Multidimensional Calculus. Lectures content. Week 10

22. Tests for alternating series.

Theorem. (**Alternating series test** or **Leibniz test**) Let $b_k \geq 0$ for all k and let $\{b_k\}$ be non-increasing. The series $\sum(-1)^k b_k$ converges if and only if $\lim\limits_{k\to\infty}(b_k) = 0$.

Example. $\sum \frac{(-1)^k}{k}$: $b_k = \frac{1}{k} \geq 0$ is decreasing and $\to 0$, hence $\sum \frac{(-1)^k}{k}$ converges (compare with harmonic series).

23. Absolute convergence of series.

Definition. We say that a series $\sum a_k$ **converges absolutely** if the series $\sum |a_k|$ converges.

Theorem. If a series $\sum a_k$ converges absolutely, then it also converges and we have $\Big|\sum\limits_{k=n_0}^{\infty} a_k\Big| \leq \sum\limits_{k=n_0}^{\infty} |a_k|$.

But not the other way around! Recall that $\sum \frac{(-1)^k}{k}$ converges, but $\sum\Big|\frac{(-1)^k}{k}\Big| = \sum \frac{1}{k} = \infty$.

Definition. We say that a series **converges conditionally** if it converges, but not absolutely.
Thus there are three possibilities now:
—$\sum a_k$ converges, $\sum |a_k|$ converges: absolute convergence (the second implies the first here)
—$\sum a_k$ converges, $\sum |a_k|$ diverges: conditional convergence
—$\sum a_k$ diverges, $\sum |a_k|$ diverges (the first implies the second)

Example. conditional convergence: $\sum \frac{(-1)^k}{k}$; absolute convergence: $\sum \frac{(-1)^k}{k^2}$; divergence: $\sum(-1)^k$.

Example. $\sum \frac{\sin(k)}{2^k}$: We do not know how to investigate this series directly. Its terms are not non-negative, therefore the tests won't work. We can't use AST, since the series is not alternating. The necessary condition won't help either, since $a_k \to 0$.
Thus we try the absolute convergence and hope that it will come out true, so that we have some conclusion: $\sum\Big|\frac{\sin(k)}{2^k}\Big| = \sum \frac{|\sin(k)|}{2^k} \leq \sum \frac{1}{2^k}$, this converges, therefore by comparison test also $\sum\Big|\frac{\sin(k)}{2^k}\Big|$ converges, hence $\sum \frac{\sin(k)}{2^k}$ converges absolutely.

Example. $\sum(-1)^k \frac{2^k}{k^3}$: absolute: $\sum\Big|(-1)^k \frac{2^k}{k^3}\Big| = \sum \frac{2^k}{k^3}$, ratio test: $\frac{a_{k+1}}{a_k} = 2\big(\frac{k}{k+1}\big)^3 \to 2 = \lambda > 1$,
thus $\sum\Big|(-1)^k \frac{2^k}{k^3}\Big|$ diverges, hence $\sum(-1)^k \frac{2^k}{k^3}$ does not converge absolutely. But we do not know whether it by itself does converge (then it would be conditional convergence) or not.
However, $\frac{2^k}{k^3} \to \infty$, thus $a_k = (-1)^k \frac{2^k}{k^3} \not\to 0$, so the series diverges.

Theorem. Consider a series $\sum\limits_{k=2n_0}^{\infty} a_k$.
If $\sum a_k$ converges absolutely, then also $\sum a_{2k}$ and $\sum a_{2k+1}$ converge and
$\sum\limits_{k=2n_0}^{\infty} a_k = \sum\limits_{k=n_0}^{\infty} a_{2k} + \sum\limits_{k=n_0}^{\infty} a_{2k+1}$.

Not true for conditional convergence, see $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k}$.

Theorem. Consider a series $\sum\limits_{k=n_0}^{\infty} a_k$.
If $\sum a_k$ converges absolutely, then for every choice of signs $\varepsilon_k = \pm 1$ also $\sum \varepsilon_k a_k$ converges.
If $\sum a_k$ converges conditionally, then there is a choice of signs $\varepsilon_k = \pm 1$ such that $\sum \varepsilon_k a_k = \infty$.

Definition. Consider a series $\sum\limits_{k=n_0}^{\infty} a_k$.

By a **rearrangement** of $\sum\limits_{k=n_0}^{\infty} a_k$ we mean any series $\sum\limits_{k=n_0}^{\infty} a_{\pi(k)}$, where π is an arbitrary bijective mapping of $\{n_0, n_0+1, n_0+2, \dots\} \subset \mathbb{Z}$ onto $\{n_0, n_0+1, n_0+2, \dots\}$, i.e. π is a permutation of $\{n_0, n_0+1, n_0+2, \dots\}$.

Theorem. Consider a series $\sum\limits_{k=n_0}^{\infty} a_k$.

If $\sum a_k$ converges absolutely, then also all its rearrangements $\sum a_{\pi(k)}$ converge and we have $\sum\limits_{k=n_0}^{\infty} a_{\pi(k)} = \sum\limits_{k=n_0}^{\infty} a_k$.

If $\sum\limits_{k=n_0}^{\infty} a_k$ converges conditionally, then $\forall c \in \mathbb{R} \cup \{\pm\infty\}$ there exists its rearrangement such that $\sum\limits_{k=n_0}^{\infty} a_{\pi(k)} = c$.

24. Sequences and series of functions.

Definition. By a **sequence of functions** we mean an ordered set
$\{f_k\}_{k=n_0}^{\infty} = \{f_{n_0}, f_{n_0+1}, f_{n_0+2}, \dots\}$, where f_k are functions.
Remark: Given a sequence of functions $\{f_k\}_{k=n_0}^{\infty}$ and $x \in \bigcap D(f_k)$, then $\{f_k(x)\}$ is a standard sequence of real (complex) numbers.

Definition. Let $\{f_k\}_{k\geq n_0}$, f be functions on a set M.
We say that $\{f_k\}$ **converges (pointwise)** to f on M, denoted $f_k \to f$ or $f = \lim\limits_{k\to\infty}(f_k)$,
if $\forall x \in M$: $\lim\limits_{k\to\infty}\big(f_k(x)\big) = f(x)$.

Example. Consider $f_k(x) = \arctan(kx)$. Then $\lim\limits_{k\to\infty}\big(f_k(x)\big) = \begin{cases} 0, & x = 0; \\ \frac{\pi}{2}, & x > 0; \\ -\frac{\pi}{2}, & x < 0. \end{cases}$

Definition. Let $\{f_k\}_{k\geq n_0}$, f be functions on a set M.
We say that $\{f_k\}$ **converges uniformly** to f on M, denoted $f_k \rightrightarrows f$,
if $\forall \varepsilon > 0 \exists N_0 \in \mathbb{N}$ such that $\forall k \geq N_0 \forall x \in M : |f(x) - f_k(x)| < \varepsilon$.

Theorem. Let $f_k \rightrightarrows f$ on M.
(i) If all f_k are continuous on M, then also f is continuous there.
(ii) If all f_k have a derivative on M, then also f has it there and $f' = \lim\limits_{k\to\infty}(f_k')$ on M.
(iii) If all f_k have antiderivative on M, then also f has it there and $\int_{x_0}^{x} f\,dx = \lim\limits_{k\to\infty}\big(\int_{x_0}^{x} f_k\,dx\big)$ for $\overline{x_0, x} \subseteq M$.

Definition. A **series of functions** is a symbol $\sum\limits_{k=n_0}^{\infty} f_k = f_{n_0} + f_{n_0+1} + f_{n_0+2} + \dots$, where f_k are functions.
Remark: Given a series of functions $\sum f_k$ and $x \in \bigcap D(f_k)$, then $\sum f_k(x)$ is a standard series of real (complex) numbers.

Definition. Consider a series of functions $\sum\limits_{k=n_0}^{\infty} f_k$.
The **region of convergence** of this series is the set $\big\{x \in \bigcap D(f_k);\ \sum f_k(x) \text{ converges}\big\}$. By defining $f(x) = \sum\limits_{k=n_0}^{\infty} f_k(x)$ we then obtain a function f on this set called the **sum of the series**, denoted $\sum\limits_{k=n_0}^{\infty} f_k = f$.
The **region of absolute convergence** of this series is the set
$\big\{x \in \bigcap D(f_k);\ \sum f_k(x) \text{ converges absolutely}\big\}$.
We say that this series **converges uniformly** to f on M, denoted $\sum f_k \rightrightarrows f$ on M, if the sequence of partial sums $\Big\{\sum\limits_{k=n_0}^{N} f_k(x)\Big\}$ converges uniformly to f on M.

Theorem. Consider series of functions $\sum f_k$ and $\sum g_k$.

If $\sum\limits_{k=n_0}^{\infty} f_k = f$ on M and $\sum\limits_{k=n_0}^{\infty} g_k = g$ on M, then $\forall a,b \in I\!R$: $\sum\limits_{k=n_0}^{\infty} (af_k + bg_k) = af + bg$ on M.

Theorem. (Weierstrass criterion) Let f_k for $k \geq n_0$ be functions on M. Let $a_k \geq 0$ satisfy $\forall x \in M \forall k \geq n_0$: $|f_k(x)| \leq a_k$.
If $\sum a_k$ converges, then $\sum f_k$ converges uniformly on M.

Example. $\sum x^k = \frac{1}{1-x}$ on $(-1,1)$, but the convergence is not uniform. It will be uniform if we restrict our attention to $[-\varrho, \varrho]$ for $\varrho \in (0,1)$.

Theorem. Let $\sum f_k \rightrightarrows f$ on M.
(i) If all f_k are continuous on M, then also f is continuous there.
(ii) If all f_k have a derivative on M, then also f has it there and $f' = \sum\limits_{k=n_0}^{\infty} f_k'$ on M.
(iii) If all f_k have an antiderivative on M, then also f has it there and $\int_{x_0}^{x} f\,dx = \sum\limits_{k=n_0}^{\infty} \int_{x_0}^{x} f_k\,dx$ for $\overline{x_0, x} \subseteq M$.

None of this is true in general for ordinary (pointwise) convergence.

Exercises. Lab 10

- Discuss the converge and absolute convergence of the following series

1) $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}$ **2)** $\sum\limits_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k^2+4}$ **3)** $\sum\limits_{k=2}^{\infty} (-1)^k \frac{k-1}{k+1}$

4) $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{2^{k^2}}$ **5)** $\sum\limits_{k=1}^{\infty} \frac{\sin k}{k^2+4}$ **6)** $\sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(1+\ln k)}$ $a \in I\!R$

- Use the known criterions to discuss for what values of $x \in I\!R$ the following series is convergent

1) $\sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(1+x^2)^k}$ **2)** $\sum\limits_{k=1}^{\infty} (\frac{x+2}{x})^k$ **3)** $\sum\limits_{k=1}^{\infty} \frac{(k+5)^4}{k!} x^k$