

MA3: Lectures content, week 13

27. Fourier series

Remark: If f is T -periodic, then also its derivative f' is T -periodic, but this is not true for its antiderivative $F(t) = \int_0^t f(u) du$. This one is T -periodic if $\int_0^T f(u) du = 0$, i.e. $a_0 = 0$.

Theorem.

Let f be a T -periodic function that is piecewise continuous on $[0, T)$ and it has a piecewise continuous derivative on $[0, T)$. Let $f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(k\omega t) + b_k \sin(k\omega t)]$.

(i) Then $f' \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} [-a_k \sin(k\omega t)k\omega + b_k \cos(k\omega t)k\omega] = \frac{a_0}{2} + \sum_{k=1}^{\infty} [b_k k\omega \cos(k\omega t) - a_k k\omega \sin(k\omega t)]$.

(ii) If $\int_0^T f(u) du = 0$ (that is, $a_0 = 0$), then

$$F(t) = \int_0^t f(u) du \sim \sum_{k=1}^{\infty} [a_k \sin(k\omega t) \frac{1}{k\omega} - b_k \cos(k\omega t) \frac{1}{k\omega}] = \sum_{k=1}^{\infty} [\frac{-b_k}{k\omega} \cos(k\omega t) + \frac{a_k}{k\omega} \sin(k\omega t)].$$

Remark: If we know that Fourier series converges to f on some interval $[a, b]$, then we unfortunately cannot claim that the convergence is uniform there. At the ends of the interval we have the so-called Gibbs problem.

Definition.

Let $f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(k\omega t) + b_k \sin(k\omega t)]$. Denote $A_k = \sqrt{a_k^2 + b_k^2}$, find φ_k so that

$b_k = A_k \cos(\varphi_k)$ and $a_k = A_k \sin(\varphi_k)$. Then $f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} A_k \sin(k\omega t + \varphi_k)$.

This series is called the **Fourier series in amplitude-phase form**.

Denote $c_0 = \frac{a_0}{2}$ a $c_k = \frac{1}{2}(a_k - j b_k)$, $c_{-k} = \frac{1}{2}(a_k + j b_k)$ for $k \in \mathbb{N}$. Then $f \sim \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$.

This series is called the **Fourier series in complex form**.

Remark $\varphi_k = \arctg(\frac{a_k}{b_k})$, or $\varphi_k = \operatorname{arccotg}(\frac{b_k}{a_k})$, or some shifts, see transformation of Cartesian coordinates to polar.

Fact.

We have $c_k = \frac{1}{T} \int_0^T f(t) e^{-jk\omega t} dt$.

28. Application of series

Example.

$$y'' + y = \begin{cases} 1, & t \in [2k, 2k+1); \\ 0, & t \in [2k-1, 2k). \end{cases}$$

Expand the right hand-side $f = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1-(-1)^k}{k\pi} \sin(k\pi t)$.

We assume that a solution can be found of the form $y = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(k\pi t) + b_k \sin(k\pi t)]$.

We substitute into the equation and obtain

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k(1 - k^2\pi^2) \cos(k\pi t) + b_k(1 - k^2\pi^2) \sin(k\pi t)] = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1-(-1)^k}{k\pi} \sin(k\pi t).$$

Comparing both sides we get $a_0 = 1$, $a_k = 0$ for $k \geq 1$ and $b_k = \frac{1-(-1)^k}{k\pi(1-k^2\pi^2)}$
 and thus $y = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1-(-1)^k}{k\pi(1-k^2\pi^2)} \sin(k\pi t) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi(1-(2k+1)^2\pi^2)} \sin((2k+1)\pi t)$.

Example.

$y'' - x^3y = 24x^2$ around $x_0 = 0$.

On the right we have a power series with $x_0 = 0$, we will try to find a solution of the form

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

We substitute into the equation and get $\sum_{k=2}^{\infty} a_k k(k-1)x^{k-2} - \sum_{k=0}^{\infty} a_k x^{k+3} = 24x^2$, hence

$$2a_2 + 6a_3x + 2a_4x^2 + \sum_{k=3}^{\infty} [a_{k+2}(k+1)(k+2) - a_{k-3}]x^k = 24x^2.$$

Comparing both sides we get $a_2 = 0$, $a_3 = 0$, $a_4 = 2$, and equations $3 \cdot 5a_5 - a_0 = 0$, $5 \cdot 6a_6 - a_1 = 0$, $6 \cdot 7a_7 - a_2 = 0$, $7 \cdot 8a_8 - a_3 = 0$, $8 \cdot 9a_9 - a_4 = 0$, $9 \cdot 10a_{10} - a_5 = 0$, $10 \cdot 11a_{11} - a_6 = 0$, etc.

We choose $a_0 = a$, $a_1 = b$, then $a_5 = \frac{a}{20}$, $a_6 = \frac{b}{30}$, $a_7 = a_8 = 0$, $a_9 = \frac{1}{36}$, $a_{10} = \frac{a}{1800}$, $a_{11} = \frac{b}{3300}$, $a_{12} = a_{13} = 0$, etc.

and thus $y(x) = a + bx + 2x^4 + \frac{a}{20}x^5 + \frac{b}{30}x^6 + \frac{1}{36}x^9 + \frac{a}{1800}x^{10} + \frac{b}{3300}x^{11} + \dots$

Example.

$$\lim_{x \rightarrow 0} \left(\frac{1 - \cos(x)}{x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{1 - [1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots]}{x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{\frac{x^2}{2} - \frac{x^4}{4!} + \dots}{x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{1}{2} - \frac{x^2}{4!} + \dots \right) = \frac{1}{2}.$$

Example.

$$\begin{aligned} \int \frac{1}{x} \sin(x) dx &= \int \frac{1}{x} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int x^{2k} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{x^{2k+1}}{2k+1} + C \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!(2k+1)} x^{2k+1} + C. \end{aligned}$$