Plošný integrál

Zadání

1. Vypočtěte obsah plochy M, která je částí paraboloidu $z = x^2 + y^2$ pod rovinou $z = 2$ (tj. $M = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2, z \leq 2\}$).

2. Vypočtěte plošný integrál z funkce f přes plochu M, jestliže

   (a) $f(x, y, z) = x$ a M je trojúhelník s vrcholy $(1, 0, 0)$, $(0, -2, 0)$, $(0, 0, 4)$;

   (b) $f(x, y, z) = x^2z + y^2z$ a M je polosféra $x^2 + y^2 + z^2 = 4$, $z \geq 0$.

3. Vypočtěte plošný integrál z vektorového pole F přes orientovanou plochu M, jestliže

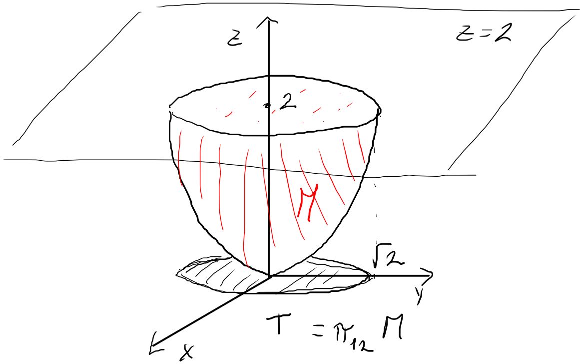
   (a) $F(x, y, z) = (yz, zx, xy)$ a $M = \{(x, y, x \sin y) \in \mathbb{R}^3 \mid x \in [0, 2], y \in [0, \pi]\}$ je orientovaná normálovým polem s kladnou třetí komponentou;

   (b) $F(x, y, z) = (x, y, 5)$ a plocha M je hranice množiny

   $$\left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + z^2 \leq 1, y \geq 0, x + y \leq 2\right\}$$

   orientovaná vnějším normálovým polem.

1. Vypočtěte obsah plochy M, která je částí paraboloidu $z = x^2 + y^2$ pod rovinou $z = 2$ (tj. $M = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2, z \leq 2\}$).

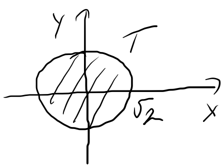


pro elementární plochu $M = M(g, T)$ danou C^1 funkcí g na základní oblasti T je

$$\text{obsah}(M) = S(g, T) = \iint_T \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}. \tag{8.4}$$

$g(x, y) = x^2 + y^2$

$$\text{obsah}(M) = \iint_T \sqrt{1 + (2x)^2 + (2y)^2}\, dA = \text{polár.}$$



$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{1 + 4\rho^2}\, \rho \, d\rho \, d\varphi =$$

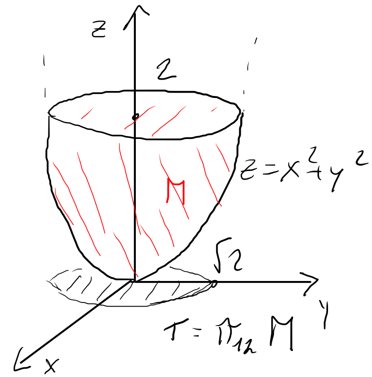
$$= 2\pi \cdot \frac{1}{8} \left[(1 + 4\rho^2)^{3/2} \cdot \frac{2}{3} \right]_0^{\sqrt{2}} = \frac{\pi}{6} (27 - 1) = \frac{13}{3} \pi$$

<u>nebo</u> ....

nebo :

Věta 8.10. *(Obecná parametrizace) Nechť $M \subset \mathbb{R}^3$ je plocha se spojitou parametrizací $\Phi\colon T \longrightarrow M$, která je třídy C^1 a prostá na vnitřku základní oblasti $T \subset \mathbb{R}^2$. Pak*

$$\text{obsah}(M) = \iint_T \left\| \frac{\partial \Phi}{\partial s} \times \frac{\partial \Phi}{\partial t} \right\|.$$



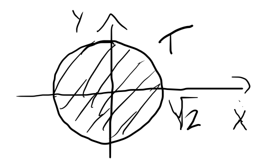
$M: \ \Phi(x,y) = (x, y, x^2+y^2), \quad (x,y) \in T$

$\Phi_x = (1, 0, 2x)$

$\Phi_y = (0, 1, 2y)$

$\Phi_x \times \Phi_y = (-2x, -2y, 1)$

$\|\Phi_x \times \Phi_x\| = \sqrt{4x^2+4y^2+1}$



$\text{obsah}(M) = \iint_T \sqrt{4(x^2+y^2)+1}\, dA = \text{polár.} = \cdots \dfrac{13}{3}\pi$

nebo $\quad M: \ \Phi(\varrho,\theta) = (\varrho\cos\theta, \varrho\sin\theta, \varrho^2), \quad 0 \le \varrho \le \sqrt{2}, \ 0 \le \theta \le 2\pi$

$\Phi_\varrho = (\cos\theta, \sin\theta, 2\varrho)$

$\Phi_\theta = (-\varrho\sin\theta, \varrho\cos\theta, 0)$

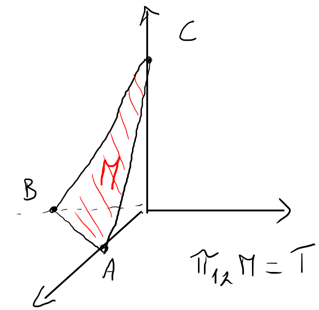
$\Phi_\varrho \times \Phi_\theta = (-2\varrho^2\cos\theta, -2\varrho^2\sin\theta, \varrho)$

$\|\Phi_\varrho \times \Phi_\theta\| = \sqrt{4\varrho^4+\varrho^2} = \varrho\sqrt{4\varrho^2+1}$

$\text{obsah}(M) = \displaystyle\int_0^{2\pi}\int_0^{\sqrt{2}} \varrho\sqrt{4\varrho^2+1}\, d\varrho\, d\theta = \cdots \frac{13}{3}\pi.$

2. Vypočtěte plošný integrál z funkce f přes plochu M, jestliže

(a) $f(x, y, z) = x$ a M je trojúhelník s vrcholy $A(1, 0, 0)$, $B(0, -2, 0)$, $C(0, 0, 4)$;

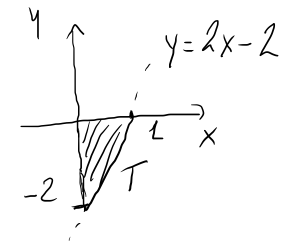


a) rovina ABC : $4x - 2y + z = 4$

M : $\phi(x, y) = (x, y, 4 - 4x + 2y)$, $(x, y) \in T$

$\phi_x = (1, 0, -4)$ $\quad (\phi_x \times \phi_y) = (4, -2, 1)$

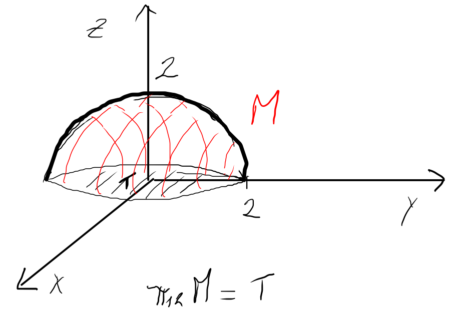
$\phi_y = (0, 1, 2)$ $\quad \|\phi_x \times \phi_y\| = \sqrt{16+4+1} = \sqrt{21}$



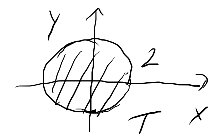
$$\iint_M x \, dS = \iint_T x\sqrt{21} = \sqrt{21}\int_0^1 \int_{2x-2}^0 x \, dy \, dx = \sqrt{21}\int_0^1 x(-2x+2) \, dx =$$

$$= \sqrt{21}\left[-\frac{2x^3}{3} + x^2\right]_0^1 = \frac{\sqrt{21}}{3}$$

(b) $f(x,y,z) = x^2z + y^2z$ a M je polosféra $x^2+y^2+z^2=4$, $z \geq 0$.



$\pi_{xy} M = T$



$M: \quad \Phi(x,y) = \langle x, y, \sqrt{4-x^2-y^2} \rangle, \quad (x,y) \in T$

$\phi_x = \langle 1, 0, \dfrac{-x}{\sqrt{4-x^2-y^2}} \rangle$

$\phi_y = \langle 0, 1, \dfrac{-y}{\sqrt{4-x^2-y^2}} \rangle$

$\phi_x \times \phi_y = \left(\dfrac{x}{\sqrt{4-x^2-y^2}}, \dfrac{y}{\sqrt{4-x^2-y^2}}, 1 \right)$

$\|\phi_x \times \phi_y\| = \sqrt{\dfrac{x^2+y^2}{4-x^2-y^2} + 1}$

$\iint_M (x^2z + y^2z)\, dS = \iint_T \cancel{\sqrt{4-x^2-y^2}}\,(x^2+y^2) \cdot \dfrac{\sqrt{4}}{\cancel{\sqrt{4-x^2-y^2}}}\, dA =$ pol. sour.

$= \int_0^{2\pi} \int_0^2 \rho^2 \cdot 2\rho \, d\rho \, d\varphi = 2\pi \cdot 2 \cdot \left[\dfrac{\rho^4}{4} \right]_0^2 = 16\pi$

<u>nebo</u>: . . . .

mebo

(b) $f(x,y,z) = x^2z + y^2z$ a M je polosféra $x^2+y^2+z^2=4$, $z \geq 0$.

$M : \phi(\theta,\varphi) : \langle 2\sin\theta\cos\varphi,\ 2\sin\theta\sin\varphi,\ 2\cos\theta \rangle$

(sfer.)

$0 \leq \theta \leq \pi/2 \qquad 0 \leq \varphi \leq 2\pi$

$\phi_\theta = \langle +2\cos\theta\cos\varphi,\ +2\cos\theta\sin\varphi,\ -2\sin\theta \rangle$

$\phi_\varphi = \langle -2\sin\theta\sin\varphi,\ +2\sin\theta\cos\varphi,\ 0 \rangle$

$\phi_\theta \times \phi_\varphi = \langle 4\sin^2\theta\cos\varphi,\ 4\sin^2\theta\sin\varphi,\ \underbrace{4\sin\theta\cos\theta\cos^2\varphi + 4\sin\theta\cos\theta\sin^2\varphi}_{4\sin\theta\cos\theta} \rangle$

$\|\phi_\theta \times \phi_\varphi\| = \sqrt{16\sin^4\theta(\cos^2\varphi+\sin^2\varphi) + 16\sin^2\theta\cos^2\theta} = 4\sin\theta$

$$\iint_M z(x^2+y^2)\,dS = \int_0^{2\pi}\int_0^{\pi/2} 2\cos\theta\left(4\sin^2\theta\right)\cdot 4\sin\theta\,d\theta\,d\varphi =$$

$$= 2\pi \cdot 32\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2} = 16\pi$$

3. Vypočtěte plošný integrál z vektorového pole F přes orientovanou plochu M, jestliže

(a) $F(x,y,z) = (yz, zx, xy)$ a $M = \{(x, y, x\sin y) \in \mathbb{R}^3 \mid x \in [0,2], y \in [0,\pi]\}$ je orientovaná normálovým polem s kladnou třetí komponentou;

$M: \quad \phi(x,y) = (x, y, x\sin y) \qquad x \in [0,2], \quad y \in [0,\pi]$

$\phi_x = (1, 0, \sin y)$

$\phi_y = (0, 1, x\cos y) \qquad \phi_x \times \phi_y = (-\sin y, -x\cos y, 1) \qquad 1 > 0 \quad$ ANO!

$$\iint_{(M)} \vec{F}\,\mathrm{d}\vec{S} = \left(\iint_D \vec{F}(\Phi)\cdot\left(\frac{\partial\Phi}{\partial s}\times\frac{\partial\Phi}{\partial t}\right)\right) = \int_0^2\int_0^\pi (yx\sin y,\ x^2\sin y,\ xy)\cdot(-\sin y, -x\cos y, 1)\,dy\,dx =$$

$$= \int_0^2\int_0^\pi \left(-xy\sin^2 y - x^3\sin y\cos y + xy\right)dy\,dx =$$

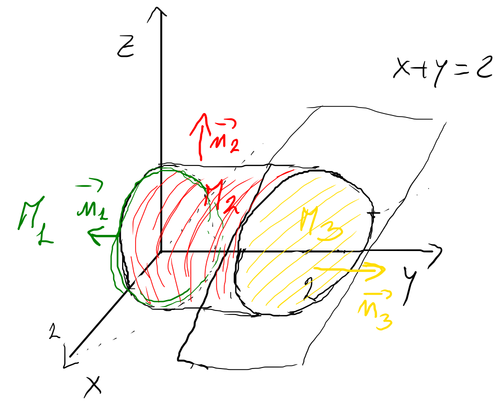
$$= \int_0^2 -x\,dx\cdot\int_0^\pi y\sin^2 y\,dy - \int_0^2 x^3\,dx\cdot\int_0^\pi \sin y\cos y + \int_0^2 x\,dx\cdot\int_0^\pi y\,dy =$$

$$= \left[-\frac{x^2}{2}\right]_0^2\cdot\underbrace{\int_0^\pi y\left(\frac{1}{2} - \cancelto{0}{\frac{\cos 2y}{2}}\right)dy}_{\left[\frac{y^2}{4}\right]_0^\pi} - \left[\frac{x^4}{4}\right]_0^2\cancelto{0}{\left[\frac{\sin^2 y}{2}\right]_0^\pi} + \left[\frac{x^2}{2}\right]_0^2\left[\frac{y^2}{2}\right]_0^\pi = -\frac{2\pi^2}{4} + \frac{2\pi^2}{2} = \frac{\pi^2}{2}$$

(b) $F(x,y,z) = (x,y,5)$ a plocha M je hranice množiny

$$\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + z^2 \le 1, y \ge 0, x + y \le 2\}$$

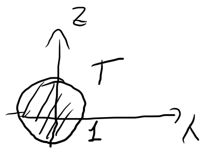
orientovaná vnějším normálovým polem.



$M = M_1 \cup M_2 \cup M_3$

$M_1 = \{(x,y,z) : x^2+z^2 \le 1, y=0\}$

$\phi_1(x,z) = (x,0,z) \quad (x,z) \in T$

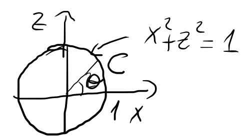


$\phi_{1x} = (1,0,0) \quad \phi_{1z} = (0,0,1)$

$\vec{m}_1 = (0,-1,0)$

$$\int_{(M_1)} (x,y,5)\, dS = \iint_T (x,y,5)\cdot(0,-1,0)\, dA = \text{pol.} \int_0^{2\pi}\int_0^1 -\rho \sin\varphi\, \rho\, d\rho\, d\varphi = 0$$

$M_2 = \{(x,y,z) \quad x^2+z^2=1 \quad 0 \le y \le 2-x\}$ $\qquad \pi_{13}(M_2)$:



$\phi_2(\theta, y) = (\cos\theta, \ y, \sin\theta) \qquad \theta \in [0,2\pi] \quad 0 \le y \le 2-\cos\theta$

$$\phi_2(\sigma,y) = (\cos\sigma,\, y,\, \sin\sigma) \qquad \sigma\in[0,2\pi] \quad 0\le y\le 2-\cos\sigma$$

$$\phi_{2\sigma} = (-\sin\sigma,\, 0,\, \cos\sigma)$$

$$\phi_{2y} = (0,\, 1,\, 0)$$

$$\phi_{2\sigma}\times\phi_{2y} = (-\cos\sigma,\, 0,\, -\sin\sigma)$$ směřující „dovnitř“

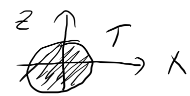
$$\vec{n}_2 = (\cos\sigma,\, 0,\, \sin\sigma) = -(\phi_{2\sigma}\times\phi_{2y})$$

$$\int_{(M_2)} (x,y,5)\,dS = \int_0^{2\pi}\int_0^{2-\cos\sigma} (\cos\sigma,\, y,\, 5)\cdot(\cos\sigma,\, 0,\, \sin\sigma)\,dy\,d\sigma =$$

$$= \int_0^{2\pi}\int_0^{2-\cos\sigma} (\cos^2\sigma + 5\sin\sigma)\,dy\,d\sigma = \int_0^{2\pi} (\cos^2\sigma + 5\sin\sigma)(2-\cos\sigma)\,d\sigma$$

$$= \int_0^{2\pi}\Big(1 + \underbrace{\cos 2\sigma}_{0} - \underbrace{\cos^3\sigma}_{0} + \underbrace{10\sin\sigma}_{0} - \underbrace{5\sin\sigma\cos\sigma}_{0}\Big)\,d\sigma = 2\pi$$

M_3: $\phi_3(x,z) = (x,\, 2-x,\, z)$ $\quad (x,z)\in T = \pi_{13} M_3$:



$$\phi_{3x} = (1,-1,0) \qquad \phi_{3z} = (0,0,1) \qquad \phi_{3x}\times\phi_{3z} = (-1,\,-1,\,0)$$ opačnou orientaci $\vec{n}_3 = (1,1,0)$

$$\int_{(M_3)} (x,y,5)\,dS = \iint_T (x,\, 2-x,\, 5)\cdot(1,1,0)\,dA = \iint_T (\cancel{x} + 2 - \cancel{x})\,dA = 2(\text{obsah } T) = 2\pi$$

$$\int_M (x,y,5)\,dS = 0 + 2\pi + 2\pi = 4\pi$$