Czech Technical University in Prague
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Calculus 2 - Exercises

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Prague 2016

Authors were supported by the project RPAPS 13101/105/1051603C005.

Exercises. Week 1

1) Find domain of $f(x,y) = \ln \frac{x^2+2x+y^2}{x^2-2x+y^2}$, $f(x,y,z) = \arcsin \frac{z}{\sqrt{x^2+y^2}}$.

2) Find domain of $f(x,y,z) = \frac{x}{|y+z|}$, $g(x,y) = \sqrt{1-|x|-|y|}$.

3) Graph $f(x,y) = x^2+y^2$, $g(x,y) = 4x^2+9y^2$, discuss level curves.

4) Verify that $\lim\limits_{(x,y)\to(0,0)} \frac{e^y \sin x}{x} = 1$, $\lim\limits_{(x,y)\to(1,0)} \frac{x \sin y}{x^2+y} = 0$, $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{2-\sqrt{4-xyz}}{xyz} = \frac{1}{4}$.

5) Approach zero along different paths to prove that the following limits do not exist:
$\lim\limits_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$, $\lim\limits_{(x,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2}$,
$\lim\limits_{(x,y)\to(0,0)} \frac{-x}{\sqrt{x^2+y^2}}$, $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^4+y^4}}$.

6) Use ε-δ- proof to verify that
$\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^3}{\sqrt{x^2+y^2}} = 0$, $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{\sqrt{x^2+y^2}} = 0$, $\lim\limits_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}} = 0$.

7) Use polar coordinates $x = r\cos\vartheta$, $y = r\sin\vartheta$ to solve the previous limits and the limits discussed during the lecture.

8) Find $c \in \mathbb{R}$ such that $f(x,y) = \begin{cases} \frac{xy}{|x|+|y|} & (x,y) \neq (0,0) \\ c & (x,y) = (0,0) \end{cases}$ is continuous everywhere, prove the existence of the limit at zero with an ε-δ- proof.

9) Get acquainted with the quadric surfaces. Draw the surfaces with the given equation for $a = b = c = 1$:
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ellipsoid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ hyperboloid of one sheet
$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ hyperboloid of two sheets
$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ cone
$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ elliptic paraboloid
$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ hyperbolic paraboloid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ elliptic cylinder
$y = ax^2$ parabolic cylinder.

Exercises. Week 2

1) Given $f(x,y) = \sinh\sqrt{3x+4y}$, find $D(f)$, f_x, f_y.

2) Given $f(x,y,z) = xy^2z^3 \ln(x+2y+3z)$, find $D(f)$, f_x, f_y, f_z.

3) Given $f(x,y,z) = e^{xy^2} + x^4y^4z^3$, verify that $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, $f_{zy} = f_{yz}$. Find f_{xyz}.

4) Is $f(x,y) = x^2 - y^2$ a solution of Laplace equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$?

5) Find linearization of $f(x,y,z) = e^x + \cos(y+z)$ at $(0, \frac{\pi}{4}, \frac{\pi}{4})$.

6) Given $f(x,y)=\ln(x-3y)$, find its linearization at $(x_0,y_0)=(7,2)$. Use the result to approximate the value of f at $(6.9,2.02)$.

7) Given $f(x,y)=xe^{xy}$, find its linearization at $(x_0,y_0)=(6,0)$. Use the result to approximate the value of f at $(5.9,0.01)$.

8) Find tangent plane to $z=\ln(2x+y)$ at $(-1,3,0)$.

9) Given the function $f(x,y)=\begin{cases}\frac{x^3y-xy^3}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$,
prove that f is continuous, f_x and f_y exist everywhere but $f_{xy}(0,0)\neq f_{yx}(0,0)$.

Exercises. Week 3

1) $z=x\sqrt{1+y^2}$, $x=te^{2t}$, $y=e^{-t}$, find $\frac{dz}{dt}$.

2) $z=\sin x\cos y$, $x=(s-t)^2$, $y=s^2-t^2$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

3) $u=xy+yz+zx$, $x=st$, $y=e^{st}$, $z=t^2$, find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$.

4) In a circular cylinder, the radius R is decreasing at a rate of 1.2 cm/s, while its height h is increasing at a rate of 3 cm/s. At what rate is the volume of the cylinder changing when $R=80$ cm and $h=150$ cm?

5) In a right circular cone, the radius R is increasing at a rate of 1.8 cm/s, while its height h is decreasing at a rate of 2.5 cm/s. At what rate are the volume and surface of the cone changing when $R=12$ cm and $h=140$ cm?

6) Prove that any function of the form $h(x,t)=f(x+at)+g(x-at)$, $a\in\mathbb{R}$, is a solution of the wave equation $\frac{\partial^2 h}{\partial t^2}=a^2\frac{\partial^2 h}{\partial x^2}$.

7) Verify that $2y^2+\sqrt[3]{xy}=3x^2+18$ defines y as a function of x around $P=(-2,4)$, find $\frac{dy}{dx}\big|_P$

8) Using the implicit function theorem find the tangent line to the given curve at the given point:
$x^2-xy+y^4=3$, $A=(1,-1)$; $x\cos y+y\cos x=1$, $B=(1,0)$,
$2y^2++\sqrt[3]{xy}=3x^2+22$, $C=(2,4)$.

9) Rewrite the Laplace equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$ using polar coordinates.

10) Solve the partial differential equation $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=1$ changing it into polar coordinates.

11) Find $\nabla f\big|_P$ for $f(x,y)=\ln(x^2+y^2)$, $P=(1,1)$.

12) Find $\nabla f\big|_P$ for $f(x,y,z)=e^{x+y}\cos z+(y+1)\sin x$, $P=(0,0\pi/2)$.

13) Find $D_{\vec{u}}f\big|_P$ for $f(x,y,z)=3e^x\cos(yz)$, $P=(0,0,0)$, $\vec{v}=<2,1,-2>$.

14) Find $D_{\vec{u}}f\big|_P$ for $f(x,y,z)=x^2+2y^2-3z^2$, $P=(0,0,0)$, $\vec{v}=<1,1,1>$.

15) Find tangent plane and normal line to the given surface at the given point:
$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$ at $P = (-2, 1, -3)$, $\quad x^2 - 2y^2 - 3z^2 + xyz = 4$ at $A = (3, -2, 1)$,
$z + 1 = xe^y \cos z$ at $B = (1, 0, 0)$.

16) Prove that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24$ are tangent at the point $P = (1, 1, 2)$.

17) Find the maximal and minimal rate of change of $f(x, y) = xe^{-y} + 3y$ at $P = (1, 0)$ in the direction in which they occur.

18) Find the maximal and minimal rate of change of $f(x, y, z) = \frac{x}{y} + \frac{y}{z}$ at $P = (4, 2, 1)$ in the direction in which they occur.

Exercises. Week 4

1) We recall that the quadratic approximation of a function $f(\vec{x})$, whose partial derivatives of second order are defined and continuous on a neighbourhood of a point $\vec{a}$, is defined as:
$Q(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a}) + \frac{1}{2}(\vec{x} - \vec{a})^T Hf(\vec{a})(\vec{x} - \vec{a})$.
Find the quadratic approximation of $f(x, y) = (1 + x^2)e^{x^2+y^2}$ at $\vec{a} = (0, 0)$,
and of $g(x, y) = xe^y + 1$ at $\vec{a} = (1, 0)$.

2) Find local maximum, minimum and saddle points for $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$.

3) Find local maximum, minimum and saddle points for $f(x, y) = 4xy - x^4 - y^4$.

4) Find local maximum, minimum and saddle points for $f(x, y) = y\sqrt{x} - y^2 - x + 6y$.

5) Find local maximum, minimum and saddle points for $f(x, y) = \frac{x^2y^2 - 8x + y}{xy}$.

6) Find two numbers $a \leq b$ such that $\int_a^b (6 - x - x^2)\, dx$ has largest value. Find a geometrical interpretation of the problem.

7) Find absolute max. and min. value of $f(x, y) = x^2 + xy + y^2 - 6x + 2$ on the rectangle $0 \leq x \leq 5$, $-3 \leq y \leq 0$.

8) Find absolute max. and min. value of $f(x, y) = 2x^3 + y^4$ on the region $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

9) Find absolute max. and min. value of $f(x, y) = x^2 + y^2 - 6x - 4y + 11$ on the region $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 4x \leq 5\}$.

10) The temperature of a heated plate is given by $T(x, y) = 4x^2 - 4xy + y^2$. A bug walks on the plate along a circle centred at $(0, 0)$ with radius 5. Find the coordinates of the hottest and coldest points reached by the bug and the temperature there.

11) Use Lagrange multipliers to find the maximum and minimum value of $f(x, y, z) = x + 3y + 5z$ on $x^2 + y^2 + z^2 = 1$. Then use the geometrical meaning of the gradient and the fact that f is a linear function to find a geometrical solution of the problem.

12) Find the points on $xy^2 = 54$ nearest to the origin.

Exercises. Week 5-6

1) Integrate $f(x,y) = xe^{(xy)}$ over the rectangle $0 \le x \le 1$, $0 \le y \le 1$.

2) Integrate $f(x,y) = \frac{1}{x+y}$ over the rectangle $1 \le x \le 2$, $0 \le y \le 1$.

3) Sketch the region of integration and evaluate the integral $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy$.

4) Evaluate $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y}\, dxdy$.

5) Evaluate $\int\int_D \frac{\sin x}{x}\, dA$, where A is the triangle with vertices $(0,0), (1,0), (1,1)$.

6) Change order of integration in the following integrals

$$\int_0^1 \int_0^{\sqrt{x}} f(x,y)\, dxdy, \qquad \int_0^{\frac{\pi}{2}} \int_0^{\sin x} f(x,y)\, dydx,$$

$$\int_0^1 \int_0^x f(x,y)\, dydx + \int_1^2 \int_0^{2-x} f(x,y)\, dydx.$$

7) Change order of integration to evaluate $\int_0^1 \int_x^1 e^{\frac{x}{y}}\, dxdy$.

8) Change order of integration to evaluate $\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\, dxdy$.

9) Rewrite the integral, first changing order of integration, then transforming it using polar coordinates

$$\int_0^1 \int_0^{2-y} f(x,y)\, dxdy,$$

$$\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\, dydx, \quad a > 0.$$

10) Evaluate $\int\int_R e^{x^2+y^2}\, dxdy$, where R is the half disk with center $(0,0)$ and radius 1 lying above the x-axis by changing the integral into polar coordinates.

11) Evaluate with the use of a double integral the area of a disk of radius one.

12) Sketch the curve and find the area of the region the curve encloses (in polar coordinates):

$$\rho = \sin\vartheta, \quad \vartheta \in [0,\pi], \qquad \rho = 1 + \sin\vartheta, \quad \vartheta \in [0, 2\pi],$$

$$\rho = \cos(2\vartheta), \quad \vartheta \in [0, 2\pi], \qquad \rho = |\vartheta| + 1, \quad \vartheta \in [-\pi, \pi].$$

13) With the use of a double integral in polar coordinates, evaluate the area enclosed by the curves with equation $\rho = 3 + 2\sin\vartheta$, $\rho = 2$.

14) Use polar coordinates to evaluate:

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dxdy, \qquad \int_0^1 \int_0^x \frac{x}{x^2+y^2}\, dydx,$$

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \frac{x^2-y^2}{\sqrt{x^2+y^2}}\,dxdy, \qquad \int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} \arctan\frac{y}{x}\,dydx,$$

$$\int_{0}^{\sqrt{2}}\int_{y}^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\,dxdy.$$

15) Find the volume of the solid bounded by $z = 0$, and the paraboloid $z = 1 - x^2 - y^2$.

16) Find the volume of the solid bounded by $z = 9$, and the paraboloid $z = x^2 + y^2$.

17) Find the volume of the solid bounded by the paraboloides $z = 4 - x^2 - y^2$ and $z = 3x^2 + 2 + 3y^2$.

18) Use polar coordinates to find the volume of a right circular cone with height h and a circular base with radius R.

19) Knowing that the average value of a function f over a region R is by definition

$$\text{Average}(f(x,y)) = \frac{1}{\text{Area}(R)}\int\int_{R} f(x,y)\,dA$$

find the average value of $f(x,y) = x\cos(xy)$ over the rectangle $R = [0,\pi]\times[0,1]$.

20) Knowing that the mass m and the center of gravity $C = (x_0, y_0)$ of a flat object occupying a region of the plane D with density $\rho(x,y)$ are defined by

$$m = \int\int_{D} \rho(x,y)\,dA,$$

$$x_0 = \frac{1}{m}\int\int_{D} x\rho(x,y)\,dA, \qquad y_0 = \frac{1}{m}\int\int_{D} y\rho(x,y)\,dA,$$

find the mass and center of gravity of

a) a triangle with vertex at $(0,0), (1,1), (4,0)$ and density $\rho(x,y) = x$,

b) the part of the plane bounded by the parabola $y = 9 - x^2$ and the x-axis, with density $\rho(x,y) = y$.

21) Use a substitution to evaluate $\int\int_R (x+2y)\sqrt[3]{x-y}\,dA$, where R is the closed region bounded by $y = x$, $y = x - 1$, $x + 2y = 0$, $x + 2y = 2$.

22) Use a substitution to evaluate $\int_0^1\int_0^{1-x} \sqrt{x+y}(y-2x)^2\,dydx$.

23) Use a substitution to evaluate $\int\int_R (x+y)\cos(\pi(x-y))\,dA$, where
$R = \{(x,y)\in\mathbb{R}^2 : 0 \le x+y, \quad x \le 1, \quad 1+y \le x \le 2+y\}$.

24) Use a substitution to evaluate $\int\int_R \frac{y}{x}e^{xy}\,dA$, where R is the closed region bounded by $xy = 2$, $xy = 4$, $y = 2x$, $y = \frac{x}{2}$.

25) Evaluate the integral $\int\int_T e^{-y^2}\,dA$ over the unbounded region $T = \{(x,y)\in\mathbb{R}^2 : 0 \le x \le y\}$.

26) Evaluate the integral $\int_2^{\infty}\int_2^{y} \frac{1-\ln x}{y^3}\,dA$.

Exercises. Week 7-8

1) Evaluate $\int_0^1\int_0^\pi\int_0^\pi y\sin z\,dxdydz$.

2) Evaluate $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^x yz\,dydzdx$.

3) Sketch the region of integration

$$\int_0^1\int_0^z\int_0^y f dxdydz, \qquad \int_0^1\int_x^{2x}\int_0^{x+y} f dzdydx,$$

$$\int_0^\pi\int_0^2\int_0^{\sqrt{4-z^2}} f dxdzdy, \qquad \int_0^3\int_0^{\sqrt{9-x^2}}\int_0^x f dydzdx,$$

4) Sketch the region of integration and evaluate

$$\int_0^1\int_0^{3-3x}\int_0^{3-3x-y} dz\,dy\,dx, \qquad \int_0^\pi\int_0^{\ln(\sin y)}\int_{-\infty}^z e^x\,dx\,dz\,dy.$$

5) Set up the integral $\iiint_E f dV$, using all possible orders of integration, where E is bounded by the surfaces:

$$a)\, x^2+z^2=4,\ y=0,\ y=6, \qquad b)\, z=0,\ z=y,\ x^2=1-y,\ 9x^2+4y^2+z^2=1.$$

6) Evaluate $\iiint_E e^x dV$ where $E=\{(x,y,z),\ 0\le y\le 1, 0\le x\le y, 0\le z\le x+y\}$.

7) Evaluate $\iiint_E y\,dV$ where E is bounded above by the plane $z=x+2y$, and lies above the region of the xy-plane enclosed by the curves $y=x^2, y=0, x=1$.

8) Evaluate $\iiint_E xy\,dV$ where E is the tetrahedron with vertex in $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, $(0,1,1)$.

9) Evaluate $\iiint_E x\,dV$ where E is bounded by the paraboloid $x=4y^2+4z^2$ and the plane $x=4$.

10) Use cylindrical coordinates to evaluate $\iiint_D x^2+y^2\,dV$, where D is the solid bounded below by the cone $z=\sqrt{x^2+y^2}$, and above by the plane $z=2$.

11) Use spherical coordinates to evaluate $\iiint_B (x^2+y^2+z^2)\,dV$, where B is the unit ball $x^2+y^2+z^2\le 1$.

12) Find the volume of the region bounded above by the sphere $z=x^2+y^2+z^2$ and below by the cone $z=\sqrt{x^2+y^2}$.

13) Find the volume of the solid bounded by the elliptic cylinder $4x^2+z^2=4$ and the planes $y=0$, $y=z+2$.

14) Sketch the region of integration and evaluate:

$$\int_0^{2\pi}\int_0^2\int_0^{4-r^2} r\,dzdrd\vartheta, \qquad \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^2\sin\varphi\,d\rho d\vartheta d\varphi.$$

15) Evaluate $\iiint_E x^2 + y^2 \, dV$, where $E = \{(x,y,z),\ x^2+y^2 \leq 4, -1 \leq z \leq 2\}$.

16) Evaluate $\iiint_E x^2 \, dV$, where E is the region inside the cylinder $x^2+y^2=1$, bounded above by the cone $z^2 = 4x^2+4y^2$, and below by $z=0$.

17) Evaluate $\iiint_E x e^{(x^2+y^2+z^2)^2} \, dV$, where E is the region bounded by the spheres centred at the origin with radius 1 and 2.

18) Change the integral to cylindrical coordinates and then evaluate it:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2+y^2)^{\frac{3}{2}} \, dz dy dz,$$

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz \, dz dx dy.$$

19) Change the integral to spherical coordinates and then evaluate it:

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2} \, dz dy dz,$$

$$\int_{0}^{3} \int_{0}^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2) \, dz dx dy.$$

Exercises. Week 9

1) Evaluate the length of the spiral with parametric equation $\vec{\varphi}(t) =< 2\cos t, 2\sin t, \frac{t}{\pi} >$, with $t \in [0, 2\pi]$.

2) Calculate the length of the cycloid with parametric equation $\vec{\varphi}(t) =< t - \sin t, 1 - \cos t >$, with $t \in [0, 2\pi]$.

3) Find the length of the curve $\rho = 1 + \cos t$, with $t \in [0, 2\pi]$.

4) Evaluate $\int_C (x+y)\, ds$, where C is the circle centred at $(1/2, 0)$ with radius $1/2$.

5) Integrate $f(x,y) = x + y^2$ over the line segment from $A = (0,0)$ to $B = (1,1)$.

6) Evaluate $\int_C y \sin z \, ds$, where C is the circular helix with parametric equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq 2\pi$.

7) Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} =< x^2, xy >$, C is the part of $\frac{x^2}{4} + \frac{y^2}{9} = 1$, with $y \geq 0$ positively oriented.

8) Find the work done by the force field $\vec{F} = x\vec{i} + y\vec{j} + (xz - y)\vec{k}$ to move a particle along the curve with parametric equations $\vec{r}(t) =< t^2, 2t, 4t^3 >$, $0 \leq t \leq 1$, from $A = (0,0,0)$ to $B = (1,2,4)$.

9) Find the work done by the force field $\vec{F} =< x^2, ye^x >$ to move a particle along the curve $x = y^2 + 1$, from $A = (1, 0)$ to $B = (2, 1)$.

10) Show that $\vec{F} =< e^x \cos y + yz, xz - e^x \sin y, xy + z >$ is conservative, then find a potential function and use it to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is any path from $A = (1, 0, 2)$ to $B = (0, \pi, 1)$.

11) Determine if the following vector fields are conservative, and evaluate a potential (if any):

$$\vec{F}(x, y, z) =< e^{yz}, xze^{yz}, xye^{yz} >, \qquad \vec{G}(x, y, z) =< 1, \sin z, y \cos z > .$$

12) Show that the integral is independent on the path, and evaluate it:

$$\int_C \tan y \, dx + x \sec^2 y \, dy, \qquad C \text{ from } (1, 0) \text{ to } (2, \frac{\pi}{4}).$$

13) Given $\vec{F}(x, y) =< x^2, y^2 >$, evaluate $\int_C \vec{F} \, d\vec{r}$, where C is the path on $y = 2x^2$ from $(1, 2)$ to $(2, 8)$. (Use both a direct computation of the line integral, and a potential function of $\vec{F}$).

14) Given $\vec{F}(x, y) =< \frac{y^2}{1+x^2}, 2y \arctan x >$, evaluate $\int_C \vec{F} \, d\vec{r}$, where C has parametric equations $\vec{r}(t) =< t^2, 2t >$, with $0 \leq t \leq 1$. (Use a potential function of $\vec{F}$).

15) Use Green's theorem to evaluate $\oint_C x^4 \, dx + xy \, dy$, where C is the contour of the triangle with vertices $A = (0, 1)$, $O = (0, 0)$, $B = (1, 0)$, positively oriented.

16) Use Green's theorem to evaluate $\oint_C \vec{F} \, d\vec{r}$, where $\vec{F} =< y^2 \cos x, x^2 + 2y \cos x >$ and C is the triangular path from $O = (0, 0)$ to $A = (2, 6)$ to $B = (2, 0)$ and back to $O = (0, 0)$ (with this orientation!).

17) Consider the path C that from $A = (-2, 0)$, along the x-axis, reaches the point $B = (2, 0)$ and then goes back to $A = (-2, 0)$ along the graph of $y = \sqrt{4 - x^2}$. Find the work done by $\vec{F} =< x^2, x^2 + 2xy >$ to move a particle along C.

18) Evaluate $\int_C (2 - x - y) \, ds$, where C is the unit circle in the xy with the center in the origin.

19) Use Green's theorem to evaluate $\oint_C (3y - e^{\sin x}) \, dx + (7x + \sqrt{y^4 + 1}) \, dy$, where C is the circle $x^2 + y^2 = 9$, positively oriented.

20) Use Green's theorem to evaluate $\oint_C < (2y^2 + \sqrt{1 + x^5}), (5x - e^{y^2}) > d\vec{r}$, where C is the circle $x^2 + y^2 = 4$, positively oriented.

21) Verify Green's theorem for $\vec{F} =< 3x - y, x + 5y >$, if C is the circle $x^2 + y^2 = 1$, positively oriented.

22) Use the generalized form of Green's theorem to evaluate $\int_C y^2 \, dx + 3xy \, dy$ where $C = C_1 \cup C_2$ is the boundary of the annulus D enclosed between the circle C_1 with radius 2 and center the origin, oriented anticlockwise, and the circle C_2 with radius 1 and center the origin, oriented clockwise.

23) Consider C, the path from $O = (0, 0)$ to $A = (2\pi, 0)$ on the curve with parametric equation

$$x(t) = t \cos t, \quad y(t) = t \sin t, \qquad 0 \leq t \leq 2\pi,$$

followed by the straight segment on the x-axis from $A = (2\pi, 0)$ back to $O = (0, 0)$. Use Green's theorem to find the area of the region D enclosed by C.

24) Use Green's theorem to find the area of the region D enclosed by the path C, if C has parametric equations $\vec{\varphi}(t) =< \sin 2t, \sin t >, \quad 0 \leq t \leq \pi$.

Exercises. Week 10

1) Find the area of the part of the plane $x + 2y + z = 4$ that lies inside the cylinder $x^2 + y^2 = 4$.

2) Find the area of the part of $2x + 3y - z = 1$ that lies above the rectangle $[1, 4] \times [2, 4]$.

3) Find the area of the part of paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

4) Find the area of the sides of the cylinder $x^2 + y^2 = 1$ enclosed between the plane $z = 0$ and the plane $x + y + z = 2$.

5) Evaluate $\int\int_S x^2 \, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

6) Evaluate $\int\int_S z \, dS$ where S is the part of the cylinder $x^2 + y^2 = 1$ between the planes $z = 0$ and $z = x + 1$.

7) Evaluate $\int\int_S yz \, dS$ where S is the surface with parametric equations $x = uv$, $y = u + v$, $z = u - v$, $u^2 + v^2 \leq 1$.

8) Evaluate $\int\int_S (x^2 z + y^2 z) \, dS$ where S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$.

9) Find the mass of a funnel S that lies on the cone $z = \sqrt{x^2 + y^2}$, $1 \leq z \leq 4$, if the density is given by the function $\rho(x, y, z) = 10 - z$. ($\text{mass}(S) = \int\int_S \rho \, dS$).

10) Evaluate $\int\int_S xy \, dS$ where S is the part of the cylinder $x^2 + z^2 = 1$ between the planes $y = 0$ and $x + y = 2$.

11) Evaluate $\int\int_S \sqrt{1 + x^2 + y^2} \, dS$ where S is the helicoid with parametric equation
$\vec{r}(u, v) =< u \cos v, u \sin v, v >$, $0 \leq u \leq 1$, $0 \leq v \leq \pi$.

12) Evaluate $\int\int_S \vec{F} \cdot d\vec{S}$, where $\vec{F} =< y, x, z >$, S is the part of the paraboloid $z = 1 - x^2 - y^2$ with $z \geq 0$.

13) Evaluate $\int\int_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = e^y \vec{i} + ye^x \vec{j} + x^2 y \vec{k}$, and S is the part of the paraboloid $z = x^2 + y^2$ that lies above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ and has upward orientation.

14) Evaluate $\int\int_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = x\vec{i} + xy\vec{j} + xz\vec{k}$, and S is the part of the plane $3x + 2y + z = 6$ that lies in the first octant with upward orientation.

15) Evaluate $\int\int_S \vec{F} \cdot d\vec{S}$, where $\vec{F} =< 0, y, -z >$, S is the union of the part of the paraboloid $y = x^2 + z^2$

with $0 \le y \le 1$ and the disk intersection of $x^2 + z^2 \le 1$ with $y = 1$, positively oriented.

16) Evaluate $\int\int_S \vec{F} \cdot d\vec{S}$, where $\vec{F} =< y, x, z^2 >$, S is the helicoid with parametric equation $\vec{r}(u, v) =< u\cos v, u\sin v, v >$, $0 \le u \le 1$, $0 \le v \le \pi$, with the orientation induced by the parameterization.

17) A fluid with density 1 flows with velocity $\vec{v} =< y, 1, z >$. Evaluate the rate of flow upward of the fluid through the part S of the paraboloid $z = 9 - \frac{(x^2+y^2)}{4}$, with $x^2 + y^2 \le 36$. (Evaluate $\int\int_S \vec{v} \cdot d\vec{S}$)

18) The temperature at a point (x, y, z), of a substance with conductivity $k = 6,5$, is given by the function $u(x, y, z) = 2y^2 + 2z^2$. Find the rate of heat flow inward across the part S of the cylinder $y^2 + z^2 = 6$, $0 \le x \le 4$. (Evaluate $\int\int_S -k\nabla u \cdot d\vec{S}$.)

Exercises. Week 11-12

1) Use Stoke's theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$, for $\vec{F} =< yz, xz, xy >$, C is any closed curve in $\mathbb{R}^3$.

2) Use Stoke's theorem to evaluate $\int\int_S \operatorname{curl} \vec{F} \cdot d\vec{S}$, where $\vec{F} =< xyz, x, e^{xy}\cos z >$, S is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$ oriented upward.

3) Use Stoke's theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} =< 2z, 4x, 5y >$, C is the intersection of $z = x + 4$ with the cylinder $x^2 + y^2 = 4$.

4) Use Stoke's theorem to evaluate $\int\int_S \operatorname{curl} \vec{F} \cdot d\vec{S}$, where $\vec{F} =< y^2 z, xz, x^2 y^2 >$, C is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 1$ oriented upward.

5) Use Stokes' Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} =< xz, 2xy, 3xy >$, C is the boundary of the part of the plane $3x + y + z = 3$ in the first octant oriented counterclockwise as viewed from above.

6) Calculate the work done by the force field $\vec{F} = (x^x + z^2)\vec{i} + (y^y + x^2)\vec{j} + (z^z + y^2)\vec{k}$ when a particle moves under its influence around the edge of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies in the first octant, in a counterclockwise direction as viewed from above.

7) Use the Divergence Theorem to calculate the flux of $\vec{F}$ across S (that is, the surface integral $\int\int_S \vec{F} \cdot d\vec{S}$), where

a) $\vec{F} = 3y^2 z^3 \vec{i} + 9x^2 yz^2 \vec{j} - 4xy^2 \vec{k}$, and S is the surface of the cube with vertices $(\pm 1, \pm 1, \pm 1)$;
b) $\vec{F} = x^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k}$, and S is the sphere $x^2 + y^2 + z^2 = 1$. .

8) Verify that the Divergence Theorem is true for the vector field $\vec{F}(x, y, z) =< 3x, xy, 2xz >$ where the region E is the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$.

9) Use the divergence theorem to evaluate $\int\int_S \vec{F} \cdot d\vec{S}$, where $\vec{F} =< x^2 y, -x^2 z, z^2 y >$, S is the surface of the rectangular box bounded by $x = 0, x = 3$, $y = 0, y = 2$, $z = 0, z = 1$.

10) Use the divergence theorem to evaluate $\int\int_S \vec{F} \cdot d\vec{S}$, where $\vec{F} =< xy, y^2 + e^{xz}, \sin(xy) >$, S is the surface of the region bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.

Exercises. Week 13

1) Find the Fourier series of the periodic extension of $f(t) = \begin{cases} 1, & t \in [0,1), \\ -1, & t \in [1,2). \end{cases}$

2) Given $f(t) = t^2 \quad t \in [-1,1]$, find its Fourier series. Justified by Jordan criterion, substitute $t = 1$ into the found series to prove that $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

3) For (the appropriate periodic extension of) $f(t) = \begin{cases} -t+1, & t \in [0,1) \\ 0, & t \in [1,2) \end{cases}$

find the Fourier series, the sine Fourier series and the cosine Fourier series. For each series determine its sum.

4) For (the appropriate periodic extension of) $f(t) = \begin{cases} t, & t \in [0,1) \\ 1, & t \in [1,2) \end{cases}$ find the Fourier series, the sine Fourier series and the cosine Fourier series. For each series determine its sum.

5) Find the Fourier series of $f(t) = |\sin t|$.

6) Find the Fourier series of the periodic extension of $f(t) = \begin{cases} \sin t, & t \in [0,\pi), \\ 0, & t \in [\pi, 2\pi). \end{cases}$