**Multidimensional Calculus. Lectures content. Week 1**

**0. $\mathbb{R}^n$**

In the Euclidian space $\mathbb{R}^n$ of all real $n$-dimensional vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ the following are defined:

$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$ - vector addition

$c\mathbf{x} = (cx_1, cx_2, \dots, cx_n)$ - multiplication of a vector by a scalar

$\mathbf{x} \cdot \mathbf{y} =< \mathbf{x}, \mathbf{y} >:= x_1y_1 + x_2y_2 + \cdots + x_ny_n$ - dot product (inner product, scalar product)

$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + \cdots + x_n^2}$ - norm (length) of a vector

$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + \dots (x_n - y_n)^2}$ - distance of $\mathbf{x}$ from $\mathbf{y}$

If $\theta$ is the angle between the nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ then $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos\theta$.

**Definition.** Given a point (vector) $\mathbf{a} \in \mathbb{R}^n$, we defined the ball with center $\mathbf{a}$ and radius $\varepsilon$ (the $\varepsilon$-neighbourhood of $\mathbf{a}$) as $B(\mathbf{a}, \varepsilon) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| < \varepsilon\}$.

**1. Functions of several variables.**

**Definition.** Let $D \subset \mathbb{R}^n$, a function $f : D \to \mathbb{R}$ is a function of several variables. We usually indicate $f$ as $f(\mathbf{x})$. If $n = 2$, $f(x, y)$ is a function of two variables, if $n = 3$, $f(x, y, z)$ is a function of three variables. $D$ is the domain of $f$ and $\{f(\mathbf{x}) : \mathbf{x} \in D\}$ is the range of $f$.

**Example.** $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$, $D = \{(x, y) : x + y + 1 \geq 0, x \neq 1\}$ is the half plane above the line $y = -x - 1$ without the points on the line $x = 1$.
**Example.** $f(x, y, z) = xy \ln z$, $D \subset \mathbb{R}^3$ is the half space $z > 0$.

**Definition.** If $f$ is a function of two variables with domain $D$, the graph of $f$ is the set $\{(x, y, z) \in \mathbb{R}^3 : z = f(x, y), (x, y) \in D\}$.

**Example.** $f(x, y) = \sqrt{9 - x^2 - y^2}$, $D$ is the disk with center $(0, 0)$ and radius 3, the graph of $f$ is the top half of the sphere with center at the origin and radius 3.

**Definition.** The level curves of a function $f$ of two variables are the curves with equation $f(x, y) = k$, where $k$ is a constant.

**Example.** $f(x, y) = 4x^2 + y^2$, elliptic paraboloid, its level curves $x^2 + y^2 = k$ are ellipses.

**Remark.** For functions of three variables we talk of level surfaces: for $f(x, y, z) = x^2+y^2+z^2$ the level surfaces are spheres.

**2. Limits and continuity.**

**Definition.** Let $f$ be a function of several variables defined on a neighborhood of $\mathbf{a} \in \mathbb{R}^n$ (a ball with center at $\mathbf{a}$) except possibly at $\mathbf{a}$. Then we say that the limit of $f(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{a}$ is $L$

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$$

if for every $\varepsilon > 0$ there exists a corresponding $\delta > 0$ such that $|f(\mathbf{x}) - L| < \varepsilon$ whenever $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$.

**Remark.** $|\cdot|$ indicates the absolute value, i.e. $|f(\mathbf{x}) - L|$ is the distance of $f(\mathbf{x})$ from $L$ in $\mathbb{R}$, while $\|\cdot\|$ indicates the norm, i.e. $\|\mathbf{x} - \mathbf{a}\|$ the distance of $\mathbf{x}$ from $\mathbf{a}$ in $\mathbb{R}^n$.

**Example.** Discuss $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$. Let's approach $(0, 0)$ along the axis:

$x -$ axis, $y = 0 \Rightarrow f(x, 0) = \frac{x^2}{x^2} = 1$ so $f(x, y) \to 1$ along $x -$ axis;

$y-$ axis, $x=0 \Rightarrow f(x,0)=\frac{-y^2}{y^2}=-1$ so $f(x,y)\to -1$ along $y-$ axis;
thus, the limit does not exist.

**Fact.** If a function approaches different limits along two different paths, then the limit does not exist (DNE).

**Example.** $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ DNE (zero along the axis, 1/2 along $y=x$).

$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ DNE (zero along the axis and along every line $y=mx$, 1/2 along $x=y^2$).

$\lim_{(x,y)\to(0,0)} \frac{8x^2y}{x^2+y^2}=0$, $\varepsilon$-$\delta$-proof with $\delta=\varepsilon/8$.

$\lim_{(x,y,z)\to(0,0,0)} \ln(x^2+y^2+z^2)=-\infty$

**Definition.** Let $f$ be a function of several variables defined on a ball $B \ni \mathbf{a}$, then $f$ is continuous at $\mathbf{a}$ if $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})=f(\mathbf{a})$.
This means that three conditions must be satisfied: 1) the function is defined at $\mathbf{a}$; 2) the limit exists; 3) the limit equals the value of the function at $\mathbf{a}$.
A function $f$ is continuous on a set $D$ if it is continuous at every point in $D$.

**Example.** $f(x,y)=\frac{8x^2y}{x^2+y^2}$ is not continuous at $(0,0)$ because it is not defined there;
$f(x,y)=\begin{cases}\frac{8x^2y}{x^2+y^2} & (x,y)\neq(0,0)\\ 1 & (x,y)=(0,0)\end{cases}$ is not continuous at $(0,0)$ because the limit is different from $f(0,0)$;
$f(x,y)=\begin{cases}\frac{8x^2y}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$ is continuous everywhere.

**Remark.** All Theorems about limits and continuity studied for functions of one variable are extendable to functions of several variables: uniqueness of limit, limit and continuity of sum, difference, product, quotient, composition of functions. In particular, if $f$ is a continuous function defined on a closed bounded set $D$ of $\mathbb{R}^n$, then $f$ obtains its maximum and minimum value in $D$.
**Example.** Find domain, discuss continuity and graph the function $f(x,y)=\ln(x^2+y^2-1)$.

## Exercises. Lab 1

**1)** Find domain of $f(x,y)=\ln\frac{x^2+2x+y^2}{x^2-2x+y^2}$.

**2)** Domain of $f(x,y,z)=\frac{x}{|y+z|}$.

**3)** Graph $f(x,y)=x^2+y^2$, $g(x,y)=4x^2+9y^2$, discuss level curves.

**4)** $\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x}=1$. $\lim_{(x,y)\to(1,0)} \frac{x\sin y}{x^2+y}=0$.

**5)** Approach zero along different paths to prove that
$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$ DNE,
$\lim_{(x,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2}$ DNE,
$\lim_{(x,y)\to(0,0)} \frac{-x}{\sqrt{x^2+y^2}}$ DNE.

**6)** Use $\varepsilon$-$\delta$- proof to verify that $\lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}}=0$

**7)** Use polar coordinates $x=r\cos\theta$, $y=r\sin\theta$ to solve the previous limits and the limits discussed during the lecture.

**8)** Find $c \in \mathbb{R}$ such that $f(x,y) = \begin{cases} \frac{xy}{|x|+|y|} & (x,y) \neq (0,0) \\ c & (x,y) = (0,0) \end{cases}$ is continuous everywhere, prove the existence of the limit at zero with an $\varepsilon$-$\delta$- proof.

**9)** Get acquainted with the quadric surfaces. Draw the surfaces with the given equation for $a = b = c = 1$:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ellipsoid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ hyperboloid of one sheet

$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ hyperboloid of two sheets

$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ cone

$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ elliptic paraboloid

$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ hyperbolic paraboloid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ elliptic cylinder

$y = ax^2$ parabolic cylinder.