Multidimensional Calculus. Lectures content. Week 3

4. Chain rule. Implicit differentiation.

Theorem. Suppose $z = f(x, y)$ is a differentiable function, and suppose $x = g(t)$ and $y = h(t)$ with both g and h differentiable functions of t. Then f is a differentiable function of t and

$$\frac{df}{dt} = \frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}, \qquad \begin{array}{cc} \multicolumn{2}{c}{z} \\ \swarrow & \searrow \\ x & y \\ \downarrow & \downarrow \\ t & t \end{array}$$

It is convenient to draw a tree diagram to visualize the dependent, intermediate and independent variables.

Example. Given $z = x^2y + 3xy^4$, $x = e^t$, $y = \sin t$, find $\frac{dz}{dt}$ and $\frac{dz}{dt}\big|_{t=0}$.

Theorem. The above theorem can be extended to function f of n variables $x_1, x_2, \ldots, x_n$ where each x_j is a function of m variables $t_1, \ldots, t_m$, then if all partial derivatives $\frac{\partial x_j}{\partial t_i}$ exist, we have

$$\frac{\partial f}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial t_i}$$

for every $i = 1, \ldots, m$.

Example. Evaluate $\frac{dw}{dt}$ if $w = f(x, y, z) = xy + z$ and $x = \cos t$, $y = \sin t$, $z = t$. (The function f is consider over a path, indeed $x = \cos t$, $y = \sin t$, $z = t$ are the parametric equations of a helix in $\mathbb{R}^3$). In this case using the tree diagram

$$\begin{array}{ccc} & w & \\ \swarrow & \downarrow & \searrow \\ x & y & z \\ \downarrow & \downarrow & \downarrow \\ t & t & t \end{array} \qquad \text{we get} \qquad \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

Example. Given $z = e^x \sin y$, $x = st^2$, $y = s^2t$, evaluate $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$. From

$$\begin{array}{cccc} \multicolumn{4}{c}{z} \\ \multicolumn{2}{c}{\swarrow} & \multicolumn{2}{c}{\searrow} \\ \multicolumn{2}{c}{x} & \multicolumn{2}{c}{y} \\ \swarrow & \searrow & \swarrow & \searrow \\ s & t & s & t \end{array} \qquad \text{we get} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}, \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}.$$

Example. Given $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 \ln s$, $z = 2r$, find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ using the tree diagram

$$\begin{array}{cccccc} \multicolumn{6}{c}{w} \\ \multicolumn{2}{c}{\swarrow} & \multicolumn{2}{c}{\downarrow} & \multicolumn{2}{c}{\searrow} \\ \multicolumn{2}{c}{x} & \multicolumn{2}{c}{y} & \multicolumn{2}{c}{z} \\ \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow \\ s & r & s & r & s & r \end{array}.$$

Theorem. *Implicit Function Theorem.* $F(x, y) = 0$ defines implicitly y as a function of x near a point (a, b) if $F(a, b) = 0$, $F_y(a, b) \neq 0$ and F_x and F_y are continuous on a disk containing (a, b).

Theorem. If $F(x,y) = 0$ defines implicitly y as a function of x then the derivative of y with respect to x can be evaluated by

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Proof. (Use chain rule.)
Example. Given $F(x,y) = x^2 + \sin y - 2y$ use implicit differentiation to find $\frac{dy}{dx}$ near $(0,0)$. (Prove first that $F(x,y) = 0$ defines implicitly y as a function of x near $(0,0)$.)

5. Directional derivative. Gradient.

If $z = f(x,y)$, the partial derivatives f_x, f_y represent the rate of change of f in the direction of the x and y axis. We may wish to find the rate of change of z in any direction. Remember that given any vector $\mathbf{v}$ its direction is determine by the unit vector $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$.
Definition. The directional derivative of f at (x_0, y_0) in the direction of the unit vector $\mathbf{u} = (a,b)$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if the limit exists.
Theorem. If f is a differentiable function of x and y, then f has directional derivative in the direction of any unit vector $\mathbf{u} = (a,b)$ and

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b.$$

Proof. (Use chain rule.)
Example. Find the directional derivative of $f(x,y) = x^3 y^4$ at $P(6,-1)$ in the direction of $\mathbf{v} =< 2,5 >= 2\mathbf{i} + 5\mathbf{j}$.

Definition. Given $f(x,y)$ we call gradient of f the vector function $\nabla f =< f_x, f_y >= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$.
Example. Given $f(x,y) = \cos x - e^{xy^2}$, evaluate ∇f and $\nabla f\,(0,1)$.

Remark. $D_{\mathbf{u}} f(x,y) = \nabla f \cdot \mathbf{u}$ ($\cdot$ inner product of two vectors).
Remark. Due to the fact that $\mathbf{v} \cdot \mathbf{u} = \|\mathbf{v}\| \cdot \|\mathbf{u}\| \cos\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{u}$, we have that $D_{\mathbf{u}}$ is maximal when $\mathbf{u}$ and ∇f are parallel. Thus:
$f(x,y)$ increases most rapidly in the direction of ∇f (at every point).
Any direction $\mathbf{u} \perp$ to ∇f is a direction of no change in f, so ∇f is $\perp$ to level curves.
The direction of $-\nabla f$ is the direction of minimal change of rate of f.
Remark. In three dimensions: $\nabla f =< f_x, f_y, f_z >$, ∇f is (still) the direction of maximal rate of increase, ∇f (if $\neq 0$) is orthogonal to level surfaces, so it is $\perp$ to the tangent plane to the level surface.
Example. Find the tangent plane to the circular paraboloid $x^2 + y^2 + z - 9 = 0$ at $P(1,2,4)$. (Notice that if $f(x,y,z) = x^2 + y^2 + z - 9$ the paraboloid is the level surface $f = 0$).

For the general case of a function $f : \mathbb{R}^n \to \mathbb{R}$, the definition and theorems concerning Directional derivatives are the following.
Definition. The directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ in the direction of a unit vector $\mathbf{u}$ at the point $\mathbf{x}$ is

$$D_{\mathbf{u}} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h}.$$

Theorem. If $\mathbf{x}$ is a point such that the gradient of f, $\nabla f =< \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} >$ is a continuous vector function at $\mathbf{x}$, then $D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$.

Definition. The linearization of a function $f(\mathbf{x})$ at a point $\mathbf{a}$ is:

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f|_{\mathbf{a}} \cdot (\mathbf{x} - \mathbf{a}).$$

Exercises. Lab 3

1) $z = x\sqrt{1+y^2}$, $x = te^{2t}$, $y = e^{-t}$, find $\frac{dz}{dt}$.

2) $z = \sin x \cos y$, $x = (s-t)^2$, $y = s^2 - t^2$, find $\frac{\partial z}{\partial s}$ $\frac{\partial z}{\partial t}$.

3) $u = xy + yz + zx$, $x = st$, $y = e^{st}$, $z = t^2$, find $\frac{\partial u}{\partial s}$ $\frac{\partial u}{\partial t}$.

4) Verify that $2y^2 + \sqrt[3]{xy} = 3x^2 + 18$ defines y as a function of x around $P(-2,4)$, find $\frac{dy}{dx}\big|_P$.

5) Find $\nabla f\big|_P$ for $f(x,y) = \ln(x^2+y^2)$, $P(1,1)$.

6) Find $\nabla f\big|_P$ for $f(x,y,z) = e^{x+y}\cos z + (y+1)\sin x$, $P(0, 0\pi/2)$.

7) Find $D_{\mathbf{u}}f\big|_P$ for $f(x,y,z) = 3e^x\cos(yz)$, $P(0,0,0)$, $\mathbf{v} =< 2, 1, -2 >$.

8) Find $D_{\mathbf{u}}f\big|_P$ for $f(x,y,z) = x^2 + 2y^2 - 3z^2$, $P(0,0,0)$, $\mathbf{v} =< 1, 1, 1 >$.

9) Find the maximal and minimal rate of change of $f(x,y) = xe^{-y} + 3y$ at $P(1,0)$ in the direction in which they occur.

10) Find the maximal and minimal rate of change of $f(x,y,z) = \frac{x}{y} + \frac{y}{z}$ at $P(4,2,1)$ in the direction in which they occur.