Substituce v dvojném integrálu

Zadání

1. Vhodnou substitucí vypočtěte $\int_M f(x,y)\,\mathrm{d}A$, jestliže

   (a) $f(x,y) = x - y$ a M je rovnoběžník s vrcholy $(1,2)$, $(4,3)$, $(3,4)$ a $(6,5)$;

   (b) $f(x,y) = \frac{y^2}{x^2+y^2}$ a $M = \{(x,y) \in \mathbb{R}^2 \,|\, 1 \leq x^2 + y^2 \leq 3\}$;

   (c) $f(x,y) = \sqrt{1 - \frac{x^2}{9} - y^2}$ a $M = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \frac{x^2}{9} + y^2 \leq 1\right\}$;

   (d) $f(x,y) = (x-y)e^{x^2-y^2}$ a M je množina ohraničená křivkami $x + y = 1$, $x + y = 3$, $x^2 - y^2 = -1$ a $x^2 - y^2 = 1$.

2. Využitím polárních souřadnic nalezněte obsah plochy ohraničené křivkami $x^2 + y^2 = 2x, x^2 + y^2 = 4x, y = x, y = 0$.

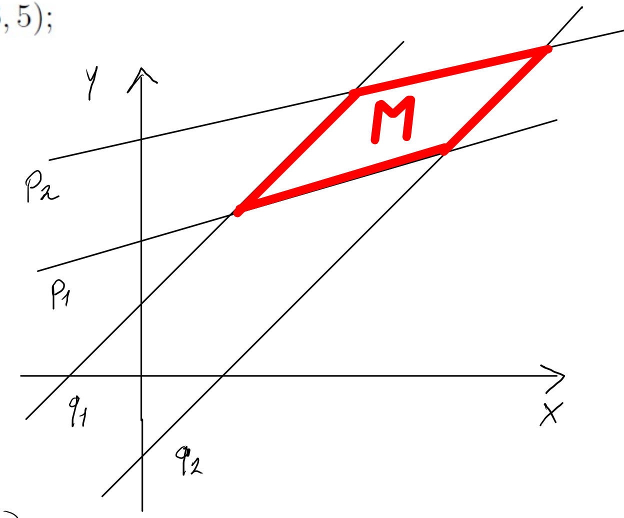
3. Přepište následující integrál

$$\int_0^1 \int_{-\sqrt{2x-x^2}}^{x} f\,dy\,dx$$

   nejprve v opačném pořadí integrace a pak v polárních souřadnicích se středem v počátku v pořadí $d\varrho\,d\varphi$.

1. Vhodnou substitucí vypočtěte $\int_M f(x,y)\,\mathrm{d}A$, jestliže

(a) $f(x,y) = x - y$ a M je rovnoběžník s vrcholy $(1,2)$, $(4,3)$, $(3,4)$ a $(6,5)$;



$p_1: 3y - x = 5$ $\qquad q_1: y - x = 1$

$p_2: 3y - x = 9$ $\qquad q_2: y - x = -1$

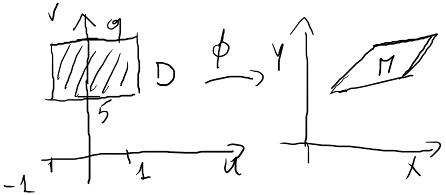
$M = \{(x,y) : -1 \le y - x \le 1,\ 5 \le 3y - x \le 9\}$

svolíme $\quad u = y - x$, $\quad v = 3y - x$ $\qquad \vec{\phi}^{-1}(x,y) = \langle y-x, 3y-x \rangle$

$|J\vec{\phi}^{-1}| = \left|\det\begin{pmatrix} -1 & 1 \\ -1 & 3 \end{pmatrix}\right| = 2 \Rightarrow |J\vec{\phi}| = \frac{1}{2}$

nebo $\vec{\phi}(u,v) = \left\langle \frac{v-u}{2}, \frac{v-3u}{2} \right\rangle$

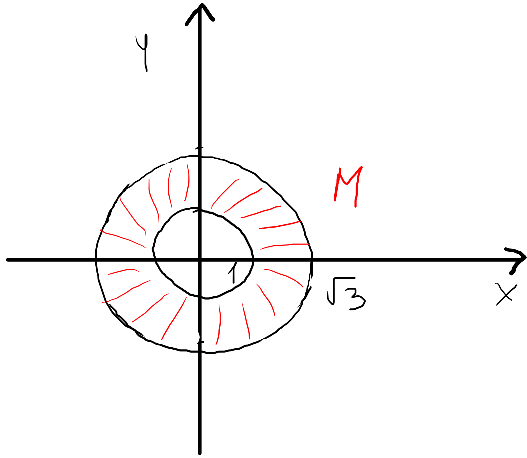
$|J\vec{\phi}| = \left|\det\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}\right| = \frac{1}{2}$



$$\iint_M f(x,y)\,dA = \iint_D f(\vec{\phi}(u,v)) \cdot |J\vec{\phi}|\,dA = \int_{-1}^{1}\int_5^9 -u \cdot \frac{1}{2}\,dv\,du =$$

$$= -\frac{1}{2}\int_{-1}^{1} u\,du \cdot \int_5^9 1\,dv = -\frac{1}{2}(0) \cdot 4 = 0$$

(b) $f(x,y)=\frac{y^2}{x^2+y^2}$ a $M=\{(x,y)\in\mathbb{R}^2 \,|\, 1\le x^2+y^2\le 3\}$;



polární $x=\rho\cos\varphi$ Jakobián $=\rho$ $x^2+y^2=\rho^2$

$y=\rho\sin\varphi$

$M=\{(\rho,\varphi):\ 1\le\rho\le\sqrt{3},\ 0\le\varphi\le 2\pi\}$

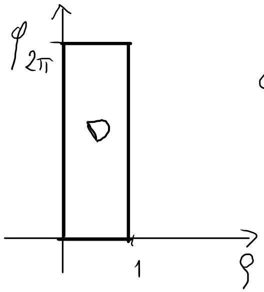
$$\iint_M f(x,y)\,dA \underset{\text{polární}}{=} \int_0^{2\pi}\int_1^{\sqrt{3}} \frac{\rho^2\sin^2\varphi}{\rho^2}\,\rho\,d\rho\,d\varphi =$$

$$= \int_0^{2\pi}\left(\frac{1}{2}-\frac{\cos 2\varphi}{2}\right)d\varphi\cdot\int_1^{\sqrt{3}}\rho\,d\rho =$$

$$= 2\pi\cdot\frac{1}{2}\cdot\left[\frac{\rho^2}{2}\right]_1^{\sqrt{3}} = \pi$$

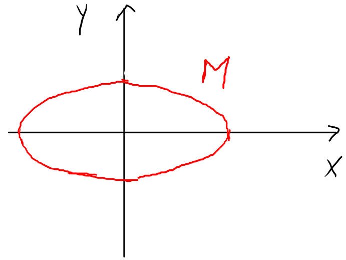
$$\left[\int_0^{2\pi}\cos 2\varphi\,d\varphi=0,\quad \sin^2\varphi=\frac{1}{2}-\frac{\cos 2\varphi}{2},\ \left(\cos^2\varphi=\frac{1}{2}+\frac{\cos 2\varphi}{2}\right)\right]$$

(c) $f(x,y)=\sqrt{1-\frac{x^2}{9}-y^2}$ a $M=\left\{(x,y)\in\mathbb{R}^2 \,\middle|\, \frac{x^2}{9}+y^2\le 1\right\}$;



$\vec{\Phi}(\rho,\varphi)=\langle 3\rho\cos\varphi, \rho\sin\varphi\rangle$

$\vec{\Phi}(D)=M$



$x=3\rho\cos\varphi$

$y=\rho\sin\varphi$

$\frac{x^2}{9}+y^2=1 \Rightarrow \rho^2=1$

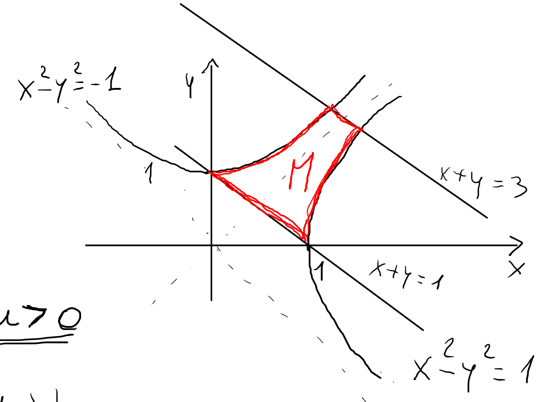
$$|J_{\vec{\Phi}}|=\left|\det\begin{pmatrix}3\cos\varphi & -3\rho\sin\varphi\\ \sin\varphi & \rho\cos\varphi\end{pmatrix}\right|=\rho$$

$$\iint_M f(x,y)\,dA=\int_0^{2\pi}\int_0^1\sqrt{1-\rho^2}\,3\rho\,d\rho\,d\varphi=\int_0^{2\pi}1\,d\varphi\cdot\int_0^1\frac{3}{2}\sqrt{1-\rho^2}\,(2\rho)\,d\rho$$

$$=2\pi\cdot\frac{3}{2}\left[\frac{[1-\rho^2]^{3/2}}{-3/2}\right]_0^1=2\pi$$

$$\left[\begin{array}{l}\text{subst.}\\ s=1-\rho^2\\ ds=-2\rho\,d\rho\\ \int -s^{1/2}\,ds=-\dfrac{s^{3/2}}{3/2}\end{array}\right]$$

(d) $f(x,y) = (x-y)e^{x^2-y^2}$ a M je množina ohraničená křivkami $x+y=1$, $x+y=3$, $x^2-y^2=-1$ a $x^2-y^2=1$.



$$M = \{(x,y) : -1 \le x^2-y^2 \le 1,\ 1 \le x+y \le 3\}$$

Svolíme $\quad u = x+y \quad\quad D = \{(u,v) : -1 \le v \le 1, 1 \le u \le 3\}$

$\quad\quad\quad\quad v = x^2-y^2 \quad\quad \vec{\phi}(D) = M \quad\quad \underline{\underline{u>0}}$

$$\vec{\phi}^{-1}(x,y) = \{x+y,\ x^2-y^2\}$$

$$\left|J_{\vec{\phi}^{-1}}\right| = \left|\det\begin{pmatrix}1 & 1\\ 2x & -2y\end{pmatrix}\right| = 2(x+y) \quad (!\ \forall (x,y)\in M \quad x+y>0\}$$

$$\left|J_{\vec{\phi}}\right| = \frac{1}{|J_{\phi^{-1}}|} = \frac{1}{2u} \quad (!\ \forall (u,v)\in D \quad u>0\}$$

$$\iint_M (x-y)\,e^{x^2-y^2}\,dA = \iint_D f(\vec{\phi}(u,v))\cdot\frac{1}{2u}\,du\,dv = \int_{-1}^{1}\int_{1}^{3}\frac{v}{u}\,e^{v}\cdot\frac{1}{2u}\,du\,dv = \int_{-1}^{1} ve^{v}\,dv\cdot\int_{1}^{3}\frac{1}{2u^2}\,du =$$

$$= \left[ve^{v}-e^{v}\right]_{-1}^{1}\cdot\left[-\frac{1}{2u}\right]_{1}^{3} = \frac{2}{3e}$$

$$\left[\ \begin{cases} x+y=u \\ (x+y)(x-y)=v \end{cases} \Rightarrow x-y=\frac{v}{u}\ \right]$$

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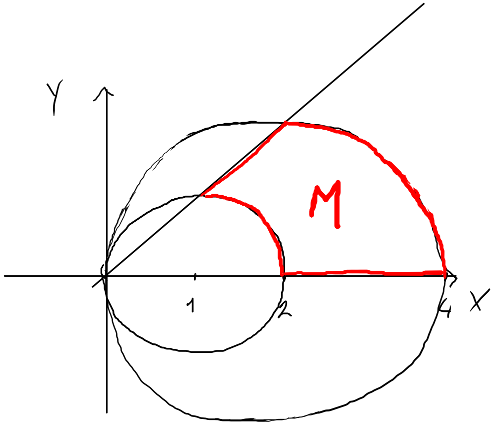

obsah $M = \iint_M 1 \, dA =$

$= \text{polární} \int_0^{\pi/4} \int_{2\cos\varphi}^{4\cos\varphi} \rho \, d\rho \, d\varphi =$

$= \int_0^{\pi/4} \left[\frac{\rho^2}{2}\right]_{2\cos\varphi}^{4\cos\varphi} d\varphi =$

$= \int_0^{\pi/4} (8\cos^2\varphi - 2\cos^2\varphi) \, d\varphi =$

$= \int_0^{\pi/4} 6\left(\frac{1}{2} + \frac{\cos 2\varphi}{2}\right) d\varphi = \left[3\varphi + 3\,\frac{\sin 2\varphi}{2}\right]_0^{\pi/4} =$

$= \frac{3}{4}\pi + \frac{3}{2}$

$x^2 + y^2 = 2x \xrightarrow{\text{polární}} \rho^2 = 2\rho\cos\varphi, \quad \rho = 2\cos\varphi$

$x^2 + y^2 = 4x \xrightarrow{\text{polární}} \rho^2 = 4\rho\cos\varphi, \quad \rho = 4\cos\varphi$

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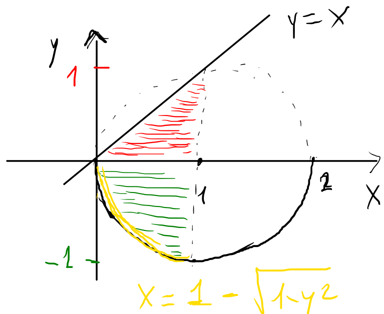
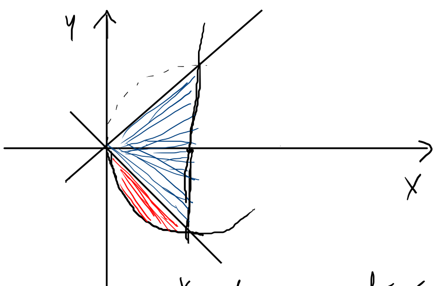
$y = -\sqrt{2x-x^2} \Rightarrow \begin{cases} y \le 0 \\ x^2 - 2x + y^2 = 0 \end{cases}$

$\begin{cases} (x-1)^2 + y^2 = 1 \\ y \le 0 \end{cases}$

$x - 1 = \pm\sqrt{1-y^2}$



$$\underline{\int_{-1}^{0} \int_{1-\sqrt{1-y^2}}^{1} f(x,y) \, dx \, dy} + \underline{\int_{0}^{1} \int_{y}^{1} f(x,y) \, dx \, dy}$$



$$\underline{\int_{-\pi/2}^{-\pi/4} \int_{0}^{2\cos\varphi} f(\rho\cos\varphi, \rho\sin\varphi) \, \rho \, d\rho \, d\varphi} + \underline{\int_{-\pi/4}^{\pi/4} \int_{0}^{\frac{1}{\cos\varphi}} f(\rho\cos\varphi, \rho\sin\varphi) \, \rho \, d\rho \, d\varphi}$$

$x = 1 \rightsquigarrow$ polární $\rho\cos\varphi = 1$, $\rho = \frac{1}{\cos\varphi}$; $x^2 + y^2 - 2x = 0 \underset{\text{polární}}{\rightsquigarrow}$ $\rho^2 = 2\rho\cos\varphi$, $\rho = 2\cos\varphi$