Dvojný integrál

1. Zaměňte pořadí integrace v následujících dvojnásobných integrálech:

   (a) $\int_0^4 \int_{3x^2}^{12x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$;

   (b) $\int_{-1}^2 \int_0^{e^{-x}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$.

2. Vypočtěte $\int_M f(x,y)\,\mathrm{d}A$, jestliže

   (a) $f(x,y) = x\cos y$ a M je množina ohraničená křivkami $y = 0$, $y = x^2$, $x = 1$;

   (b) $f(x,y) = \frac{y}{x^2+y^2}$ a M je množina ohraničená křivkami $y = x$, $y = 2x$, $x = 1$, $x = 2$;

   (c) $f(x,y) = |y - \sin x|$, $M = [0,\pi] \times [0,1]$.

3. Nalezněte obsah plochy ohraničené křivkami $y = -3x+6$, $y = 0$ a $y = 4x - x^2$, kde $x \in [0,2]$.

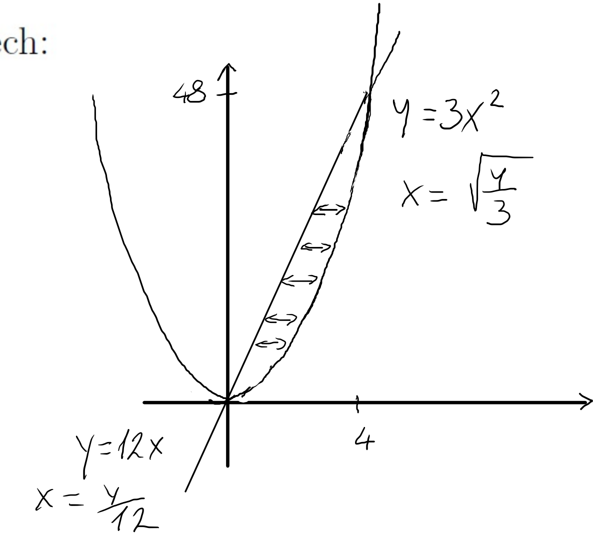
4. Vypočtěte následující integrály tak, že napíšete obě pořadí integrace a jedno z nich dopočtete.

(a) $\displaystyle\iint_D \min\{x,y\}$, $\quad D = \langle 0,a\rangle^2$, $a > 0$; $\quad$ (b) $\displaystyle\iint_D e^{y^2+1}\,dxdy$, kde D je trojúhelník s vrcholy $(0,0)$, $(-2,4)$ a $(8,4)$.

1. Zaměňte pořadí integrace v následujících dvojnásobných integrálech:

(a) $\int_0^4 \int_{3x^2}^{12x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x;$

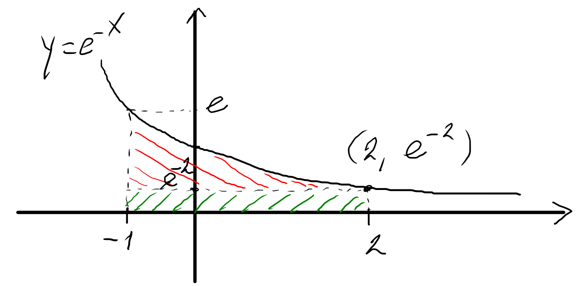
$$= \int_0^{48} \int_{y/12}^{\sqrt{y/3}} f(x,y)\,dx\,dy$$



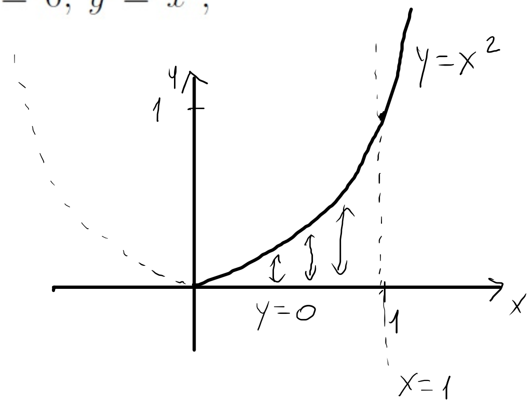
(b) $\int_{-1}^2 \int_0^{e^{-x}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$

$$= \int_0^{e^{-2}} \int_{-1}^{2} f(x,y)\,dx\,dy + \int_{e^{-2}}^{e} \int_{-1}^{-\ln y} f(x,y)\,dx\,dy$$

$$y = e^{-x} \Rightarrow -x = \ln y \Rightarrow x = -\ln y$$



(a) $f(x,y) = x\cos y$ a M je množina ohraničená křivkami $y = 0$, $y = x^2$, $x = 1$;



$$\int_0^1 \int_0^{x^2} x \cos y \, dy \, dx =$$

$$\int_0^1 x \, [\sin y]_{y=0}^{y=x^2} \, dx = \int_0^1 x \sin x^2 \, dx =$$

$$= -\left[\frac{\cos x^2}{2}\right]_0^1 = \frac{-\cos 1 + 1}{2}$$

$$\int x \sin x^2 \, dx = \left| \begin{matrix} \text{subst. } & u = x^2 \\ & du = 2x\,dx \end{matrix} \right| = \int \frac{\sin u}{2} \, du$$

$$= \frac{-\cos u}{2} = -\frac{\cos x^2}{2} \ (+C)$$

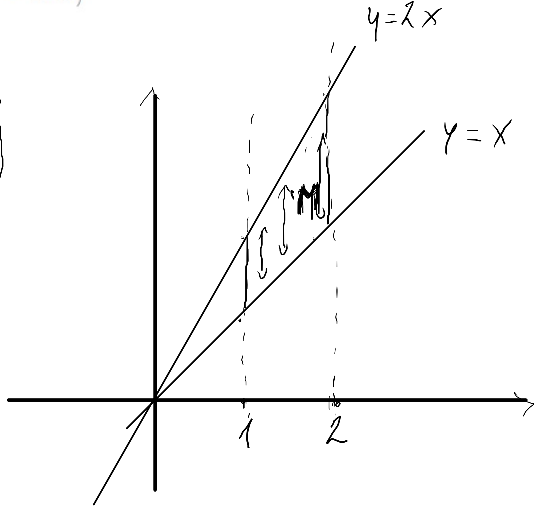
$$\left[\text{nebo } \int_0^1 \int_{\sqrt{y}}^1 x \cos y \, dx \, dy = \int_0^1 \left[\frac{x^2}{2}\right]_{x=\sqrt{y}}^{x=1} \cos y \, dy = \int_0^1 \left(\frac{1}{2} - \frac{y}{2}\right) \cos y \, dy = \int_0^1 \left(\frac{1}{2}\cos y - \frac{y}{2}\cos y\right) dy = \right]$$

$$= \left[\frac{1}{2}\sin y - \frac{1}{2} y \sin y - \frac{1}{2}\cos y\right]_0^1 = \cancel{\frac{1}{2}\sin 1} - \cancel{\frac{1}{2}\sin 1} - \frac{1}{2}\cos 1 + \frac{1}{2}$$

$$\int y \cos y = \text{per partes} \left| \begin{matrix} y & \cos y \\ 1 & \sin y \end{matrix} \right| =$$

$$= y \sin y + \cos y$$

(b) $f(x,y) = \frac{y}{x^2+y^2}$ a M je množina ohraničená křivkami $y = x$, $y = 2x$, $x = 1$, $x = 2$;



$$\int_1^2 \int_x^{2x} \frac{y}{x^2+y^2}\, dy\, dx =$$

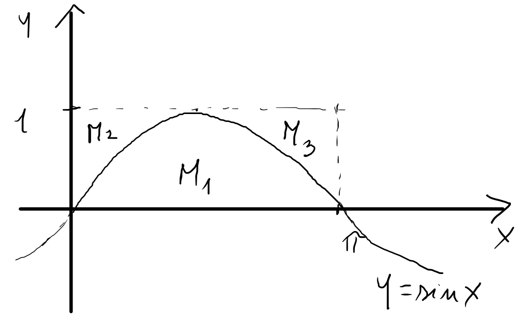
$$\left[\int \frac{y}{x^2+y^2}\, dy = \left| \begin{array}{l} \text{sub. } u = x^2+y^2 \\ du = 2y\, dy \end{array} \right| = \int \frac{1}{2} \frac{1}{u}\, du = \frac{1}{2} \ln(x^2+y^2) \right]$$

$$= \int_1^2 \frac{1}{2} \Big[\ln(x^2+y^2) \Big]_{y=x}^{y=2x} dx =$$

$$= \int_1^2 \frac{1}{2} \left(\ln(5x^2) - \ln(2x^2) \right) dx =$$

$$= \int_1^2 \frac{1}{2} \ln\left(\frac{5x^2}{2x^2}\right) dx = \int_1^2 \frac{1}{2} \ln\frac{5}{2}\, dx = \frac{1}{2} \ln\frac{5}{2} \Big[x\Big]_1^2 = \frac{1}{2} \ln\frac{5}{2}$$

(c) $f(x,y) = |y - \sin x|$, $M = [0,\pi] \times [0,1]$.



$M = M_1 \cup M_2 \cup M_3$

na M_1, $y \le \sin x$, $y - \sin x \le 0$

na $M_2 \cup M_3$, $y \ge \sin x$, $y - \sin x \ge 0$

$$\iint_M |y - \sin x| \, dA = \iint_{M_2 \cup M_3} |y - \sin x| \, dA + \iint_{M_1} |y - \sin x| \, dA =$$

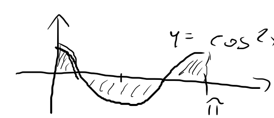
$$= \int_0^\pi \int_{\sin x}^1 (y - \sin x) \, dy \, dx + \int_0^\pi \int_0^{\sin x} (-y + \sin x) \, dy \, dx =$$

$$= \int_0^\pi \left(\frac{y^2}{2} - y \sin x \right)\Big|_{y=\sin x}^{y=1} dx + \int_0^\pi \left[-\frac{y^2}{2} + y \sin x \right]_{y=0}^{y=\sin x} dx =$$

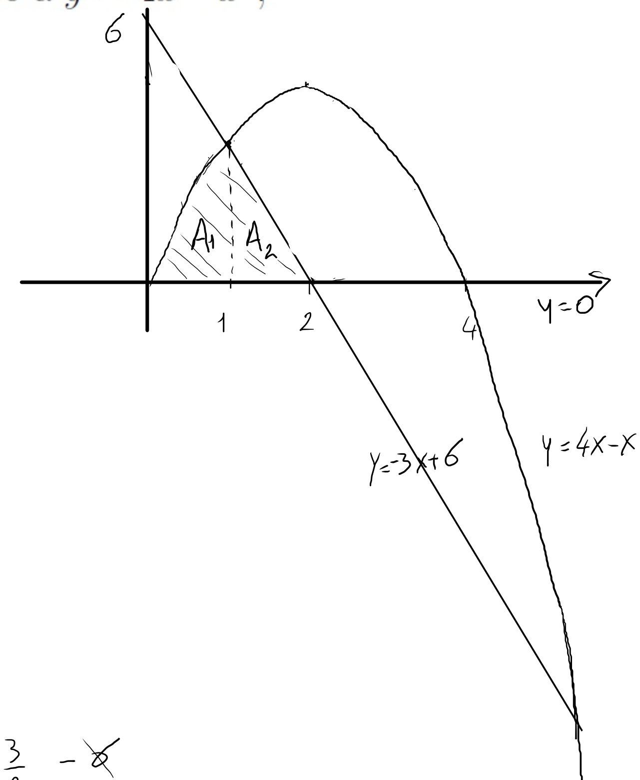
$$= \int_0^\pi \left[\frac{1}{2} - \sin x - \cancel{\frac{\sin^2 x}{2}} + \cancel{\sin^2 x} - \cancel{\frac{\sin^2 x}{2}} + \sin^2 x \right] dx =$$

$$= \int_0^\pi \left(\frac{1}{2} - \sin x + \frac{1}{2} - \frac{\cos 2x}{2} \right) dx = \left[x + \cos x - \frac{\sin 2x}{4} \right]_0^\pi = \pi - 2$$

$$\left[\int_0^\pi \cos 2x \, dx = 0 \right]$$



Nalezněte obsah plochy ohraničené křivkami $y = -3x + 6$, $y = 0$ a $y = 4x - x^2$, kde $x \in [0, 2]$.



$$\begin{cases} y = 4x - x^2 \\ y = -3x + 6 \end{cases} \qquad x^2 - 7x + 6 = 0$$

$$x = \frac{7 \pm \sqrt{49-24}}{2} \begin{matrix} \nearrow 6 \\ \searrow 1 \end{matrix}$$

$$\text{Obsah}(A_1 \cup A_2) = \iint_{A_1 \cup A_2} 1 \, dA =$$

$$= \int_0^1 \int_0^{4x-x^2} dy\, dx + \int_1^2 \int_0^{-3x+6} dy\, dx =$$

$$= \int_0^1 (4x - x^2)\, dx + \int_1^2 (-3x+6)\, dx =$$

$$= \left[2x^2 - \frac{x^3}{3}\right]_0^1 + \left[-\frac{3x^2}{2} + 6x\right]_1^2 = 2 - \frac{1}{3} - \cancel{6} + \cancel{12} + \frac{3}{2} - \cancel{6}$$

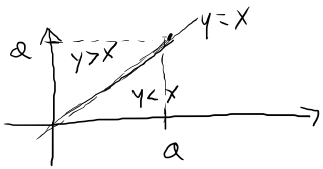
$$= \frac{12 - 2 + 9}{6} = \frac{19}{6}$$

4. Vypočtěte následující integrály tak, že napíšete obě pořadí integrace a jedno z nich dopočtete.

(a) $\iint_D \min\{x, y\}, \quad D = \langle 0, a\rangle^2,\ a > 0;$

$$\int_0^a\int_0^x y\,dy\,dx + \int_0^a\int_x^a x\,dy\,dx = \int_0^a \left[\frac{y^2}{2}\right]_0^x dx + \int_0^a \left[xy\right]_{y=x}^{y=a} dx =$$

$$= \int_0^a \frac{x^2}{2}\,dx + \int_0^a (ax - x^2)\,dx = \int_0^a \left(ax - \frac{x^2}{2}\right) dx = \left[\frac{ax^2}{2} - \frac{x^3}{6}\right]_0^a = \frac{a^3}{6}$$

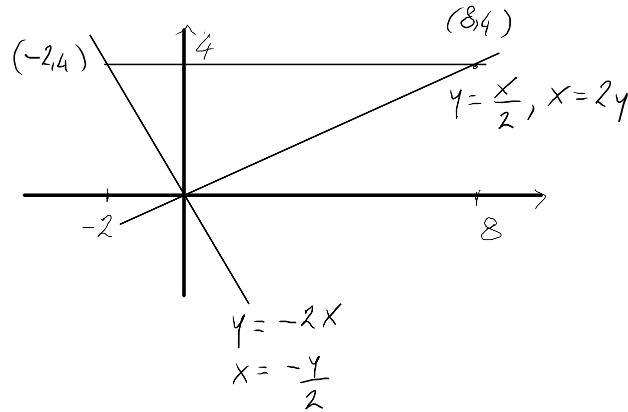


nebo $\int_0^a\int_0^y x\,dx\,dy + \int_0^a\int_y^a y\,dx\,dy$

(b) $\iint_D e^{y^2+1}\,dxdy$, kde D je trojúhelník s vrcholy $(0, 0)$, $(-2, 4)$ a $(8, 4)$

$$\int_0^4\int_{-y/2}^{2y} e^{y^2+1}\,dx\,dy = \int_0^4 \left[x\right]_{-y/2}^{2y} e^{y^2+1}\,dy = \int_0^4 \left(2y + \frac{y}{2}\right) e^{y^2+1}\,dy =$$

$$= \int_0^4 \frac{5}{4}(2y)\, e^{y^2+1}\,dy = \frac{5}{4}\left[e^{y^2+1}\right]_0^4 = \frac{5}{4}\left(e^{17} - 1\right)$$



nebo $\int_{-2}^0\int_{-2x}^4 e^{y^2+1}\,dy\,dx + \int_0^8\int_{x/2}^4 e^{y^2+1}\,dy\,dx = ?$

$\int e^{y^2}\,dy = ?$