.

Example 8.2.5 From the proof of 8.1.3 it follows that FILT has locally determined cofiltered bilimits:

Let $D : \underline{\mathrm{I}} \longrightarrow \mathsf{FILT}^r$ be a pseudofunctor with $\underline{\mathrm{I}}$ a small cofiltered category. Let (R_i, ρ^w) be a pseudocone with vertex $\underline{\mathrm{L}}$ over D in FILT^r. By Claims B. and C. from the proof of 8.1.3, if $\varepsilon_i : L_i \cdot R_i \Longrightarrow 1_{\underline{\mathrm{L}}}$ is a colimit cocone for W_D, then (R_i, ρ^w) is a bilimit pseudocone.

Conversely, if (R_i, ρ^w) is a bilimit pseudocone for D in FILT^r, we can without loss of generality, assume that that (R_i, ρ^w) is the pseudocone from the proof of 8.1.3. By Claim B. we conclude that $\varepsilon_i : L_i \cdot R_i \Longrightarrow 1_{\underline{\mathrm{L}}}$ is a colimit cocone for W_D. □

Example 8.2.6 Other examples of 2-quasicategories with locally determined cofiltered bilimits are: $\aleph_0$-ACC, GDOM and SC — cf. 8.1.4. □

The following characterization of 2-functors which preserve cofiltered bilimits follows immediately from Definition 8.2.4:

Theorem 8.2.7 *Suppose that* K *and* L *are sub-2-quasicategories of* FILT *with locally determined cofiltered bilimits. Let* $F : \mathsf{K}^r \longrightarrow \mathsf{L}^r$ *be a 2-functor and let* $D : \underline{\mathrm{I}} \longrightarrow \underline{\mathrm{K}}^r$ *be a pseudofunctor with* $\underline{\mathrm{I}}$ *a cofiltered small category. Let* (R_i, ρ^w) *be a pseudocone on* D *and let* $W_D : \underline{\mathrm{I}}^{op} \longrightarrow \mathsf{K}(\underline{\mathrm{L}}, \underline{\mathrm{L}})$ *denote the filtered diagram from 8.2.3. Then the following are equivalent:*

1. *The 2-functor* F *preserves a bilimit of* D.
2. *The functor* $F_{\underline{\mathrm{L}},\underline{\mathrm{L}}} : \mathsf{K}(\underline{\mathrm{L}}, \underline{\mathrm{L}}) \longrightarrow \mathsf{L}(F(\underline{\mathrm{L}}), F(\underline{\mathrm{L}}))$ *preserves a colimit of* W_D.

Definition 8.2.8 Let $F : \mathsf{K} \longrightarrow \mathsf{L}$ be a 2-functor. We say that F is *locally finitary*, if the functor $F_{a,b} : \mathsf{K}(a, b) \longrightarrow \mathsf{L}(Fa, Fb)$ is finitary for each pair of 0-cells a, b.

Theorem 8.2.9 *Let* K *and* L *be sub-2-quasicategories of* FILT *with locally determined cofiltered bilimits. If* $F : \mathsf{K} \longrightarrow \mathsf{L}$ *is a locally finitary 2-functor, then* $F^r : \mathsf{K}^r \longrightarrow \mathsf{L}^r$ *preserves cofiltered bilimits. Consequently,* F *has a least fixed point.*

Proof. Immediate from 8.2.7 and 8.2.1. □

Examples 8.2.10 The proofs that the following 2-functors are locally finitary are given in [Ad97]:

1. *Product with a fixed object*, i.e. a 2-functor: $\_ \times \underline{\mathrm{K}} : \mathsf{K} \longrightarrow \mathsf{K}$ where $\underline{\mathrm{K}}$ is a fixed 0-cell in K and K is any sub-2-quasicategory of FILT closed under products with $\underline{\mathrm{K}}$.
2. *Binary Product*, i.e. a 2-functor: $\_ \times \_ : \mathsf{K} \times \mathsf{K} \longrightarrow \mathsf{K}$ where K is any sub-2-quasicategory of FILT closed under products in K.
3. *Functor space*, i.e. a 2-functor: $[\_, \_] : \mathsf{K}^* \times \mathsf{K} \longrightarrow \mathsf{K}$ where K is any cartesian closed sub-2-quasicategory of FILT and K^* denotes the 2-quasicategory with all 1-cells reversed (but the 2-cells are not reversed).

By Theorem 8.2.9 we conclude that each of the above 2-functors preserves cofiltered bilimits whenever K has locally determined cofiltered bilimits. $\square$

It is trivial that also the following "abstract constructors" are locally finitary:

Lemma 8.2.11 *Let* K *be a sub-2-quasicategory of* FILT. *Then the following 2-functors are locally finitary:*

1. Diagonal *2-functor:* $\Delta : \mathsf{K} \longrightarrow \mathsf{K} \times \mathsf{K}$.

2. Tupling *2-functor:* $[\underline{\mathrm{A}}, \_] : \mathsf{K} \longrightarrow \mathsf{K}$, *where* $\underline{\mathrm{A}}$ *is a small category and where* K *is closed in* CAT *under cotensors with* $\underline{\mathrm{A}}$.

We can therefore conclude that recursive equations $\underline{\mathrm{X}} = F(\underline{\mathrm{X}})$ formed using 2-functors from Examples 8.2.10 and Lemma 8.2.11 have least solutions.

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Index

Glossary of Symbols