Multidimensional Calculus. Lectures content. Week 6

11. Triple integrals.

Preceding with a construction similar to the one that brought to the definition of the double integral, we define the triple integral of a function $f : \mathbb{R}^3 \to \mathbb{R}$ over a region $B \subset \mathbb{R}^3$, and then in general the multiple integrals of a function $f : \mathbb{R}^n \to \mathbb{R}$ over a region $R \subset \mathbb{R}^n$.

Definition. Consider a function $f(x, y, z)$ defined over a box $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$. Consider a partition $a_1 = x_1 < x_2 < \cdots < x_m = b_1$, of $[a_1, b_1]$, a partition $a_2 = y_1 < y_2 < \cdots < y_n = b_2$, of $[a_2, b_2]$, and a partition $a_3 = z_1 < z_2 < \cdots < z_l = b_3$, of $[a_3, b_3]$, then the set of all mnl sub-boxes $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ forms a partition of B. We call norm of this partition of B the length δ of the longest diagonal of all sub-boxes B_{ijk}. Also the volume of the ijk-sub-box is $\Delta V_{ijk} = (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})$. Choose a point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ inside each B_{ijk}, we now can define the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_{ijk}.$$

The function f is said to be Riemann integrable over B, and the triple integral of f over B is

$$\iiint_B f(x, y, z)\, dV = \lim_{\delta\to 0}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_{ijk}$$

if the limit exists.

Definition. The volume of a closed bounded region D in space is Volume$(D) = \iiint_D dV$.

Remark. Fubini's theorem is extended to triple integrals over a box region $B = [a, b] \times [c, d] \times [r, s]$, and there are six different ways to express an integral over B.

Example. Express in other five ways the integral $\int_0^1 \int_0^2 \int_0^{1-y} dz\, dx\, dy$.

Definition. The mass M of a solid occupying a region D, is defined in terms of the density δ as $M = \iiint_D \delta\, dV$.

Example. A solid cube in the first octant bounded by $x = 1$, $y = 1$, $z = 1$, has density $\delta(x, y, z) = x + y + z + 1$, find its mass.

12. Cylindrical and spherical coordinates. Substitution in multiple integral.

In three dimensional space we have the possibility to introduced two more useful coordinate systems other than the cartesian one, cylindrical coordinates are particularly useful for describing regions symmetric with respect to the z-axis, spherical coordinates for regions symmetric with respect to the origin.

Definition. Cylindrical coordinates. A point $P(x, y, z) \in \mathbb{R}^3$ can be assigned cylindrical coordinates (r, θ, z), where $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and (r, θ) are the polar coordinates of the projection of P on the xy-plane.

Definition. Spherical coordinates. A point $P(x, y, z) \in \mathbb{R}^3$ can be assigned spherical coordinates (ρ, θ, ϕ), where $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$ and ρ is the distance of P from the origin, θ correspond to the θ of the polar coordinates of the projection of P on the xy-plane, ϕ is the angle between the z-axis and the segment OP, O the origin.

Theorem. (Substitution.) Given $U \subset \mathbb{R}^n$, $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ a C^1-transformation that maps U onto $\Theta(U)$, then

$$\int_{\Theta(U)} f(\mathbf{x})\, d\mathbf{x} = \int_U f(\Theta(\mathbf{u}))|\det J_\Theta(\mathbf{u})|\, d\mathbf{u}$$

where $|\det J_\Theta|$ is the absolute value of the determinant of the Jacobian of the transformation.

Remark. Expressing an integral in cylindrical coordinates will imply that $dV = r\, dr\, d\theta\, dz$, while if we calculate the determinant of the Jacobian of the transformation related to the change of variable from cartesian to spherical coordinates we find out that in this case $dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$.

Example. Use cylindrical coordinates to find the volume of the solid bounded above by $x^2 + y^2 + z^2 = 4$, on the sides by $x^2 + y^2 = 1$, below by $z = 0$.

Example. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

13. Line integral of scalar functions.

Definition. A set $C \subset \mathbb{R}^n$ is called an arch, or curve, if there exists a continuous, continuously differentiable (C^1) vector function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$, $\mathbf{r}(t) =< r_1(t), \ldots, r_n(t) >$ such that, for a certain interval $[a, b] \subset \mathbb{R}$, $\mathbf{r} : [a, b] \to C$ is onto, and $\mathbf{r}'(t) \neq \mathbf{0}$ for every $t \in (a, b)$. In this case $\mathbf{r}$ is called a parametrization of C, $\mathbf{r}(a)$ and $\mathbf{r}(b)$ the starting and ending points of C.

Remark. $\mathbf{r}'(t) =< r_1'(t), \ldots, r_n'(t) >$ is a tangent vector to C at $\mathbf{r}(t)$. A unit vector tangent to C at $r(t)$ is $\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.

Example. A segment from point $\mathbf{a}$ to point $\mathbf{b}$ has parametrization $\mathbf{r}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$, $0 \le t \le 1$, $r'(t) = \mathbf{b} - \mathbf{a}$.

Example. A circle in $\mathbb{R}^2$ with center at (a, b) and radius ρ has parametrization $\mathbf{r}(t) =< a + \rho\cos t, b + \rho\sin t >$, $0 \le t \le 2\pi$, $\mathbf{r}'(t) =< -\rho\sin t, \rho\cos t >$, $\|\mathbf{r}'(t)\| = \rho$.

Definition. A piecewise-smooth curve (path) in $\mathbb{R}^n$ is a finite union of arches, such the ending point of every arch (except the last one) is the starting point of the next arch. A curve is called simple, if it does not intersect itself, closed if initial point and end point coincide, positively oriented (in $\mathbb{R}^2$) if the orientation induced by the parametrization is the anticlockwise one.

Theorem. Let C be a smooth space curve with parametric equations given by the vector function $\mathbf{r}(t)$, $a \le t \le b$ (for example $\mathbf{r}(t) =< r_1(t), r_2(t), r_3(t) >$, if $C \subset \mathbb{R}^3$, or $\mathbf{r}(t) =< r_1(t), r_2(t) >$ if $C \subset \mathbb{R}^2$).
Given a scalar function $f(\mathbf{x})$, ($f(x, y, z)$ in $\mathbb{R}^3$ or $f(x, y)$ in $\mathbb{R}^2$) continuous on a region containing C, the line integral of f along C is

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\,dt.$$

Remark. For $f = 1$ the above integral represents the length of C.

Example. Calculate the length of the spiral with parametric equations $x = 2\cos t$, $y = 2\sin t$, $z = \frac{3}{2\pi}t$, $t \in [0, 2\pi]$.

Remark. The line integral does not depend on the parametrization, but a change in the orientation of the curve corresponds with a change of sign of the line integral.

Remark. If $C = C_1 \cup C_2$ is a piecewise smooth curve then $\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$.

Example. Evaluate $\int_C (2 + x^2y)\,ds$, where C is the upper half of the unit circle in $\mathbb{R}^2$.

Exercises. Lab 6

1) Evaluate $\displaystyle\int_0^1 \int_0^\pi \int_0^\pi y \sin z\,dxdydz$.

2) Evaluate $\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^x yz\,dydzdx$.

3) Use cylindrical coordinates to evaluate $\displaystyle\iiint_D x^2 + y^2\,dV$, where D is the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$, and above by the plane $z = 2$.

4) Use spherical coordinates to evaluate $\displaystyle\iiint_B (x^2 + y^2 + z^2)\,dV$, where B is the unit ball $x^2 + y^2 + z^2 \leq 1$.

5) Find the volume of the region bounded above by the sphere $z = x^2+y^2+z^2$ (center?, radius?) and below by the cone $z = \sqrt{x^2 + y^2}$.

6) The area of the side sides of the cylinder $x^2 + y^2 = 1$ enclosed between the plane $z = 0$ and the plane $x + y + z = 2$ is given by the lane integral $\int_C (2 - x - y)\,ds$, where C is the unit circle in the xy-plane. Evaluate the area.

7) Evaluate $\int_C (x + y)\,ds$, where C is the circle centered at $(1/2, 0)$ with radius $1/2$.

8) Integrate $f(x, y) = x + y^2$ over the line segment from $A(0, 0)$ to $B(1, 1)$.

9) Evaluate $\int_C y \sin z\,ds$, where C is the circular helix with parametric equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq 2\pi$.