Multidimensional Calculus. Lectures content. Week 9

18. Series of real numbers.

Definition. A **series** is a symbol $\sum\limits_{k=n_0}^{\infty} a_k = a_{n_0} + a_{n_0+1} + a_{n_0+2} + \dots,$
where $n_0 \in \mathbb{Z}$, $a_k \in \mathbb{R}$ (series of real numbers).

Definition. Let $\sum\limits_{k=n_0}^{\infty} a_k$ be a series.
We define its **partial sums** by $s_N = \sum\limits_{k=n_0}^{N} a_k$ for $N \geq n_0$.
We say that the given series **converges** if $\{s_N\}_{N=n_0}^{\infty}$ converges.
We say that the given series **converges to** A, denoted $\sum\limits_{k=n_0}^{\infty} a_k = A$, if $\lim\limits_{N\to\infty}(s_N) = A$.
We say that the given series **diverges** if $\{s_N\}_{N=n_0}^{\infty}$ diverges.
We say that the given series **diverges to** ∞, denoted $\sum\limits_{k=n_0}^{\infty} a_k = \infty$, if $\lim\limits_{N\to\infty}(s_N) = \infty$.
We say that the given series **diverges to** $-\infty$, denoted $\sum\limits_{k=n_0}^{\infty} a_k = -\infty$, if $\lim\limits_{N\to\infty}(s_N) = -\infty$.

Example. $\sum\limits_{k=1}^{\infty} \frac{1}{2^k}$: $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$, induction: $s_N = 1 - \frac{1}{2^N}$, hence $s_N \to 1$ and $\sum\limits_{k=1}^{\infty} \frac{1}{2^k} = 1$ (series converges).

Example. $\sum\limits_{k=1}^{\infty} 1$: $s_1 = 1$, $s_2 = 1+1 = 2$, $s_3 = 1+1+1 = 3$, induction: $s_N = N$, hence $s_N \to \infty$ and $\sum\limits_{k=1}^{\infty} 1 = \infty$ (series diverges).

Example. $\sum\limits_{k=0}^{\infty} (-1)^k$: $s_0 = 1$, $s_1 = 1-1 = 0$, $s_2 = 1-1+1 = 1$, induction: $s_N = \begin{cases} 1, & N \text{ even;} \\ 0, & N \text{ odd,} \end{cases}$
thus $\lim\limits_{N\to\infty}(s_N)$ DNE and $\sum\limits_{k=0}^{\infty} (-1)^k$ diverges.

19. Summing up a series.

Definition. Let $a, q \in \mathbb{R}$. The series $\sum\limits_{k=n_0}^{\infty} a\,q^k$ is called a **geometric series**.

Fact. (i) For $N \in \mathbb{N}_0$ we have $\sum\limits_{k=0}^{N} q^k = \dfrac{1-q^{N+1}}{1-q}$;
for $N \in \mathbb{N}$, $N \geq n_0$ we have $\sum\limits_{k=n_0}^{N} q^k = q^{n_0}\dfrac{1-q^{N+1-n_0}}{1-q} = \dfrac{q^{n_0} - q^{N+1}}{1-q}$.
(ii) We have $\sum\limits_{k=0}^{\infty} q^k \begin{cases} = \frac{1}{1-q}, & |q| < 1; \\ = \infty \text{ (diverges)}, & q \geq 1; \\ \text{diverges}, & q \leq -1. \end{cases}$

More generally, $\sum\limits_{k=n_0}^{\infty} q^k = \frac{q^{n_0}}{1-q}$ for $|q|<1$.

Definition. Let $a,q \in I\!R$. The series $\sum\limits_{k=n_0}^{\infty} (a+qk)$ is called an **arithmetic series**.

Fact. (i) For $N \in I\!N_0$ we have $\sum\limits_{k=0}^{N} (a+qk) = (N+1)a + \frac{1}{2}N(N+1)q$.
(ii) An arithmetic series converges only if $a=q=0$.

Summing up a series: we can sum up directly only two kinds:
1) geometric series (might be in disguise):

Example. $\sum\limits_{k=2}^{\infty} \frac{5\cdot 3^{k-1}}{2^{2k+1}} = \sum\limits_{k=2}^{\infty} \frac{5\cdot 3^{-1}\cdot 3^k}{2^1\cdot (2^2)^k} = \frac{5}{6}\sum\limits_{k=2}^{\infty}\frac{3^k}{4^k} = \frac{5}{6}\sum\limits_{k=2}^{\infty}\big(\frac{3}{4}\big)^k = \frac{5}{6}\big(\frac{3}{4}\big)^2\sum\limits_{k=0}^{\infty}\big(\frac{3}{4}\big)^k$
$= \langle\!\langle \big|\frac{3}{4}\big| < 1 \rangle\!\rangle = \frac{5}{6}\big(\frac{3}{4}\big)^2 \frac{1}{1-\frac{3}{4}} = \frac{15}{8}$.

Note: For a geometric series $\sum\limits_{k=n_0}^{\infty} q^k = q^{n_0}\sum\limits_{k=0}^{\infty} q^k$ is true in general.
Or substitution: $\sum\limits_{k=2}^{\infty}\big(\frac{3}{4}\big)^k = \left\langle\!\!\left\langle \begin{matrix} n=k-2 \implies k=n+2 \\ 2\mapsto 0,\ \infty\mapsto\infty \end{matrix} \right\rangle\!\!\right\rangle = \sum\limits_{n=0}^{\infty}\big(\frac{3}{4}\big)^{n+2} = \big(\frac{3}{4}\big)^2\sum\limits_{n=0}^{\infty}\big(\frac{3}{4}\big)^n$.
This an be used for any series, sometimes I use notation $\langle\!\langle k-2 \mapsto k^* \rangle\!\rangle$.
2) telescopic series (might be in disguise):

Example. $\sum\limits_{k=3}^{\infty}\frac{1}{k(k-1)} = \sum\limits_{k=3}^{\infty}\big(\frac{1}{k-1}-\frac{1}{k}\big) = \big(\frac{1}{2}-\frac{1}{3}\big)+\big(\frac{1}{3}-\frac{1}{4}\big)+\big(\frac{1}{4}-\frac{1}{5}\big)+\dots$.
Induction: $s_N = \frac{1}{2}-\frac{1}{N} \to \frac{1}{2}$, hence $\sum\limits_{k=3}^{\infty}\frac{1}{k(k-1)} = \frac{1}{2}$.

Remark: formulas for finite sums:
$\sum\limits_{k=1}^{n} 1 = n$, $\sum\limits_{k=1}^{n} k = \frac{1}{2}n(n+1)$, $\sum\limits_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$, etc.

Theorem. Let series $\sum\limits_{k=n_0}^{\infty} a_k$, $\sum\limits_{k=n_0}^{\infty} b_k$ converge.
Then also the series $\sum\limits_{k=n_0}^{\infty}(a_k+b_k)$ converges and $\sum\limits_{k=n_0}^{\infty}(a_k+b_k) = \sum\limits_{k=n_0}^{\infty} a_k + \sum\limits_{k=n_0}^{\infty} a_k$.
For $c\in I\!R$ also $\sum\limits_{k=n_0}^{\infty}(c\,a_k)$ converges and $\sum\limits_{k=n_0}^{\infty}(c\,a_k) = c\Big(\sum\limits_{k=n_0}^{\infty} a_k\Big)$.

20. Convergence of series.

Theorem. Let $n_0 < n_1$, consider a series $\sum\limits_{k=n_0}^{\infty} a_k$. $\sum\limits_{k=n_0}^{\infty} a_k$ converges if and only if $\sum\limits_{k=n_1}^{\infty} a_k$ converges.
Then we also have $\sum\limits_{k=n_0}^{\infty} a_k = \sum\limits_{k=n_0}^{n_1-1} a_k + \sum\limits_{k=n_1}^{\infty} a_k$.

If we are only interested in convergence of a series and not its sum (if it exists), then we leave out the index specification.

Theorem. (**necessary condition** for convergence) If a series $\sum a_k$ converges, then necessarily $\lim\limits_{k\to\infty}(a_k)=0$.
Equivalently: If $\lim\limits_{k\to\infty}(a_k)=0$ is not true, then the series $\sum a_k$ necessarily diverges.

Theorem. Consider a series $\sum a_k$. If $a_k\geq 0$ for all k, then either $\sum a_k$ converges, or $\sum a_k=\infty$.

21. Tests for series with non-negative numbers.

Theorem. (**integral test**) Let $f\geq 0$ be a non-increasing function on $[n_0,\infty)$ for $n_0\in\mathbb{Z}$.
The series $\sum\limits_{k=n_0}^{\infty} f(k)$ converges if and only if $\int\limits_{n_0}^{\infty} f(x)\,dx$ converges.
Moreover we then have $\int\limits_{n_0}^{\infty} f(x)\,dx \leq \sum\limits_{k=n_0}^{\infty} f(k)\leq f(n_0)+\int\limits_{n_0}^{\infty} f(x)\,dx$.

Example. $\sum\limits_{k=3}^{\infty}\dfrac{1}{k\ln^2(k)}$: $\int\limits_{x=3}^{\infty}\dfrac{dx}{x\ln^2(x)} = \left|\begin{matrix} y=\ln(x)\\ dy=\frac{dx}{x}\end{matrix}\right| = \int\limits_{x=\ln(3)}^{\infty}\dfrac{dy}{y^2}<\infty$. Therefore the series converges.
Moreover, $\sum\limits_{k=3}^{\infty}\frac{1}{k\ln^2(k)}\in\left[\frac{1}{\ln(3)},\frac{1}{3\ln^2(3)}+\frac{1}{\ln(3)}\right]\sim[0.91,1.19]$.
Trick: $\sum\limits_{k=3}^{\infty}\frac{1}{k\ln^2(k)}=\sum\limits_{k=3}^{9}\frac{1}{k\ln^2(k)}+\sum\limits_{k=10}^{\infty}\frac{1}{k\ln^2(k)}$

$$\in\Big[\sum_{k=3}^{9}\frac{1}{k\ln^2(k)}+\int_{10}^{\infty}\frac{dx}{x\ln^2(x)},\sum_{k=3}^{9}\frac{1}{k\ln^2(k)}+\frac{1}{10\ln^2(10)}+\int_{10}^{\infty}\frac{dx}{x\ln^2(x)}\Big]\sim[1.059,1.078].$$

Corollary. (p**-test**) $\sum\frac{1}{k^p}$ converges if and only if $p>1$.

Example. $\sum\limits_{k=1}^{\infty}\dfrac{1}{k^2}=\dfrac{\pi^2}{6}$.
$\sum\limits_{k=1}^{\infty}\dfrac{1}{k}=\infty$ (**harmonic series**).

Theorem. (**comparison test**) Let $\exists n_0$ so that $0\leq a_k\leq b_k$ for $k\geq n_0$.
If $\sum b_k$ converges, then also $\sum a_k$ converges.
If $\sum a_k$ diverges, then also $\sum b_k$ diverges (i.e. $\sum a_k=\infty\implies\sum b_k=\infty$).
Remark: Symbolically (and roughly) $a_k\leq b_k\implies\sum a_k\leq\sum b_k$.

Theorem. (**limit comparison test**) Let $\exists n_0\in\mathbb{Z}$ so that $a_k\geq 0, b_k\geq 0$ for $k\geq n_0$.
If $a_k\sim b_k$, i.e. $\lim\limits_{k\to\infty}\left(\frac{a_k}{b_k}\right)=A>0$, then $\sum a_k\sim\sum b_k$, i.e. $\sum a_k$ converges if and only if $\sum b_k$ converges.

Example. $\sum\frac{1}{k^2+1}$: $0\leq\frac{1}{k^2+1}\leq\frac{1}{k^2}$ and $\sum\frac{1}{k^2}$ converges, therefore by CT also $\sum\frac{1}{k^2+1}$ converges.
Remark: Also IT and LCT would work here.

Example. $\sum\frac{1}{2k^2-1}$: $\frac{1}{2k^2-1}\geq\frac{1}{2k^2}\geq 0$, $\sum\frac{1}{2k^2}=\frac{1}{2}\sum\frac{1}{k^2}$ converges, but the inequality goes the wrong way, so no conclusion possible.
Guess $\frac{1}{2k^2-1}\sim\frac{1}{2k^2}$ for large k, confirm: $\lim\limits_{k\to\infty}\left(\frac{\frac{1}{2k^2-1}}{\frac{1}{2k^2}}\right)=1>0$,
$\sum\frac{1}{2k^2}=\frac{1}{2}\sum\frac{1}{k^2}$ converges, hence by LCT also $\sum\frac{1}{2k^2-1}$ converges.

Example. $\sum \frac{1}{k\ln^2(k)}$: Two comparisons seem reasonable, $\frac{1}{k^2} \leq \frac{1}{k\ln^2(k)} \leq \frac{1}{k}$, but both are in the wrong direction, so nothing here.
Limit comparison: No candidate, $\frac{1}{k\ln^2(k)} \sim \frac{1}{k}$ or $\frac{1}{k\ln^2(k)} \sim \frac{1}{k^2}$ definitely not true.
Thus comparison tests won't help.

Theorem. Let $a_k \geq 0$ for all k.
ratio test: (i) If $\exists q < 1$ and $\exists n_0 \in \mathbb{Z}$ such that $\forall k \geq n_0$: $\frac{a_{k+1}}{a_k} \leq q$, then $\sum a_k$ converges.
(ii) If $\exists n_0 \in \mathbb{Z}$ such that $\forall k \geq n_0$: $\frac{a_{k+1}}{a_k} \geq 1$, then $\sum a_k$ diverges $(= \infty)$.
limit ratio test: Let $\lambda = \lim\limits_{k\to\infty} \left(\frac{a_{k+1}}{a_k}\right)$, assuming that the limit converges.
(i) If $\lambda < 1$, then $\sum a_k$ converges.
(ii) If $\lambda > 1$, then $\sum a_k$ diverges $(= \infty)$.
root test: (i) If $\exists q < 1$ and $\exists n_0 \in \mathbb{Z}$ such that $\forall k \geq n_0$: $\sqrt[k]{a_k} \leq q$, then $\sum a_k$ converges.
(ii) If $\exists n_0 \in \mathbb{Z}$ such that $\forall k \geq n_0$: $\sqrt[k]{a_k} \geq 1$, then $\sum a_k$ diverges $(= \infty)$.
limit root test: Let $\varrho = \lim\limits_{k\to\infty} \left(\sqrt[k]{a_k}\right)$, assuming that the limit converges.
(i) If $\varrho < 1$, then $\sum a_k$ converges.
(ii) If $\varrho > 1$, then $\sum a_k$ diverges $(= \infty)$.

Example. $\sum \frac{k!}{2^k}$: Limit ratio test $\lambda = \lim\limits_{k\to\infty} \left(\frac{a_{k+1}}{a_k}\right) = \lim\limits_{k\to\infty} \left(\frac{(k+1)!}{k!}\,\frac{2^k}{2^{k+1}}\right) = \lim\limits_{k\to\infty} \left(\frac{1}{2}(k+1)\right) = \infty > 1$.
Thus $\sum \frac{k!}{2^k}$ diverges.

Example. $\sum \frac{2}{\ln^k(k+1)}$: Limit root test $\varrho = \lim\limits_{k\to\infty} \left(\sqrt[k]{a_k}\right) = \lim\limits_{k\to\infty} \left(\frac{\sqrt[k]{2}}{\ln(k+1)}\right) = \frac{1}{\infty} = 0 < 1$.
Thus $\sum \frac{2}{\ln^k(k+1)}$ converges.

Example. $\sum \left(\frac{k}{k+1}\right)^k$: Limit root test $\varrho = \lim\limits_{k\to\infty} \left(\sqrt[k]{a_k}\right) = \lim\limits_{k\to\infty} \left(\frac{k}{k+1}\right) = 1$, no conclusion.
Similarly limit ratio fails. Integral test without chance, comparison as well.
But: $a_k = \left(1 - \frac{1}{k+1}\right)^k \to e^{-1} \neq 0$, hence $\sum \left(\frac{k}{k+1}\right)^k$ diverges.

Exercises. Lab 9

Discuss the convergence of the following series and, if possible, find its sum.

1) $\sum\limits_{k=1}^{\infty} \frac{2^{2k+1}}{3^{k-1}}$ **2)** $\sum\limits_{k=3}^{\infty} \frac{2(-3)^{k+1}}{2^{3k-4}}$ **3)** $\sum\limits_{k=2}^{\infty} \frac{2}{k^2-1}$

4) $\sum\limits_{k=1}^{\infty} \sqrt{\frac{k+1}{k+2}}$ **5)** $\sum\limits_{k=1}^{\infty} \frac{3^k}{k+2}$ **6)** $\sum\limits_{k=1}^{\infty} \frac{3^k}{(k+1)!}$

7) $\sum\limits_{k=1}^{\infty} \frac{k+1}{k^3+k+13}$ **8)** $\sum\limits_{k=1}^{\infty} \frac{3k+1}{2^k}$ **9)** $\sum\limits_{k=1}^{\infty} (1 - \frac{3}{a})\ \ a \in \mathbb{R}$

10) $\sum\limits_{k=1}^{\infty} \frac{k!}{2^k}$ **11)** $\sum\limits_{k=1}^{\infty} (\frac{k}{k+1})^k$ **12)** $\sum\limits_{k=1}^{\infty} (\frac{k}{k+1})^{k^2}$