Czech Technical University in Prague

Calculus 2

A collection of solved exercises

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Contents

1 Sets in $\mathbb{R}^n$ and their properties

Exercise 1.1. *Construct examples of a nonempty set M in $\mathbb{R}^2$, that*

(i) has no interior point,

(ii) has no boundary point,

(iii) has no exterior point,

(iv) has no accumulation point,

(v) has no isolated point.

Solution:
(i) Any countable set (for example $\mathbb{Q}^2$), a circle, a line (and many of other examples).
(ii) From the condition $\partial M = \emptyset$, it follows that $M^\circ \subseteq M \subseteq \overline{M} = \partial M \cup M^\circ = M^\circ$, thus $M^\circ = M = \overline{M}$ and the set M is both open and closed. The only such nonempty set is $M = \mathbb{R}^2$.
(iii) The exterior set is equal to $\mathbb{R}^2 \setminus \overline{M}$, thus, we need to have $\overline{M} = \mathbb{R}^2$ (such set is called *dense*). We may again choose for example $M = \mathbb{Q}^2$.
(iv) any finite set; $\mathbb{N}^2$ (and many of other examples).
(v) any open set (and many of other examples).

Exercise 1.2. *Determine the interior, boundary and closure of the following sets:*

(i) $M = \{(a,b) \in \mathbb{R}^2 \mid a^2 + 2a + b^2 \leq 3, a^2 - 4a + b^2 \leq 0\}$;

(ii) $M = \mathbb{Q}^3 \subseteq \mathbb{R}^3$, *where $\mathbb{Q}$ is the set of all rational numbers.*

Solution:
(i) The given sets may be adjusted and transform on a clearer shape. Completing the squares, we rewrite the first inequality: $(a+1)^2 + b^2 \leq 4$ or $\|(a,b) - (-1,0)\|^2 \leq 4$. Similarly, the second inequality means $\|(a,b) - (2,0)\|^2 \leq 4$. The set M can be seen as

$$M = A \cap B \quad \text{, where} \quad A = \overline{U}_2(-1,0) \quad \text{a} \quad B = \overline{U}_2(2,0),$$

that is, as the intersection of two closed balls. For $\varepsilon > 0$ and $x_0 \in \mathbb{R}^2$ we use the following notation

$$\overline{U}_\varepsilon(x_0) := \{x \in \mathbb{R}^2 \mid \|x - x_0\| \leq \varepsilon\}$$

Closure of M: now we know that both sets A and B are the closure of certain sets, thus are close. The set M is their intersection, thus is also closed, therefor is the closure of itself (i.e. $\overline{M} = M$).

Interior of M: We use the relation $M^\circ = (A \cap B)^\circ = A^\circ \cap B^\circ$. Because $A^\circ = \left(\overline{U}_2(-1,0)\right)^\circ = U_2(-1,0)$ (it is easy to prove). Similarly it holds for B , thus we get

$$M^\circ = \{(a,b) \in \mathbb{R}^2 \mid a^2 + 2a + b^2 < 3, a^2 - 4a + b^2 < 0\}.$$

Boundary of M:

$$\partial M = \overline{M} \setminus M^\circ =$$
$$\{(a,b) \in \mathbb{R}^2 \mid (a^2 + 2a + b^2 = 3 \ \& \ a^2 - 4a + b^2 \leq 0) \vee (a^2 + 2a + b^2 \leq 3 \ \& \ a^2 - 4a + b^2 = 0)\}.$$

Important observation: Let $f : D \to \mathbb{R}$ (where $D \subseteq \mathbb{R}^n$) be a continuous function. Then the set

$$f^{-1}(-\infty, 0) = \{a \in D \mid f(a) < 0\}$$

(i.e. the open interval $(-\infty, 0) \subseteq \mathbb{R}$) is an open set within the set D, or $f^{-1}(-\infty, 0) = D \cap U$ for a certain open set U. Similarly

$$f^{-1}(-\infty, 0\rangle = \{a \in D \mid f(a) \leq 0\}$$

(i,e, the close interval $(-\infty, 0) \subseteq \mathbb{R}$ is a closed set within the set D, or $f^{-1}(-\infty, 0\rangle = D \cap F$ for a certain closed set F.

Because the functions $f(a, b) = a^2 + 2a + b^2 - 3$ and $g(a, b) = a^2 - 4a + b^2$ are continuous on $\mathbb{R}^2$, the set

$$M = \{(a, b) \in \mathbb{R}^2 \mid a^2 + 2a + b^2 \leq 3, a^2 - 4a + b^2 \leq 0\}$$

is closed (or $\overline{M} = M$) and the set

$$N = \{(a, b) \in \mathbb{R}^2 \mid a^2 + 2a + b^2 < 3, a^2 - 4a + b^2 < 0\}$$

is open (or $N^\circ = N$).

Attention Usually, with this approach we may not get exactly the interior of M°, but only a subset of it $N \subseteq M^\circ$. Equality should not generally occur, for example, for $A = \{x \in \mathbb{R} \mid -x^2 \leq 0\}$ we have

$$\{x \in \mathbb{R} \mid -x^2 < 0\} = \mathbb{R} \setminus \{0\} \subsetneqq \mathbb{R} = A^\circ.$$

For equality we need to use the implicit function theorem. From that theorem, then follows this statement:

Theorem: Let $U \subseteq \mathbb{R}^n$ be an open set, and $f : U \to \mathbb{R}$ a continuous differentiable function on U. Let us put

$$A = \{x \in U \mid f(x) \leq 0\}.$$

If, for every $x_0 \in A$ such that $f(x_0) = 0$, we have $f'(x_0) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)_{|x_0} \neq 0$, then

$$A^\circ = \{x \in U \mid f(x) < 0\}.$$

In our case, for $f(a, b) = a^2 + 2a + b^2 - 3$, we really have $f'(a, b) = \left(\frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}\right) = \left(2a + 2, 2b\right)$. If by chance we had that $f'(a, b) = 0$, then we have $a = -1$ and $b = 0$ and thus $f(-1, 0) = -4 \neq 0$. The condition of the previous theorem are fulfilled, and thus we really have that $\{(a, b) \in \mathbb{R}^2 \mid a^2+2a+b^2 < 3\}$ is the interior of the set $\{(a, b) \in \mathbb{R}^2 \mid a^2+2a+b^2 \leq 3\}$.

(ii) We realize that in every neighborhood of any $r \in \mathbb{R}$ lies as well a rational number as some irrational number. Furthermore, if we have $|r_i - s_i| < \varepsilon$ for $i = 1, 2, 3$ (where $r_i, s_i \in \mathbb{R}$ and $\varepsilon > 0$) then $\|(r_1, r_2, r_3) - (s_1, s_2, s_3)\| \leq \sqrt{3} \cdot \varepsilon$. In particular, thus in any neighborhood of the point $x \in \mathbb{R}^3$ lies an element of $\mathbb{Q}^3$, and as well an element of$\mathbb{R}^3 \setminus \mathbb{Q}^3$. Thus, we may immediately write

$$\overline{\mathbb{Q}^3} = \mathbb{R}^3, \quad (\mathbb{Q}^3)^\circ = \emptyset \text{ a } \partial\mathbb{Q}^3 = \overline{\mathbb{Q}^3} \setminus (\mathbb{Q}^3)^\circ = \mathbb{R}^3.$$

Exercise 1.3. *Find examples of sets for which the following holds:*

(i) A_n , $n \in \mathbb{N}$ are *open, but* $\bigcap_n A_n$ is not *open,*

(ii) $(A \cup B)^\circ \supsetneqq A^\circ \cup B^\circ$,

(iii) A_n , $n \in \mathbb{N}$ are *closed, but* $\bigcup_n A_n$ is not *closed,*

(vi) $\overline{A \cap B} \subsetneqq \overline{A} \cap \overline{B}$.

Solution:

(i) $A_n = (-\frac{1}{n}, \frac{1}{n}) \subseteq \mathbb{R}$, $n \in \mathbb{N}$. Then, we have $\bigcap_n A_n = \{0\}$.

(iii) $A_n = (-\infty, -\frac{1}{n}\rangle \subseteq \mathbb{R}$, $n \in \mathbb{N}$. Then, we have $\bigcup_n A_n = (-\infty, 0)$.

(ii), (iv) $A = \{(x, y) \in \mathbb{R}^2 \mid x < 0\}$, $B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x\}$. Then, we have

$$(A \cup B)^\circ = \mathbb{R}^2 \supsetneq \mathbb{R}^2 \setminus y - \text{axis} = A^\circ \cup B^\circ$$

and

$$\overline{A \cap B} = \emptyset \subsetneq \text{osa } y = \overline{A} \cap \overline{B}.$$

2 Limits of functions of several variables

Exercise 2.1. *Evaluate the following limits*

(i) $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y}$

(ii) $\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{x^2-y^2}$

(iii) $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+2y^2}$

(iv) $\lim_{(x,y)\to(0,0)} (x^2 + y^2)^{x^2y^2}$

Solution:

(i) The function $f(x, y) = \frac{\sin(x+y)}{x+y}$ has domain

$$D_f = \mathbb{R}^2 \setminus \{(t, -t) \mid t \in \mathbb{R}\}.$$

The point $(0, 0)$ is an accumulation point of the set D_f (thus, it has a meaning to ask for the limit value of f at this point). The limit can be evaluated using the theorem on limits of a composed function $f(x, y) = h(g(x, y))$, where $g(x, y) = x + y$ and $h(z) = \frac{\sin(z)}{z}$.

For a correct use of the theorem on limits of a composition of functions, we still must first verify that one of the following holds:

- either on a neighborhood of the point $(0, 0)$ we have $g(x, y) \neq 0$
- or the function h is continuous at $z = 0$.

The first case, we can ensure restricting the domain for the function g, i.e. we take $D_g := D_f$, and the second continuously redefining the function h at $z = 0$. Now, we have

$$\lim_{(x,y)\to(0,0)} g(x, y) = \lim_{(x,y)\to(0,0)} x + y = 0 \quad \text{(because } g \text{ is a sum of continuous functions)},$$

$$\lim_{z\to 0} h(z) = \lim_{z\to 0} \frac{\sin(z)}{z} = 1$$

so that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x + y)}{x + y} = \lim_{(x,y)\to(0,0)} h(g(x, y)) = 1.$$

(ii) For the function $f(x,y)=\frac{x^2+xy+y^2}{x^2-y^2}$, we evaluate the domain:

$$D_f=\mathbb{R}^2\setminus\{(x,y)\in\mathbb{R}^2 \mid x=\pm y\}.$$

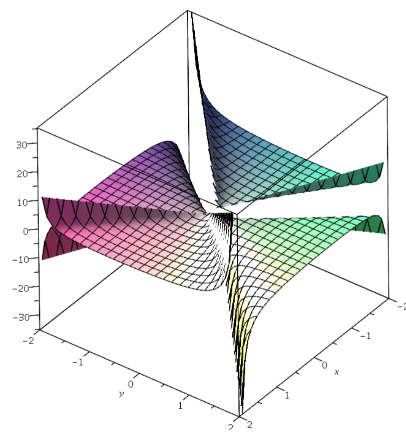
Restricting f on the x-axis (i.e. $y=0$) we get

$$\lim_{x\to 0} f(x,0)=\lim_{x\to 0}\frac{x^2}{x^2}=1.$$

On the other hand, restricting f on the y-axis (i.e. $x=0$), we get

$$\lim_{y\to 0} f(0,y)=\lim_{y\to 0}\frac{y^2}{-y^2}=-1.$$

Thus, the given limit does not exist.



(iii) For the function $f(x,y)=\frac{2xy}{x^2+2y^2}$, we evaluate the domain:

$$D_f=\mathbb{R}^2\setminus\{(0,0)\}.$$

Restricting f on the x-axis (i.e. $y=0$), we get

$$\lim_{x\to 0} f(x,0)=0.$$

On the other hand, restricting f on the line $x=y$, we get

$$\lim_{x\to 0} f(x,x)=\lim_{y\to 0}\frac{2x^2}{x^2+2x^2}=\frac{2}{3}.$$

Thus, the given limit does not exist.

(iv) For the function $f(x,y)=(x^2+y^2)^{x^2y^2}=e^{x^2y^2\ln(x^2+y^2)}$, we evaluate the domain:

$$D_f=\mathbb{R}^2\setminus\{(0,0)\}.$$

It is enough to find out the limit $\lim\limits_{(x,y)\to(0,0)} x^2y^2\ln(x^2+y^2)$. We use the estimate

$$0\le\left|x^2y^2\ln(x^2+y^2)\right|\le(x^4+2x^2y^2+y^4)\cdot\left|\ln(x^2+y^2)\right|=(x^2+y^2)^2\cdot\left|\ln(x^2+y^2)\right|.$$

Using the theorem on limits of composed functions (for example L'Hospital rule), for

$$g(x,y) = x^2 + y^2 \quad \text{a} \quad h(z) = z^2 \ln z$$

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0 \;\text{ and }\; \lim_{z\to 0_+} h(z) = 0$$

we get

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)^2 \ln(x^2+y^2) = \lim_{(x,y)\to(0,0)} h(g(x,y)) = 0.$$

From the squeeze theorem, then we get

$$\lim_{(x,y)\to(0,0)} x^2y^2 \ln(x^2+y^2) = 0,$$

thus

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)^{x^2y^2} = e^0 = 1.$$

Exercise 2.2. *Evaluate the following limits*

(i) $\lim_{(x,y)\to(0,1)} \frac{\sqrt{x^2+(y-1)^2+1}-1}{x^2+(y-1)^2}$

(ii) $\lim_{(x,y,z)\to(1,1,1)} \frac{xz^2-y^2z}{xyz-1}$

(iii) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$

(iv) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^3}$

Solution:

(i) For the function $f(x,y) = \frac{\sqrt{x^2+(y-1)^2+1}-1}{x^2+(y-1)^2}$, we evaluate the domain:

$$D_f = \mathbb{R}^2 \setminus \{(0,1)\}.$$

The point $(0,1)$ is thus an accumulation point of D_f. To get rid of the roots, we use a trick (i.e. the equality $(a-b)(a+b) = a^2 - b^2$)

$$\lim_{(x,y)\to(0,1)} \frac{\sqrt{x^2+(y-1)^2+1}-1}{x^2+(y-1)^2} = \lim_{(x,y)\to(0,1)} \frac{\sqrt{x^2+(y-1)^2+1}-1}{x^2+(y-1)^2} \cdot \frac{\sqrt{x^2+(y-1)^2+1}+1}{\sqrt{x^2+(y-1)^2+1}+1} =$$

$$= \lim_{(x,y)\to(0,1)} \frac{x^2+(y-1)^2}{\left(x^2+(y-1)^2\right)\cdot\left(\sqrt{x^2+(y-1)^2+1}+1\right)} = \frac{1}{2}\ .$$

(ii) For the function $f(x,y,z) = \frac{xz^2-y^2z}{xyz-1}$, we evaluate the domain:

$$D_f = \{(x,y,z) \in \mathbb{R}^3 \mid xyz \neq 1\},$$

thus the point $(1,1,1)$ is an accumulation point of D_f. Considering that the degree of the polynomial in the denominator is the same as the degree of the polynomial in the numerator, we try to prove that the limit does not exist.

Restricting f to the line $x = y = z$ (without the point $(x, y, z) = (1, 1, 1)$), we get $f(x, x, x) = \frac{x^3 - x^3}{x^3 - 1} = 0$, thus

$$\lim_{\substack{(x,y,z)\to(1,1,1)\\ x=y=z}} f(x, y, z) = \lim_{x\to 1} f(x, x, x) = 0 \ .$$

On the other hand, restricting f to the line $x = y = 1$ (again excluding the point $(x, y, z) = (1, 1, 1)$) we get $f(1, 1, z) = \frac{z^2 - z}{z - 1} = z$, thus

$$\lim_{\substack{(x,y,z)\to(1,1,1)\\ x=y=1}} f(x, y, z) = \lim_{z\to 1} f(1, 1, z) = \lim_{z\to 1} z = 1 \ .$$

We conclude that the given limit does not exist.

(iii) For the function $f(x, y) = \frac{x^2 y^2}{x^2 + y^2}$, we evaluate the domain:

$$D_f = \mathbb{R}^2 \setminus \{(0, 0)\},$$

thus the point $(0, 0)$ is an accumulation point of D_f. The degree of the polynomial in the numerator is higher than the degree of the polynomial in the denominator, thus we will try to prove that the limit exists and it is equal to zero (this value we may choose as a candidate restricting the function f for example to the coordinates axis).

We use an estimate and the squeeze theorem. The following inequality holds:

$$|x| \le \sqrt{x^2 + y^2} = \|(x, y)\|,$$

and this important inequality is suitable to prove our limit. Similarly $|y| \le \|(x, y)\|$, thus we get

$$0 \le \left| \frac{x^2 y^2}{x^2 + y^2} \right| \le \frac{\|(x, y)\|^2 \cdot \|(x, y)\|^2}{x^2 + y^2} = \|(x, y)\|^2.$$

¿From the definition of limit, we get that $\lim\limits_{(x,y)\to(0,0)} \|(x, y)\|^2 = 0$ and from the squeeze theorem, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x^2 + y^2} = 0 \ .$$

(iv) The difference between this and the previous example is the degree of y in the denominator, but this implies that now the denominator obtains value zero not only at the point $(0, 0)$. For the function $f(x, y) = \frac{x^2 y^2}{x^2 + y^3}$, we evaluate the domain:

$$D_f = \{(x, y) \in \mathbb{R}^2 \mid y \ne -\sqrt[3]{x^2}\},$$

and the point $(0, 0)$ is an accumulation point.

Restricting f to the line $x = 0$ (without the point $(x, y) = (0, 0)$), we get $f(0, y) = 0$, thus

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x, y) = \lim_{y\to 0} f(0, y) = 0 \ .$$

Therefore, if the limit exists, it must be equal to 0. The polynomial in the numerator obtains zero value on the axis $x = 0$ and $y = 0$, while the polynomial in the denominator is zero on the curve $y = -\sqrt[3]{x^2}$. At any point $(x_0, y_0) \in \mathbb{R}^2$ so that $y_0 = -\sqrt[3]{x_0^2}$ and $x_0 \ne 0$ thus we get

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ 0\ne y_0 = -\sqrt[3]{x_0^2}}} \left| \frac{x^2 y^2}{x^2 + y^3} \right| = \frac{x_0^2 y_0^2}{0} = +\infty.$$

Now, if our function f had at $(0,0)$ limit 0, it had to be bounded in some neighborhood of $(0,0)$, i.e. it should exists $K > 0$ and $\varepsilon > 0$, so that $\left|\frac{x^2y^2}{x^2+y^3}\right| \leq K$ for every $(x,y) \in U_\varepsilon(0,0) \cap D_f$.

In the neighborhood $U_\varepsilon(0,0)$ we also find one points $(x_0, -\sqrt[3]{x_0^2})$, at which the limit of f is unbounded. We conclude that the given limit does not exist.

3 Directional derivatives, gradient, total derivative

Exercise 3.1. *Find the gradient of the function $f(x,y) = e^x \sin y$ at the point $a_0 = (1, \frac{\pi}{4})$ and the rate of change of f at the point a_0 in the direction of the vector $\mathbf{v} = (-1, 2)$.*

Solution:
The gradient $\mathrm{grad} f_{|a_0}$ is the matrix of the derivative $f'_{|a_0}$ of the function f at the point a_0. A sufficient condition for the existence of the derivative function of f at the point a_0 is the existence of continuous partial derivatives in some neighborhood of the point a_0 (that is our case).

$$\mathrm{grad} f_{|a_0} = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})_{|a_0} = (e^x \sin y, e^x \cos y)_{|a_0} = (e\frac{\sqrt{2}}{2}, e\frac{\sqrt{2}}{2})$$

The rate of change $\frac{\partial f}{\partial \mathbf{v}}_{|a_0}$ of the function f at the point a_0 in the direction of the vector $\mathbf{v}$ is given by

$$\frac{\partial f}{\partial \mathbf{v}}_{|a_0} = \mathrm{grad} f_{|a_0} \cdot \mathbf{v} = (e\frac{\sqrt{2}}{2}, e\frac{\sqrt{2}}{2}) \cdot (-1, 2) = e\frac{\sqrt{2}}{2}.$$

Exercise 3.2. *Find the unit vector in the direction of the greatest rate of change of the function $f(x,y,z) = xe^y + z^2$ at the point $a_0 = (1, \ln 2, \frac{1}{2})$.*

Solution:

$$\mathrm{grad} f_{|a_0} = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})_{|a_0} = (e^y, xe^y, 2z)_{|a_0} = (2, 2, 1)$$

The unit vector in the direction of the greatest rate of change of the function f at the point a_0 is

$$\frac{1}{\|\mathrm{grad} f_{|a_0}\|} \cdot \mathrm{grad} f_{|a_0} = \frac{1}{\|(2,2,1)\|} \cdot (2,2,1) = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}).$$

Exercise 3.3. *A very tired mountaineer climbs up a slope, that is the graph of the function $f(x,y) = e^{xy} + \ln x$. At the moment he is standing at the point $A = (1, 1, ?) \in \mathbb{R}^3$. In which of the two directions*

$$\mathbf{U} = (1, 2, ?) \quad \textit{and} \quad \mathbf{V} = (2, 1, ?)$$

(on the tangent plane to the graph of the function f at the point A) should continue climbing, so that he would have the least effort?

Solution:
For $a = (1,1) \in \mathbb{R}^2$, $f(a) = e$, thus $A = (1,1,e)$. We evaluate the gradient (derivative) of the function f:

$$\text{grad } f_{|(1,1)} = f'_{|(1,1)} = (ye^{xy} + \frac{1}{x}, xe^{xy})_{|(1,1)} = (e+1, e)$$

The equation of the tangent plane to the graph of the function f at the point a is

$$z = f(1,1) + f'_{|(1,1)}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} = e + (e+1, e)\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} = (e+1)x + ey - (e+1)$$

The vectors U and V lie on the plane if they are orthogonal to the normal vector to the plane$\mathbf{N} = (e+1, e, -1)$ (i,e, if $\mathbf{U} \cdot \mathbf{N} = 0 = \mathbf{V} \cdot \mathbf{N}$.) Thus, we have $\mathbf{U} = (1,2,3e+1)$ and $\mathbf{V} = (2,1,3e+2)$. The steepness of the climb is determined by an angle which the vectors make with the plane $z = 0$, thus $\arctan\left(\frac{3e+1}{\sqrt{1^2+2^2}}\right)$ for $\mathbf{U}$ and $\arctan\left(\frac{3e+2}{\sqrt{1^2+2^2}}\right)$ for $\mathbf{V}$. The easier ascent is thus in the direction of the vector $\mathbf{U}$.

Notes:

(1) We could use the implicitly given graph $\Phi(x,y,z) = 0$ for $\Phi(x,y,z) := f(x,y) - z$. 56/5000 Then we got straight as normal vector gradient of Φ, $\mathbf{N} = \text{grad } \Phi_{|A}$.

(2) If we set $\mathbf{u} = (1,2)$ and $\mathbf{v} = (2,1)$, then we have $\mathbf{U} = \big(\mathbf{u}, f'_{|a}(\mathbf{u})\big)$ and $\mathbf{V} = \big(\mathbf{v}, f'_{|a}(\mathbf{v})\big)$. Moreover, if $\|\mathbf{u}\| = \|\mathbf{v}\|$ (as in our case), then for the steepness of the climb in the direction of vectors $mathbfU$ and $mathbfV$ we need only to compare the last component, ie. values $f'_{|a}(\mathbf{u})$ and $f'_{|a}(\mathbf{v})$.

Exercise 3.4. *Assume that the height of the terrain in $\mathbb{R}^3$ is described by the graph of $f\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \frac{1}{x^2+2y^2+1}$. At point A given by $x = 2$ and $y = 1$ we drop a ball. Determine the direction (when viewed from above, i.e. in R^2), in which the ball will roll down. Next, determine whether it is more inclined the tangent plane at point A or at point B given by $x = 0$ and $y = 1$ (i.e. compare angles that these planes form with the plane $z = 0$).*

Solution:
The ball will roll in the direction of steepest descent, i.e. opposite to the direction of the gradient of f,

$$\text{grad}(f)_{|A} = \left(-\frac{2x}{(x^2+2y^2+1)^2}, -\frac{4y}{(x^2+2y^2+1)^2}\right)_{|A} = \left(-\frac{4}{7^2}, -\frac{4}{7^2}\right)$$

thus in the direction, for example, of the vector $\vec{v} = (1,1)$ (the same direction is determine by any positive multiple of this vector).

The normal vector to the tangent plane to the graph of the function at the given point is $\left(\frac{\partial\ f}{\partial\ x}, \frac{\partial\ f}{\partial\ y}, -1\right)$. We thus consider the normal vectors

$$\vec{n}_A = \left(-\frac{4}{7^2}, -\frac{4}{7^2}, -1\right)$$

and

$$\vec{n}_B = \left(0, -\frac{4}{3^2}, -1\right)$$

with norm $\|\vec{n}_A\| = \sqrt{\frac{2^5}{7^4} + 1}$ and $\|\vec{n}_B\| = \sqrt{\frac{2^4}{3^4} + 1}$. The normal vector to the plane $z = 0$ is $\vec{e} = (0,0,1)$. The angles between the planes are given by

$$\cos(\alpha) = \frac{|\vec{n}_A \cdot \vec{e}|}{\|\vec{n}_A\| \cdot \|\vec{e}\|} = \frac{1}{\sqrt{\frac{2^5}{7^4} + 1}}$$

and

$$\cos(\beta) = \frac{|\vec{n}_B \cdot \vec{e}|}{\|\vec{n}_B\| \cdot \|\vec{e}\|} = \frac{1}{\sqrt{\frac{2^4}{3^4}+1}}$$

Confronting these quantites, we get

$$\cos(\alpha) \overset{?}{>} \cos(\beta) \;\Leftrightarrow\; \frac{1}{\sqrt{\frac{2^5}{7^4}+1}} \overset{?}{>} \frac{1}{\sqrt{\frac{2^4}{3^4}+1}} \;\Leftrightarrow\; \sqrt{\frac{2^4}{3^4}+1} \overset{?}{>} \sqrt{\frac{2^5}{7^4}+1} \;\Leftrightarrow$$

$$\Leftrightarrow\; \frac{2^4}{3^4} \overset{?}{>} \frac{2^5}{7^4} \;\Leftrightarrow\; 7 \cdot 7^3 \overset{?}{>} (2 \cdot 3) \cdot 3^3 \;.$$

Last relation holds because $\cos(\alpha) > \cos(\beta)$ and thus $\alpha < \beta$ and at point B the plane is more inclined.

Exercise 3.5. *Find the tangent plane to the graph of the function* $f(x, y) = 2x^2 + y^2$ *at the point* $a_0 = (1, 1)$.

Solution:
The graph of the function f is $\{(a, z) \in \mathbb{R}^3 \mid z = f(a)\}$. The tangent plane $T_{(a_0, f_{a_0})}$ to the graph of f at the point $(a_0, f_{a_0}) = (1, 1, 3)$ is given by the equation

$$z = f(a_0) + \mathrm{grad} f_{|a_0} \cdot (a - a_0).$$

We have: $\mathrm{grad} f_{|a_0} = (4x, 2y)_{|a_0} = (4, 2)$, thus the tangent plane has equation

$$z = 3 + (4, 2) \cdot (x - 1, y - 1) = 3 + 4(x - 1) + 2(y - 1)$$

or

$$4x + 2y - z = 3.$$

Exercise 3.6. *Find the tangent plane to the graph of the function* $f(x, y) = xy + \sin(x + y)$ *at the point* $(1, -1, ?)$.

Solution:
The graph of the function f is the set $\Gamma_f = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y) \;\&\; (x, y) \in D_f\}$. The tangent plane $T_{(x_0, y_0, z_0)}$, to the graph of f at the point $(x_0, y_0, z_0) = (1, -1, -1)$, where $z_0 = f(x_0, y_0) = -1$ is given by the equation

$$z = f(x_0, y_0) + \mathrm{grad} f_{|(x_0, y_0)} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}.$$

We have:

$$\mathrm{grad} f_{|(x_0, y_0)} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)_{\Big|(x_0, y_0)} = \Big(y + \cos(x + y), x + \cos(x + y)\Big)_{\Big|(1, -1)} = (2, 0),$$

thus the tangent plane has equation

$$z = -1 + (2, 0) \cdot \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix} = -1 + 2(x - 1)$$

or

$$2x - z = 3.$$

Exercise 3.7. *Find the derivative (rate of change) of the function*

(i) $f(x,y,z) = z^3 - x^2y$ *at the point* $a = (1,6,2)$, *in the direction of the vector* $\vec{v} = (3,4,12)$,

(ii) $f(x,y) = e^x \cos y + 2y$ *at the point* $a = (0,0)$, *in the direction of the vector* $\vec{v} = (-1,2)$.

Solution:
The (directional) derivative of a function f at the point a in the direction of the unit vector $\vec{v}$ is defined like

$$\frac{\partial f}{\partial \vec{v}}_{|a} := \lim_{t\to 0} \frac{f(a + t\cdot\vec{v}) - f(a)}{t} \ .$$

If, on the other hand, the derivative f'_a of the function f exists at the point a (i.e. the total derivative exists), then it holds:

$$\frac{\partial f}{\partial \vec{v}}_{|a} = f'_{|a}(\vec{v}) = \mathrm{grad} f_{|a} \cdot \vec{v} = \frac{\partial f}{\partial x_1}_{|a} \cdot v_1 + \cdots + \frac{\partial f}{\partial x_n}_{|a} \cdot v_n$$

where $\vec{v} = (v_1, \ldots, v_n)$.

(i) For $f(x,y,z) = z^3 - x^2y$ and $a = (1,6,2)$, we get

$$\mathrm{grad} f_{|a} = (2xy, x^2, 3z^2)_{|a} = (12,1,12)$$

and

$$\frac{\partial f}{\partial \vec{v}}_{|a} = (12,1,12) \cdot \begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix} = 184 \ .$$

If we take the directional derivative, in the direction of $\vec{v}$, then it is necessary to normalize the vector, i.e. we use the vector $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$ and then we have $\frac{\partial f}{\partial \vec{u}}_{|a} = \frac{1}{\|\vec{v}\|} \cdot \frac{\partial f}{\partial \vec{v}}_{|a} = \frac{184}{13}$.

(ii) For $f(x,y) = e^x \cos y + 2y$ and $a = (0,0)$, we get

$$\mathrm{grad} f_{|a} = (e^x \cos y, -e^x \sin y + 2)_{|a} = (1,2)$$

and

$$\frac{\partial f}{\partial \vec{v}}_{|a} = (1,2) \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 3 \ .$$

With $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$, then we have $\frac{\partial f}{\partial \vec{u}}_{|a} = \frac{1}{\|\vec{v}\|} \cdot \frac{\partial f}{\partial \vec{v}}_{|a} = \frac{3}{\sqrt{5}}$.

Exercise 3.8. *Find the equation of the tangent plane to the ellipsoid* $\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1$, *that*

(i) is parallel to the plane $4x + 2y + z = 3$,

(ii) cuts on the coordinates axis segments of the same length.

Solution:
When a set M is given as a level surface of a continuous differentiable function (i.e. with an equation $f(x,y,z) = 0$), then the tangent plane to M is orthogonal to the gradient of the function f (if the gradient is non-zero), i.e. the gradient is its normal vector.

In our case, we take $f(x,y,z) = \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} - 1$. So, the normal vector to the tangent plane is

$$f'_{|a_0} = \text{grad}(f)_{|a_0} = \left(\frac{2x}{25}, \frac{y}{8}, \frac{2z}{9}\right).$$

(i) The tangent plane must be parallel to the plane $\rho : 4x + 2y + z = 3$, that has normal vector $\mathbf{n}_\rho = (4, 2, 1)$. This holds exactly when

$$\left(\frac{2x}{25}, \frac{y}{8}, \frac{2z}{9}\right) = \text{grad}(f)_{|a_0} = \lambda \cdot \mathbf{n}_\rho = \lambda \cdot (4, 2, 1)$$

for some $\lambda \in \mathbb{R}$. Thus $x = 50\lambda$, $y = 16\lambda$ and $z = \frac{9}{2}\lambda$.

At the same time, it also holds that

$$\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1 \ .$$

After substitution, we get $100\lambda^2 + 16\lambda^2 + \frac{9}{4}\lambda^2 = 1$, thus $\lambda = \pm 2/\sqrt{473}$.

The tangent plane we are looking for, must have normal vector $\mathbf{n}_\rho$, thus has equation of the form $4x + 2y + z = c$, where the unknown value of $c \in \mathbb{R}$ can be calculated substituting the found point $(x_0, y_0, z_0) = \pm\frac{1}{\sqrt{473}} \cdot (100, 32, 9)$, which the tangent plane must pass through. The result is

$$4x + 2y + z = \sqrt{473}$$

and

$$4x + 2y + z = -\sqrt{473}.$$

(ii) We follow a similar approach. The plane that cuts on the coordinate axis segments of the same length, has normal vector $\mathbf{n} = (1, 1, 1)$. Thus,

$$\left(\frac{2x}{25}, \frac{2y}{16}, \frac{2z}{9}\right) = \text{grad}(f)_{|u_0} = \lambda \cdot \mathbf{n} = \lambda \cdot (1, 1, 1)$$

for some $\lambda \in \mathbb{R}$. We get $\lambda = \pm 2/\sqrt{25}$ and the possible tangent planes are

$$x + y + z = 5\sqrt{2}$$

and

$$x + y + z = -5\sqrt{2}.$$

Exercise 3.9. *Find an equation of the tangent plane to the ellipsoid $x^2 + 2y^2 + z^2 = 1$, that is parallel to the plane $\rho : 4x + 2y + z = 0$.*

Solution:
We use the following Theorem (a consequence of the Implicit Function Theorem) **Theorem:** Let G be an open subset of $\mathbb{R}^n$, $f : G \to \mathbb{R}$ a continuously differentiable function on G. Let the point $u_0 \in G$ be such that $f(u_0) = 0$ and $f'(u_0) \neq 0$. Then the tangent plane to the hypersurface (called *variety*)

$$M = \{u \in G \mid f(u) = 0 \ \ \& \ \ f'(u_0) \neq 0\}$$

at the point u_0 has equation

$$f'(u_0)(u - u_0) = 0.$$

In our case $f(x,y,z) = x^2 + 2y^2 + z^2 - 1$ and $G = \mathbb{R}^3$. Since for $u_0 = (x,y,z)$ the derivative $f'(u_0) = \text{grad}(f)_{|u_0} = (2x, 4y, 2z)$ is zero only for $u_0 = (0,0,0)$ (a $f(0,0,0) = -1 \neq 0$), we can use the previous theorem, and the normal vector to the tangent plane at the point $u_0 \in M$ is exactly $\text{grad}(f)_{|u_0}$. This plane is parallel to ρ, that has normal vector $\mathbf{n}_\rho = (4,2,1)$, only if $(2x,4y,2z) = \text{grad}(f)_{|u_0} = \lambda \cdot \mathbf{n}_\rho = \lambda \cdot (4,2,1)$ for some $\lambda \in \mathbb{R}$, thus $(x,y,z) = (2\lambda, \lambda/2, \lambda/2)$. At the same time, should also hold that $x^2 + 2y^2 + z^2 = 1$. After substitution, we get $(2\lambda)^2 + 2(\lambda/2)^2 + (\lambda/2)^2 = 1$ thus $\lambda = \pm 2/\sqrt{19}$.

The tangent plane that we are looking for must have normal vector $\mathbf{n}_\rho$, thus equation $4x+2y+z = c$, where the value $c \in \mathbb{R}$ can be found substituting the coordinates of the found point $u_0 = \pm\frac{1}{\sqrt{19}} \cdot (4,1,1)$, that the plane must contain. The result is

$$4x + 2y + z = \sqrt{19}$$

a

$$4x + 2y + z = -\sqrt{19}.$$

Exercise 3.10. *Find the angle between the surfaces*

$$x^2 + y^2 + z^2 = 8 \quad \textit{and} \quad (x-1)^2 + (y-2)^2 + (z-3)^2 = 6$$

at the point $a_0 = (2,0,2)$.

Solution:
The angle between the two surfaces is equal to the angle formed by the tangent planes to the surfaces at the given point, but, at the same time, this is equal to the angle between the normal lines to the surfaces (i.e the normal lines to the tangent planes). Like in the previous example, we get

$$\mathbf{n}_1 = (2x, 2y, 2z)_{|a_0} = (4,0,4)$$

and

$$\mathbf{n}_2 = \Big(2(x-1), 2(y-2), 2(z-3)\Big)_{|a_0} = (2,-4,-2).$$

For the angle $\alpha \in \langle 0, \frac{\pi}{2} \rangle$, then we have

$$\cos\alpha = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \cdot \|\mathbf{n}_2\|} = 0$$

thus, $\alpha = \frac{\pi}{2}$.

Exercise 3.11. *Find the angle between the graph of the functions*

$$f(x,y) = \ln(\sqrt{x^2+y^2}) \quad a \quad g(x,y) = \sin(xy)$$

at the point $(1,0,0)$.

Solution:
The angle between the graph of the functions is given as the angle between the tangent planes to the graphs of the functions at the given point, and this is equal to the angle between the normal vectors, i.e. the gradients. The graphs are given implicitly:

for f, it is $\Gamma_f = \{(x,y,z) \in \mathbb{R}^3 \mid F(x,y,z) = 0 \ \& \ (x,y) \neq (0,0)\}$, where

$$F(x,y,z) = \frac{1}{2}\ln(x^2+y^2) - z$$

and for g we have $\Gamma_g = \{(x,y,z) \in \mathbb{R}^3 \mid G(x,y,z) = 0\}$, where

$$G(x,y,z) = \sin(xy) - z.$$

The normal vectors to the tangent planes are

$$\vec{n}_1 = \text{grad } F_{|(1,0,0)} = \Big(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, -1\Big)_{|(1,0,0)} = (1,0,-1)$$

$$\vec{n}_2 = \text{grad } G_{|(1,0,0)} = \Big(y\cos(xy), x\cos(xy), -1\Big)_{|(1,0,0)} = (0,1,-1)$$

The angle $\alpha \in \langle 0, \frac{\pi}{2}\rangle$ is given by

$$\cos\alpha = \frac{\Big|\vec{n}_1 \cdot \vec{n}_2\Big|}{\|\vec{n}_1\| \cdot \|\vec{n}_2\|} = \frac{1}{2},$$

thus $\alpha = \frac{\pi}{3}$.

Exercise 3.12. *Determine if the function*

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & , \ (x,y) \neq (0,0) \\ 0 & , \ (x,y) = (0,0) \end{cases}$$

is differentiable (has a total derivative) at the point $(0,0)$.

Solution:
By definition, the derivative of f at the point $a_0 = (0,0)$ is the linear function $f'_{|a_0} = L : \mathbb{R}^2 \to \mathbb{R}$, so that

$$\lim_{a \to a_0} \frac{f(a) - f(a_0) - L(a - a_0)}{\|a - a_0\|} = 0 \ .$$

If the derivative exists, then it is uniquely determined by the partial derivatives at the given point:

$$\frac{\partial f}{\partial x}_{|(0,0)} = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \to 0} \frac{t}{t} = 1,$$

$$\frac{\partial f}{\partial y}_{|(0,0)} = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t \to 0} \frac{0}{t} = 0.$$

For $a = (x,y)$, then we have

$$L(a - a_0) = (1,0) \cdot \begin{pmatrix} x - 0 \\ y - 0 \end{pmatrix} = x \ .$$

Thus,

$$\lim_{a \to a_0} \frac{f(a) - f(a_0) - L(a - a_0)}{\|a - a_0\|} = \lim_{(x,y)\to(0,0)} \frac{\frac{x^3}{x^2+y^2} - 0 - x}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{-xy^2}{(x^2+y^2)^{3/2}}.$$

If we now restrict the former result, for example for $x = y$, we get

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y}} \frac{-xy^2}{(x^2+y^2)^{3/2}} = \lim_{x\to 0} \frac{-x^3}{(2x^2)^{3/2}} = \lim_{x\to 0} \frac{-x}{\sqrt{8}\cdot |x|} \ .$$

This limit does not exist, and thus the previous limit does not exist (and for sure it is not equal to zero, as we needed to prove). The derivative (the total derivative) does not exist at the point $(0, 0)$.

Exercise 3.13. *Find the partial derivatives of the function $z = f(x, y)$, satisfying (defined by) the equation $z^3 - 3xyz = 2$, for every (x, y) in the appropriate domain of the function. First consider a general point, then the point $(x, y, z) = (1, 1, 2)$.*

Solution:
We are not able to express the function z in a simple explicit form, nevertheless, we can find its partial derivatives. On both side of the equation we evaluate $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, using the chain rule ($z$ is depending on the variable x and y):

$$0 = \frac{\partial\, 2}{\partial x} = \frac{\partial(z^3 - 3xyz)}{\partial x} = 3z^2 \frac{\partial z}{\partial x} - 3yz - 3xy\frac{\partial z}{\partial x}$$

$$0 = \frac{\partial\, 2}{\partial y} = \frac{\partial(z^3 - 3xyz)}{\partial y} = 3z^2 \frac{\partial z}{\partial y} - 3xz - 3xy\frac{\partial z}{\partial y}$$

and thus we calculate the partial derivatives:

$$\frac{\partial z}{\partial x} = \frac{yz}{z^2 - xy}$$

$$\frac{\partial z}{\partial y} = \frac{xz}{z^2 - xy}.$$

In order to do so, we supposed that $3z^2 - 3xy \neq 0$. This expression is exactly the partial derivative with respect to $\tilde{x}$ of the function $\Phi : \mathbb{R}^3 \to \mathbb{R}$, $\Phi(\tilde{x}, \tilde{y}, \tilde{z}) = \tilde{z}^3 - 3\tilde{x}\tilde{y}\tilde{z} - 2$ of three independ variables, that defines the original equation as $\Phi\big(x, y, z(x, y)\big) = 0$ (i.e.. implicitly defined function). Thus,

$$\frac{\partial \Phi}{\partial \tilde{z}} = 3\tilde{z}^2 - 3\tilde{x}\tilde{y} \ .$$

At the point $z(1, 1) = 2$, that satisfies the implicit equation and at which $3z^2 - 3xy \neq 0$, then, we get

$$\frac{\partial z}{\partial x}_{|(1,1)} = \frac{1\cdot 2}{2^2 - 1\cdot 1} = \frac{2}{3}$$

$$\frac{\partial z}{\partial y}_{|(1,1)} = \frac{1\cdot 2}{2^2 - 1\cdot 1} = \frac{2}{3} \ .$$

Exercise 3.14. *Find the derivative of the composed function $f \circ g$, where*

(i) $g : \mathbb{R}^2 \to \mathbb{R}^3$, $g(s, t) = \begin{pmatrix} st \\ s\cos t \\ s\sin t \end{pmatrix}$ *a* $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = x^2 + y^2 + z^2$,

(ii) $g : \mathbb{R}^2 \to \mathbb{R}^3,\ g(s,t) = \begin{pmatrix} st \\ e^{st} \\ t^2 \end{pmatrix}$ *a* $f : \mathbb{R}^3 \to \mathbb{R},\ f(x,y,z) = xy + yz + zx.$

Solution:
(i) We can expressed the function $h(s,t) = (f \circ g)(s,t) = (st)^2 + (s \sin t)^2 + (s \cos t)^2 = s^2t^2 + s^2$ and then take its derivative

$$h'(s,t) = \left(\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right) = (2st^2 + 2s, 2s^2t)$$

or, we may choose to use the theorem on differentiation of composed functions:

$$h'(s,t) = (f \circ g)'(s,t) = f'(g(s,t)) \circ g'(s,t) =$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x}, & \frac{\partial f}{\partial y}, & \frac{\partial f}{\partial z} \end{pmatrix}_{|g(s,t)} \cdot \begin{pmatrix} \frac{\partial g_1}{\partial s} & \frac{\partial g_1}{\partial t} \\ \frac{\partial g_2}{\partial s} & \frac{\partial g_2}{\partial t} \\ \frac{\partial g_3}{\partial s} & \frac{\partial g_3}{\partial t} \end{pmatrix} = \begin{pmatrix} 2x, & 2y, & 2z \end{pmatrix}_{|g(s,t)} \cdot \begin{pmatrix} t & s \\ \cos t & -s \sin t \\ \sin t & s \cos t \end{pmatrix} =$$

$$= \begin{pmatrix} 2st, & 2s \cos t, & 2s \sin t \end{pmatrix} \cdot \begin{pmatrix} t & s \\ \cos t & -s \sin t \\ \sin t & s \cos t \end{pmatrix} = (2st^2 + 2s, 2s^2t)$$

where $g_i(s,t)$ are the components of the function g. While taking the derivatives, it is necessary to choose for f the same order of variables as the order of the single components of g_i in the matrix-derivative of g (thus, for example, if we choose to take the derivatives with respect to, in the order, y, z, x, then the order of the components in the matrix of the derivative of g will be, from the top, g_2, g_3 and g_1.)

(ii) We follow a similar approach: $h(s,t) = (f \circ g)(s,t) = ste^{st} + t^2e^{st} + st^3$

$$h'(s,t) = \left(\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right) = \Big((t + st^2 + t^3)e^{st} + t^3, (s + s^2t + 2t + st^2)e^{st} + 3st^2\Big)$$

or

$$h'(s,t) = f'(g(s,t)) \circ g'(s,t) = \begin{pmatrix} y+z, & z+x, & x+y \end{pmatrix}_{|g(s,t)} \cdot \begin{pmatrix} t & s \\ te^{st} & se^{st} \\ 0 & 2t \end{pmatrix} =$$

$$= \begin{pmatrix} e^{st} + t^2, & st + t^2, & e^{st} + st \end{pmatrix} \cdot \begin{pmatrix} t & s \\ te^{st} & se^{st} \\ 0 & 2t \end{pmatrix} =$$

$$= \Big((t + st^2 + t^3)e^{st} + t^3, (s + s^2t + 2t + st^2)e^{st} + 3st^2\Big).$$

Exercise 3.15. *Find the derivative of the composed function* $f \circ g$, *where*

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x,y) = \begin{pmatrix} xy \\ x^2 + y^2 \end{pmatrix},$$

$$g : \mathbb{R}^2 \to \mathbb{R}^2, \quad g(\alpha,\beta) = \begin{pmatrix} \cos\alpha \\ -\sin(\alpha\beta) \end{pmatrix}.$$

Solution:

We indicate the component functions of f as $f_1(x,y) = xy$ and $f_2(x,y) = x^2 + y^2$. For the matrix representing the derivative of f, we have

$$f' = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}$$

thus, in each row we find the gradient of the single corresponding component.

Similarly, for $g_1(\alpha,\beta) = \cos\alpha$ and $g_2(\alpha,\beta) = -\sin(\alpha\beta)$ we have

$$g' = \begin{pmatrix} \frac{\partial g_1}{\partial \alpha} & \frac{\partial g_1}{\partial \beta} \\ \frac{\partial g_2}{\partial \alpha} & \frac{\partial g_2}{\partial \beta} \end{pmatrix} = \begin{pmatrix} -\sin\alpha & 0 \\ -\beta\cos(\alpha\beta) & -\alpha\cos(\alpha\beta) \end{pmatrix} .$$

Thus, the derivative of $f \circ g$ is

$$(f\circ g)' = f'_{|g} \circ g' = \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}\Bigg|_{\substack{x=\ \cos\alpha \\ y=-\sin(\alpha\beta)}} \cdot \begin{pmatrix} -\sin\alpha & 0 \\ -\beta\cos(\alpha\beta) & -\alpha\cos(\alpha\beta) \end{pmatrix} =$$

$$= \begin{pmatrix} -\sin(\alpha\beta) & \cos\alpha \\ 2\cos\alpha & -2\sin(\alpha\beta) \end{pmatrix} \cdot \begin{pmatrix} -\sin\alpha & 0 \\ -\beta\cos(\alpha\beta) & -\alpha\cos(\alpha\beta) \end{pmatrix} =$$

$$= \begin{pmatrix} \sin(\alpha\beta)\sin\alpha - \beta\cos(\alpha\beta)\cos\alpha & -\alpha\cos(\alpha\beta)\cos\alpha \\ -2\cos\alpha\sin\alpha + 2\beta\sin(\alpha\beta)\cos(\alpha\beta) & 2\alpha\sin(\alpha\beta)\cos(\alpha\beta) \end{pmatrix} =$$

$$= \begin{pmatrix} \sin(\alpha\beta)\sin\alpha - \beta\cos(\alpha\beta)\cos\alpha & -\alpha\cos(\alpha\beta)\cos\alpha \\ -\sin(2\alpha) + \beta\sin(2\alpha\beta) & \alpha\sin(2\alpha\beta) \end{pmatrix} .$$

The components can also be evaluated using the chain rule, directly, without evaluating the product of the two matrices. The components of $f \circ g$ are $(f\circ g)_i = f_i\big(x(\alpha,\beta), y(\alpha,\beta)\big)$, where the variables x and y are dependent on α and β, as $x(\alpha,\beta) = g_1(\alpha,\beta)$ and $y(\alpha,\beta) = g_2(\alpha,\beta)$. The matrix representing the derivative of the composed function has the form:

$$(f\circ g)' = \begin{pmatrix} \frac{\partial (f\circ g)_1}{\partial \alpha} & \frac{\partial (f\circ g)_1}{\partial \beta} \\ \frac{\partial (f\circ g)_2}{\partial \alpha} & \frac{\partial (f\circ g)_2}{\partial \beta} \end{pmatrix}$$

and, from the chain rule, we have for example

$$\frac{\partial (f\circ g)_2}{\partial \alpha} = \frac{\partial f_2\big(\cos\alpha, -\sin(\alpha\beta)\big)}{\partial \alpha} = \frac{\partial f_2}{\partial x} \cdot \frac{\partial \cos\alpha}{\partial \alpha} + \frac{\partial f_2}{\partial y} \cdot \frac{\partial(-\sin(\alpha\beta))}{\partial \alpha} =$$

$$= 2x\cdot(-\sin\alpha) + 2y\cdot(-\beta\cos(\alpha\beta)) = -2\cos\alpha\cdot\sin\alpha + 2\beta\sin(\alpha\beta)\cdot\cos(\alpha\beta) .$$

Exercise 3.16. *Find the derivative of the function* $\Phi(x,y,z) = \begin{pmatrix} f(x+y,z) \\ f\left(\frac{x}{y}, \frac{y}{z}\right) \end{pmatrix}$, *where* $f : \mathbb{R}^2 \to \mathbb{R}$ *is a continuous differentiable function.*

Solution:

We indicate the components of the function Φ as Φ_1 and Φ_2. We need to find the matrix

$$\Phi' = \begin{pmatrix} \frac{\partial \Phi_1}{\partial x} & \frac{\partial \Phi_1}{\partial y} & \frac{\partial \Phi_1}{\partial z} \\ \frac{\partial \Phi_2}{\partial z} & \frac{\partial \Phi_2}{\partial y} & \frac{\partial \Phi_2}{\partial z} \end{pmatrix} .$$

In evaluating each entry, we use the chain rule. We indicate the variables of f, for example as $f(u,v)$.

Now, we can write

$$\frac{\partial \Phi_1}{\partial x} = \frac{\partial f(x+y,z)}{\partial x} = \frac{\partial f}{\partial u}(x+y,z)\cdot\frac{\partial (x+y)}{\partial x} + \frac{\partial f}{\partial v}(x+y,z)\cdot\frac{\partial z}{\partial x} =$$

$$= \frac{\partial f}{\partial u}(x+y,z)\cdot 1 + \frac{\partial f}{\partial v}(x+y,z)\cdot 0 = \frac{\partial f}{\partial u}(x+y,z)$$

and similarly

$$\frac{\partial \Phi_1}{\partial y} = \frac{\partial f(x+y,z)}{\partial y} = \frac{\partial f}{\partial u}(x+y,z)\cdot 1 + \frac{\partial f}{\partial v}(x+y,z)\cdot 0 = \frac{\partial f}{\partial u}(x+y,z)$$

$$\frac{\partial \Phi_1}{\partial z} = \frac{\partial f(x+y,z)}{\partial y} = \frac{\partial f}{\partial u}(x+y,z)\cdot 0 + \frac{\partial f}{\partial v}(x+y,z)\cdot 1 = \frac{\partial f}{\partial v}(x+y,z)$$

for the second row, we have

$$\frac{\partial \Phi_2}{\partial x} = \frac{\partial f\left(\frac{x}{y},\frac{y}{z}\right)}{\partial x} = \frac{\partial f}{\partial u}\left(\frac{x}{y},\frac{y}{z}\right)\cdot\frac{1}{y} + \frac{\partial f}{\partial v}\left(\frac{x}{y},\frac{y}{z}\right)\cdot 0 = \frac{1}{y}\cdot\frac{\partial f}{\partial u}\left(\frac{x}{y},\frac{y}{z}\right)$$

$$\frac{\partial \Phi_2}{\partial y} = \frac{\partial f\left(\frac{x}{y},\frac{y}{z}\right)}{\partial y} = \frac{\partial f}{\partial u}\left(\frac{x}{y},\frac{y}{z}\right)\cdot\left(-\frac{x}{y^2}\right) + \frac{\partial f}{\partial v}\left(\frac{x}{y},\frac{y}{z}\right)\cdot\frac{1}{z}$$

$$\frac{\partial \Phi_2}{\partial z} = \frac{\partial f\left(\frac{x}{y},\frac{y}{z}\right)}{\partial z} = \frac{\partial f}{\partial u}\left(\frac{x}{y},\frac{y}{z}\right)\cdot 0 + \frac{\partial f}{\partial v}\left(\frac{x}{y},\frac{y}{z}\right)\cdot\left(-\frac{y}{z^2}\right) = -\frac{y}{z^2}\cdot\frac{\partial f}{\partial v}\left(\frac{x}{y},\frac{y}{z}\right).$$

Thus, all together, we get

$$\Phi' = \begin{pmatrix} \frac{\partial f}{\partial u}(x+y,z) & \frac{\partial f}{\partial u}(x+y,z) & \frac{\partial f}{\partial v}(x+y,z) \\ \frac{1}{y}\cdot\frac{\partial f}{\partial u}\left(\frac{x}{y},\frac{y}{z}\right) & -\frac{x}{y^2}\frac{\partial f}{\partial u}\left(\frac{x}{y},\frac{y}{z}\right) + \frac{1}{z}\frac{\partial f}{\partial v}\left(\frac{x}{y},\frac{y}{z}\right) & -\frac{y}{z^2}\cdot\frac{\partial f}{\partial v}\left(\frac{x}{y},\frac{y}{z}\right) \end{pmatrix}.$$

4 The Taylor's polynomial, local extremes

Exercise 4.1. *Find the Taylor polynomial of second order for the function* $f(x,y) = e^{x^2+y^2} - \cos(x-y)$*, at the critical point* $a_0 = (0,0)$*, and use it to determine if the function at* a_0 *has a minimum, a maximum or a saddle point.*

Solution:
The Taylor polynomial of degree 2, that approximates the function f at the point $a_0 \in \mathbb{R}^n$, is defined by:

$$T_2(a_0+\mathbf{h}) = f(a_0) + f'(a_0)\mathbf{h} + \frac{1}{2!}f''(a_0)(\mathbf{h},\mathbf{h})$$

where $\mathbf{h} = (h_1,\ldots,h_n) \in \mathbb{R}^n$.

We have:

$$f'_{|(0,0)} = \left(2xe^{x^2+y^2} + \sin(x-y), 2ye^{x^2+y^2} - \sin(x-y)\right)_{|(0,0)} = (0,0)$$

thus, $(0,0)$ is a critical point for f,

$$f''_{(0,0)} = \begin{pmatrix} 2e^{x^2+y^2} + 4x^2e^{x^2+y^2} + \cos(x-y) & 4xye^{x^2+y^2} - \cos(x-y) \\ 4xye^{x^2+y^2} - \cos(x-y) & 2e^{x^2+y^2} + 4y^2e^{x^2+y^2} + \cos(x-y) \end{pmatrix}_{|(0,0)} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

For $\mathbf{h} = (h_1, h_2) \in \mathbb{R}^2$, we have

$$T_2(\mathbf{h}) = f(0,0) + f'_{|(0,0)}\mathbf{h} + \frac{1}{2!}f''_{|(0,0)}(\mathbf{h},\mathbf{h}) = \frac{1}{2}(h_1,h_2)\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \frac{3}{2}h_1^2 - h_1h_2 + \frac{3}{2}h_2^2.$$

By Sylvester's criterion ($\Delta_1 = 3 > 0$, $\Delta_2 = 8 > 0$), the matrix $f''_{(0,0)}$ is positive definite, thus, at the point $a = (0,0)$ there is a local minimum.

Exercise 4.2. *Find the Taylor polynomial of second order for the function f at the point a_0:*

(i) $f(x,y) = e^{2xy} - y^2$, $a_0 = (0,0)$,

(ii) $f(x,y,z) = xy^2z^3$, $a_0 = (1,2,1)$.

For the function in (i) determine if the function, at the given point, has a point of minimum, maximum or a saddle.

Solution:

(i) We have

$$f'_{|(0,0)} = \left(2ye^{2xy}, 2xe^{2xy} - 2y\right)_{|(0,0)} = (0,0)$$

thus, $(0,0)$ is a critical point for f,

$$f''_{(0,0)} = \begin{pmatrix} 4y^2e^{2xy} & 2e^{2xy} + 4xye^{2xy} \\ 2e^{2xy} + 4xye^{2xy} & 4x^2e^{2xy} - 2 \end{pmatrix}_{|(0,0)} = \begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix}$$

For $\mathbf{h} = (h_1, h_2) \in \mathbb{R}^2$, we have

$$T_2(\mathbf{h}) = f(0,0) + f'_{|(0,0)}\mathbf{h} + \frac{1}{2!}f''_{|(0,0)}(\mathbf{h},\mathbf{h}) = 1 + \frac{1}{2}(h_1,h_2)\begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 1 + 2h_1h_2 - h_2^2.$$

The quadratic form

$$g(h_1,h_2) = 2h_1h_2 - h_2^2 = h_2(2h_1 - h_2)$$

of the second derivative is indefinite (for example $g(1,1) = 1 > 0$, and $g(0,1) = -1 < 0$). At the point $a = (0,0)$, f has a saddle.

(ii) We have

$$f'(a_0) = (y^2z^3, 2xyz^3, 3xy^2z^2)_{|a_0} = (4,4,12)$$

and

$$f''(a_0) = \begin{pmatrix} 0 & 2yz^3 & 3y^2z^2 \\ 2yz^3 & 2xz^3 & 6xyz^2 \\ 3y^2z^2 & 6xyz^2 & 6xy^2z \end{pmatrix}_{|a_0} = \begin{pmatrix} 0 & 4 & 12 \\ 4 & 2 & 12 \\ 12 & 12 & 24 \end{pmatrix}.$$

Thus

$$T_2(a_0 + \mathbf{h}) = 4 + (4,4,12)\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} + \frac{1}{2}(h_1,h_2,h_3)\begin{pmatrix} 0 & 4 & 12 \\ 4 & 2 & 12 \\ 12 & 12 & 24 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} =$$

$$= 4 + 4h_1 + 4h_2 + 12h_3 + 4h_1h_2 + 12h_1h_3 + h_2^2 + 12h_2h_3 + 12h_3^2.$$

Exercise 4.3. *Find the Taylor polynomial of second order for the function f at the point a_0: $f(x,y,z) = xe^y \cos z$, $a_0 = (0,0,0)$.*

Solution:
We have

$$f'(a_0) = (e^y \cos z, xe^y \cos z, -xe^y \sin z)_{|a_0} = (1, 0, 0)$$

and

$$f''(a_0) = \begin{pmatrix} 0 & e^y \cos z & -e^y \sin z \\ e^y \cos z & xe^y \cos z & -xe^y \sin z \\ -e^y \sin z & -xe^y \sin z & -xe^y \cos \end{pmatrix}_{|a_0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$T_2(a_0 + \mathbf{h}) = h_1 + h_1 h_2.$$

Exercise 4.4. *Find the Taylor polynomial of second order for the function* $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = e^{xy} - 2xy$$

at the point $a = (0, 0)$ *and use the found polynomial to determine if the function has at this critical point a minimum, a maximum or a saddle point.*

Solution:
The function is symmetric in x and y, this simplifies the calculations.

$$f'_{|(0,0)} = \left(ye^{xy} - 2y, xe^{xy} - 2x \right)_{|(0,0)} = (0, 0)$$

$$f''_{(0,0)} = \begin{pmatrix} y^2 e^{xy} & e^{xy} + xye^{xy} - 2 \\ e^{xy} + xye^{xy} - 2 & x^2 e^{xy} \end{pmatrix}_{|(0,0)} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

For $\mathbf{h} = (h_1, h_2) \in \mathbb{R}^2$ we have

$$T_2(\mathbf{h}) = f(0,0) + f'_{|(0,0)}\mathbf{h} + \frac{1}{2!} f''_{|(0,0)}(\mathbf{h}, \mathbf{h}) = 1 + \frac{1}{2}(h_1, h_2) \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 1 - h_1 h_2.$$

The quadratic form

$$Q(\mathbf{h}) := f''_{|(0,0)}(\mathbf{h}, \mathbf{h}) = -2h_1 h_2$$

of the second derivative is indefinite (for example $Q(1, 1) = -1 > 0$ and $Q(-1, 1) = 1 < 0$). At the point $a = (0, 0)$ we thus have a saddle.

Exercise 4.5. *Find and classify the local extrema of the following functions:*

(i) $f(x, y) = x^3 - y^3 - 2xy + 6$

(ii) $f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$

Solution:
(i) The given function is a polynomial, thus has derivatives of every order. We look for critical pointsof f, i.e. points where the given function has zero gradient (a necessary condition for a point to be an extreme point).

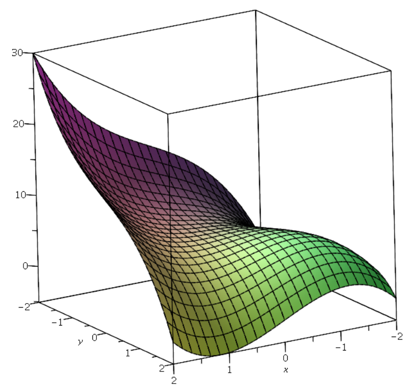
$$f'_{|(x,y)} = (3x^2 - 2y, -3y^2 - 2x)$$

Thus, $f'_{|(x,y)} = 0$ exactly for $3x^2 = 2y$ and $-3y^2 = 2x$, that gives the points $(x,y) = (0,0)$, or $(x,y) = (-\frac{2}{3}, \frac{2}{3})$. At the found critical points, we evaluate the second derivative of f.

$$f''_{|(x,y)} = \begin{pmatrix} 6x & -2 \\ -2 & -6y \end{pmatrix}$$

For $(x,y) = (0,0)$, $f''_{|(0,0)} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$. Thus, for $\mathbf{h} = (h_1, h_2)^T \in \mathbb{R}^2$we have $f''_{|(0,0)}(\mathbf{h}, \mathbf{h}) = -4h_1h_2$ and this is an indefinite form. Thus, f has a saddle at the point $(0,0)$.

For $(x,y) = (-\frac{2}{3}, \frac{2}{3})$, $f''_{|(-\frac{2}{3},\frac{2}{3})} = \begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix}$. By Sylvester's criterion ($\Delta_1 = -4 < 0$, $\Delta_2 = 16 - 4 = 12 > 0$) the form is negative definite, thus f has a local maximum at the point. This maximum is not global because the function is unbounded from below (for example, we may consider the restriction $f(x,0) = x^3 + 6$).



(ii) We proceed like in the previous case:

$$f'_{|(x,y)} = (4x + 3y - 5, 3x + 8y + 2)$$

Thus, $f'_{|(x,y)} = 0$ exactly for $(x,y) = (2,-1)$. The second derivative is

$$f''_{|(x,y)} = \begin{pmatrix} 4 & 3 \\ 3 & 8 \end{pmatrix}$$

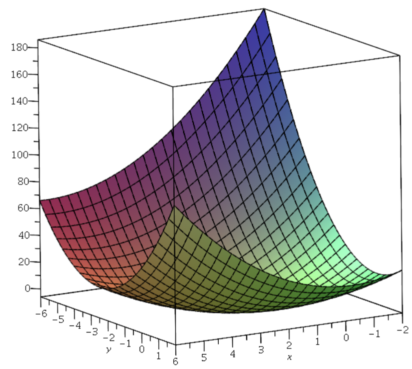
and, by Sylvester's criterion is positive definite, thus, at $(2,-1)$, f has a local minimum $f(2,-1) = -6$. This minimum is actually a global minimum, and we can prove this, for example, by completing the square:

$$f(x,y) = 2x^2 + 3xy + 4y^2 - 5x + 2y =$$

$$= 2\Big(x^2 + 2 \cdot x \cdot \frac{3}{4}y + 2 \cdot x \cdot (-\frac{5}{4}) + 2 \cdot \frac{3}{4}y \cdot (-\frac{5}{4}) + (\frac{3}{4}y)^2 + (-\frac{5}{4})^2\Big) + \frac{15}{4}y - \frac{9}{8}y^2 - \frac{25}{8} + 4y^2 + 2y =$$

$$= 2\Big(x + \frac{3}{4}y - \frac{5}{4}\Big)^2 + \frac{23}{8}y^2 + \frac{23}{4}y - \frac{25}{8} = 2\Big(x + \frac{3}{4}y - \frac{5}{4}\Big)^2 + \frac{23}{8}(y+1)^2 - 6$$

Thus, really $f(x,y) \geq -6$ and the equality holds for $x + \frac{3}{4}y - \frac{5}{4} = 0$ and $y + 1 = 0$, or $(x,y) = (2,-1)$.



5 Global extrems and extremes of functions with constraints

Exercise 5.1. *Find the absolute maximum and minimum of* $f(x,y) = x - y + 3$ *with constraint* $3x^2 + 5xy + 3y^2 = 1$.

Solution:

We use the following theorems:

Theorem: A continuous function on a closed bounded (called *compact*) set attains its maximum and minimum value.

Theorem: Let $U \subseteq \mathbb{R}^n$ be an open set $k \leq n$ and $f : U \to \mathbb{R}$ and $\Phi : U \to \mathbb{R}^k$ are continuous differentiable functions on U. We define

$$M = \{a \in U \mid \Phi(a) = 0 \ \&\ \Phi'_{|a} \text{ is regular}\}.$$

If $a_0 \in M$ is a local extrema of the function f restricted on M, then there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ (called *Lagrange multipliers*), so that

$$f'_{|a_0} = \sum_{i=1}^{k} \lambda_i \cdot (\Phi'_i)_{|a_0},$$

where Φ_i are the components of the function Φ, i.e. $\Phi(a) = (\Phi_1(a), \ldots, \Phi_k(a))$.

(*Regularity* of the derivative means that the matrix representing the derivative has maximal rank, thus rank k, i.e. its row are linearly independent. The set M is then called a *manifold* and it is possible to assign it a dimension - using the implicit function theorem - and namely $\dim M = n - k$. The dimension is then equal to the dimension n of the original space $\mathbb{R}^n$ minus the number k rank of the derivative of Φ.)

In our case, we set $U = \mathbb{R}^2$ and $\Phi(x,y) = 3x^2 + 5xy + 3y^2 - 1$. Since

$$\Phi'_{|(x,y)} = \Big(6x + 5y, 5x + 6y\Big)$$

then $\Phi'_{|(x,y)}$ *is not* regular (i.e. in this case $\Phi'_{|(x,y)} = 0$) exactly for $(x,y) = (0,0)$. Thus, it cannot happen that $\Phi(x,y) = 0$ and $\Phi'_{|(x,y)} = 0$. Thus, at every point of the set

$$M = \{(x,y) \in \mathbb{R}^2 \mid 3x^2 + 5xy + 3y^2 = 1\}$$

we have regular $\Phi'_{|(x,y)}$. For the point $a = (x, y) \in M$, that is a local extreme of f on M, there exists $\lambda \in \mathbb{R}$ so that

$$(1, -1) = f'_{|a} = \lambda \Phi'_{|a} = \lambda\Big(6x + 5y, 5x + 6y\Big)$$

and

$$3x^2 + 5xy + 3y^2 = 1.$$

Summing the first two equations, we get $x = -y$, and, after substituting into the constraint, we get as possible extrema:

$$(1, -1), \quad (-1, 1)$$

and the function attains there values

$$f(1, -1) = 5, \quad f(-1, 1) = 1.$$

We still need to find out if the set M is bounded (the closure of M follows from the fact that $M = \Phi^{-1}(\{0\})$, that means that M is the pre-image of the closed set $\{0\}$ under the continuous function Φ).

Completing the square

$$1 = 3x^2 + 5xy + 3y^2 = 3\Big(x + \frac{5}{6}y\Big)^2 + \frac{11}{12}y^2$$

we find out that we are dealing with a bounded set (it is an ellipse (rotated)). This also follow from the fact that the quadratic form $Q(x, y) = 3x^2 + 5xy + 3y^2$ is positive definite (use Sylvester's criterion).

The continuous function f, on the closed and bounded set M attains its maximum and minimum at the points $(1, -1)$ a $(-1, 1)$.

Exercise 5.2. *A circular disk $x^2 + y^2 \leq 1$ is heated at the temperature $T(x, y) = x^2 + 2y^2 - x$. Find the hottest and coldest point on the disk.*

Solution:
We use the same approach as in the previous example. We look for the extrema of T on the closed bounded set $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. We separate the search of the local extrema on the open set

$$A^\circ = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

and the extrema on the constraint

$$\partial A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

If $a = (x, y) \in A^\circ$ is an extreme of T on A, then, it is also an extreme of T on A°. Thus, it must hold that

$$T'_{|a} = (2x - 1, 4y) = 0$$

it follows that $a = (\frac{1}{2}, 0)$, and then really $a \in A^\circ$.

If $a = (x, y) \in \partial A$ is an extreme of T on A, then it is also an extreme of T on (the constraint) $\partial A = \{(x, y) \in \mathbb{R}^2 \mid \Phi(x, y) = 0\}$, where $\Phi(x, y) = x^2 + y^2 - 1$. Thus, it must exists $\lambda \in \mathbb{R}$, so that

$$(2x - 1, 4y) = T'_{|a} = \lambda \Phi'_{|a} = \lambda(2x, 2y)$$

and

$$x^2 + y^2 = 1.$$

We get $a = \pm(1,0)$ or $a = (-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$. Now, we know that the only possible candidates to be extrema are the points

$$(\frac{1}{2},0),\ (1,0),\ (-1,0),\ (-\frac{1}{2},\frac{\sqrt{3}}{2}) \text{ a } (-\frac{1}{2},-\frac{\sqrt{3}}{2}).$$

Since T attains on the (closed and bounded) set A its extrema, comparing the values of the function

$$T\Big(\frac{1}{2},0\Big) = -\frac{1}{4}, \quad T(1,0) = 0, \quad T(-1,0) = 2, \quad T\Big(-\frac{1}{2},\frac{\sqrt{3}}{2}\Big) = \frac{9}{4} = T\Big(-\frac{1}{2},-\frac{\sqrt{3}}{2}\Big)$$

we get that T attains its minimum at $(\frac{1}{2},0)$ and maximum at $(-\frac{1}{2},\pm\frac{\sqrt{3}}{2})$.

Exercise 5.3. *Find the absolute minimim and maximum of the function $f(x,y) = x^2 - xy + y^2$ on the set $|x| + |y| \leq 1$.*

Solution:
The set $A = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1\}$ is a square and it is obviously bounded and closed (is the pre-image of the closed interval $(-\infty, 1\rangle$ under the continuous $\Psi(x,y) = |x| + |y|$).

We again separate the case of finding the local extrema of the given function on the open set

$$A^\circ = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| < 1\}$$

and the case of finding extrema on the constraint

$$\partial A = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| = 1\},$$

that this time we cannot express using a condition involving a differentiable function. The constraint is formed by four open segments (the sides of the square) and four points (the corners of the square). We may simplify our search using symmetries $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ that preserve both ∂A, and the given function f. So, it must hold that $\varphi(\partial A) = \partial A$ and $f \circ \varphi = f$.

WE may choose any of the following three (non-identical) symetry:

$$(x,y) \mapsto (-x,-y) \qquad \textit{(symmetry with respect to the origin)}$$

$$(x,y) \mapsto (y,x) \qquad \textit{(symmetry with respect to the line } x = y\textit{)}$$

$$(x,y) \mapsto (-y,-x) \qquad \textit{(symmetry with respect to the line } x = -y\textit{)}$$

Extreme on A°:

$f'_{|a} = (2x - y, 2y - x) = 0$ only for $a = (0,0) \in A^\circ$, with value $f(0,0) = 0$.

Extreme on ∂A:

Due to the symmetries, it is enough to find the extreme on

$$U_1 = \{(x,y) \in \mathbb{R}^2 \mid x > 0,\ y > 0\} \text{ with constraint } \Phi_1(x,y) = x + y - 1$$

and on

$$U_2 = \{(x,y) \in \mathbb{R}^2 \mid x > 0,\ y < 0\} \text{ with constraint } \Phi_2(x,y) = x - y - 1$$

i.e. the sides of the square A without the ending points (and then just one vertex $(1,0)$ of the squre A like a single constraint).

For extreme on U_1, it must exists $\lambda_1 \in \mathbb{R}$ so that

$$(2x - y, 2y - x) = f'_{|a} = \lambda_1 \Phi'_{1|a} = \lambda_1(1, 1)$$

and

$$x + y = 1,$$

thus $(x, y) = (\frac{1}{2}, \frac{1}{2}) \in U_1$ and $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$.

similarly, for an extreme on U_2 it must exists $\lambda_2 \in \mathbb{R}$ so that

$$(2x - y, 2y - x) = f'_{|a} = \lambda_2 \Phi'_{2|a} = \lambda_2(1, -1)$$

and

$$x - y = 1,$$

thus $(x, y) = (\frac{1}{2}, -\frac{1}{2}) \in U_2$ and $f(\frac{1}{2}, -\frac{1}{2}) = \frac{3}{4}$.

It remains to consider the point $(1, 0)$ where the function has value $f(1, 0) = 1$.

Thus, the function attains its minimum at the point $(0, 0)$, and its maximum at the vertexes of the square (that we obtained from the point $(1, 0)$ using the symmetries).

Exercise 5.4. *Find the extrema of the function $f(x, y, z) = x - y + 3z$ with constraint $x^2 + y^2 + 4z^2 = 4$.*

Solution:
We use the same approach as in the previous example. We set $\Phi(x, y, z) = x^2 + y^2 + 4z^2 - 4$. Since

$$\Phi'_{|(x,y,z)} = (2x, 2y, 8z)$$

then $\Phi'_{|(x,y,z)} = 0$ exactly for $(x, y, z) = (0, 0, 0)$, that, on the other hand, cannot satisfy the constraint. Thus, at each point of the set

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + 4z^2 = 4\}$$

we have $\Phi'_{|(x,y,z)} \neq 0$. For a point $a = (x, y, z) \in M$, local extreme of f on M, it must exists $\lambda \in \mathbb{R}$ so that

$$(1, -1, 3) = f'_{|a} = \lambda \Phi'_{|a} = \lambda(2x, 2y, 8z)$$

and

$$x^2 + y^2 + 4z^2 = 4.$$

Thus, it must hold $\lambda \neq 0$, and evaluating

$$x = \frac{1}{2\lambda} \quad y = -\frac{1}{2\lambda} \quad z = \frac{3}{8\lambda}$$

and substituting into the constraint, we find the solution $a = \pm\frac{2}{\sqrt{17}}(4, -4, 3)$ and $\lambda = \pm\frac{\sqrt{17}}{16}$. Since f attains its extrema on M (since M is bounded and closed), the found points are the (absolute) extrema and the function attains there value $f(a) = \pm 2\sqrt{17}$.

Exercise 5.5. *Find three positive numbers with maximal product and given fixed sum equal to 100.*

Solution:

We are looking for positive numbers, but in order to use the theorem on continuous functions attaining their maximum and minimum it is necessary to work with a *closed* bounded set. We will thus look for points of maximum of the function

$$f(x, y, z) = xyz$$

on the set

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0 \ \& \ x + y + z = 100\},$$

(That is a triangle with its boundary), i.e. we look for *non-negative* numbers. The set A is obviously closed and bounded.

We distinguish again the search of the extrema on the open set

$$U = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z > 0\},$$

that is, on the set $A \cap U = \{(x, y, z) \in U \mid \Phi(x, y, z) = 0\}$ with the constraint $\Phi(x, y, z) = x+y+z-100$ (the triangle without its contour) and on the set $A \setminus U$ (the boundary of the triangle).

On the boundary of the triangle, the function f has value zero and thus it attains here its minimum (because f is non-zero on the remaining part of the set A).

For a point $a = (x, y, z) \in A \cap U$, extreme of f, then it must exist $\lambda \in \mathbb{R}$ so that

$$(yz, xz, xy) = f'_{|a} = \lambda \Phi'_{|a} = \lambda(1, 1, 1)$$

and

$$x + y + z = 100,$$

thus $a = \frac{100}{3}(1, 1, 1)$ and $f\left(\frac{100}{3}, \frac{100}{3}, \frac{100}{3}\right) = \left(\frac{100}{3}\right)^3$ and this point is the only point of maximum of the function f on A.

Exercise 5.6. *Find the smallest and the biggest value of the function $f(x, y, z) = xyz$ on the set M defined by the conditions*

$$x + y + z = 5 \qquad \textit{and} \qquad xy + yz + zx = 8.$$

Solution:
This time we have two constraints and it is necessary to prove their independence (on the points of the set M), i.e. the linear independence of the gradients of the constraints at corresponding points.

We set

$$\Phi_1(x, y, z) = x + y + z - 5$$

and

$$\Phi_2(x, y, z) = xy + yz + zx - 8.$$

Then, we have $M = \{a \in \mathbb{R}^3 \mid \Phi_1(a) = 0 \ \& \ \Phi_2(a) = 0\}$.

M is *closed*:

The sets $\{a \in \mathbb{R}^3 \mid \Phi_i(a) = 0\}$ represent the pre-image of the one point (thus close) set $\{0\}$ under the continuous functions Φ_i and, thus, are closed. The set M is their intersection, thus it is also closed.

M is *bounded*:

We may solve for one variable in the first equation (for example $z = 5 - x - y$), then substitute into the second equation and rewrite it completing the square:

$$xy + (x + y)(5 - x - y) = 8$$

$$x^2 + y^2 + xy - 5x - 5y = -8$$

$$\left(x + \frac{y}{2} - \frac{5}{2}\right)^2 + \frac{3}{4}\left(y - \frac{5}{3}\right)^2 = \frac{1}{3}$$

otherwise, using a more elegant approach:

$$5^2 = (x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = x^2 + y^2 + z^2 + 2 \cdot 8$$

$$x^2 + y^2 + z^2 = 5^2 - 2 \cdot 8 (= 9).$$

In either case, we see that the variables are bounded, thus the set M is bounded.

independence of constraints:
We need to prove that, for $a = (x, y, z)$, it holds:

$$\Phi_1(a) = 0 \quad \& \quad \Phi_2(a) = 0 \qquad \Longrightarrow \qquad \text{grad}(\Phi_1)_{|a} \text{ and } \text{grad}(\Phi_2)_{|a} \text{ are linearly independent.}$$

We have

$\text{grad}(\Phi_1)_{|a} = (1, 1, 1)$

$\text{grad}(\Phi_2)_{|a} = (y + z, z + x, x + y)$.

These vectors are linearly independent exactly when $y + z = z + x = x + y$, i.e. when $x = y = z$. If it must hold that $\Phi_1(a) = 0$ and $\Phi_2(a) = 0$, then we get that $3x = 5$ and $3x^2 = 8$, that is impossible. For points in M, we really have independence of constraints .

Now, finally, we may use the Lagrange multipliers method:
For a point $a = (x, y, z) \in M$, absolute (and also local) extreme of f on M, there exist $\lambda, \mu \in \mathbb{R}$ so that

$$(yz, zx, xy) = \text{grad}(f)_{|a} = \lambda \cdot \text{grad}(\Phi_1)_{|a} + \mu \cdot \text{grad}(\Phi_2)_{|a} = \lambda(1, 1, 1) + \mu(y + z, z + x, x + y)$$

a

$$x + y + z = 5 \qquad \text{and} \qquad xy + yz + zx = 8.$$

If we subtract the first two equations

$$yz = \lambda - \mu(y + z)$$

$$zx = \lambda - \mu(z + x)$$

we get $z(y - x) = \mu(y - x)$, that gives the condition $x = y$ or $z = \mu$. In a symmetric way, we get the condition $y = z$ or $x = \mu$. From this, it easily follows that either $x = y$, or $y = z$, or $x = \mu = z$, i.e. two coordinates are always equal. We solve one case, and we get the rest with a permutation of coordinates.

For example, from the condition $x = y$, substituting into the constraints, we get the solutions $(x, y, z) = (2, 2, 1)$ or $(x, y, z) = \left(\frac{4}{3}, \frac{4}{3}, \frac{7}{3}\right)$. It is not necessary to evaluate the values of the parameters λ and μ, the possible candidate points can only be:

$$a = (2, 2, 1), (1, 2, 2), (2, 1, 2) \quad \text{where} \quad f(a) = 4$$

and

$$a = \left(\frac{4}{3}, \frac{4}{3}, \frac{7}{3}\right), \left(\frac{4}{3}, \frac{7}{3}, \frac{4}{3}\right), \left(\frac{7}{3}, \frac{4}{3}, \frac{4}{3}\right) \quad \text{where} \quad f(a) = \frac{112}{27}.$$

Since the function f is continuous and the set M is bounded and closed, f attains at the first point its minimum and at the second its maximum (because $\frac{112}{27} > 4$).

Exercise 5.7. *Find points on the ellipse* $M:\ \frac{x^2}{4}+\frac{y^2}{9}=1$ *with maximal and minimal distance from the line* $p:\ 3x+y-9=0$.

Solution:
We may solve the exercise in several ways:

(1) We may use the function that explicitly expresses the distance of the point $(x,y)\in\mathbb{R}^2$ from the line with given equation $\alpha x'+\beta y'+\gamma=0$, and thus $f(x,y)=\frac{|\alpha x+\beta y+\gamma|}{\sqrt{\alpha^2+\beta^2}}$.

Deriving the formula: We consider the $\mathbb{R}^3$ case (the $\mathbb{R}^2$ case is analogous). Consider the plane ρ in $\mathbb{R}^3$ with equation $\alpha x'+\beta y'+\gamma z'+\delta=0$. Its normal vector is $n=(\alpha,\beta,\gamma)$ and the equation, for the point $a'=(x',y',z')\in\mathbb{R}^3$, then can be written using the scalar product as $n\cdot a'=-\delta$. We now choose a certain point $b\in\mathbb{R}^3$ on the plane ρ. The distance of the point $a=(x,y,z)\in\mathbb{R}^3$ from the plane ρ is now given by the length of the orthogonal projection of the vector $a-b$ in the direction of the normal vector n, thus

$$\left|(a-b)\cdot\frac{n}{\|n\|}\right|.$$

Since the point b is on the plane ρ, it holds $n\cdot b=-\delta$. We can thus write

$$\left|(a-b)\cdot\frac{n}{\|n\|}\right|=\frac{|a\cdot n-b\cdot n|}{\|n\|}=\frac{|a\cdot n+\delta|}{\|n\|}=\frac{|\alpha x+\beta y+\gamma z+\delta|}{\sqrt{\alpha^2+\beta^2+\gamma^2}}.$$

We then look for the maximum and minimum of the function

$$f(x,y)=\frac{|3x+y-9|}{\sqrt{3^2+1^2}}$$

with constraint $\frac{x^2}{4}+\frac{y^2}{9}=1$. Since f is not differentiable everywhere, we can help ourselves in one of the following ways

- we choose to look for minimum and maximum, instead of f, of the function

  $$g(x,y)=10\cdot\Big(f(x,y)\Big)^2=(3x+y-9)^2$$

  (we simplify the form of f and deal with a simpler function with "equivalent" corresponding point of maximum and minimum)

- we realize that M has no intersection with the line p, that means that M lies all in one of the open half-plane determined by p (since M is a *connected* set - it is arch-wise connected). In this case, the expression $3x+y-9$, at every point of M is always positive or always negative. Looking for the extrema of f is then equivalent to looking for the extrema of the function

  $$h(x,y)=3x+y-9.$$

We choose the latter way.

For points on the ellipse M, the given constraint $\Phi(x,y):=\frac{x^2}{4}+\frac{y^2}{9}-1(=0)$ is actually $\operatorname{grad}(\Phi)=(\frac{x}{2},\frac{2y}{9})\neq 0$. For a point $a=(x,y)\in M$, absolute extreme of h on the ellipse M, there exists $\lambda\in\mathbb{R}$ so that

$$(3,\ 1)=\operatorname{grad}(h)_{|a}=\lambda\cdot\operatorname{grad}(\Phi)_{|a}=\lambda\left(\frac{x}{2},\frac{2y}{9}\right)$$

and

$$\frac{x^2}{4}+\frac{y^2}{9}=1.$$

¿From the first two equations, we get

$$\lambda\frac{x}{2}=3\cdot\lambda\frac{2y}{9},$$

thus $\lambda = 0$, or $y = \frac{3}{4}x$.

If $\lambda = 0$, then $3x + y - 9 = 0$ and thus we look for the intersection of the ellipse with the line p, that we know is empty.

Thus, it remains the case $y = \frac{3}{4}x$, that, after substituting into the equation of the ellipse, gives the equation:

$$1 = \frac{x^2}{4} + \frac{(\frac{3}{4}x)^2}{9} = \frac{5}{16}x^2$$

thus the points $(x, y) = \pm\left(\frac{4}{\sqrt{5}}, \frac{3}{\sqrt{5}}\right)$. At these points, the function f, distance from the line, attains values $\frac{9-3\sqrt{5}}{\sqrt{10}}$ and $\frac{9+3\sqrt{5}}{\sqrt{10}}$.

(2) We may use an "intuitive" approach, that is just a similar version of the previous method:

Proposition: If the set M (defined by the equation) is

- *closed,*
- *bounded* and
- *has a tangent at every point,*

then, at points of M, that are closest or farthest from p, the tangent line must be parallel to p .

In our case, the ellipse M is a level surface of the function $\Phi(x, y) := \frac{x^2}{4} + \frac{y^2}{9} - 1$, so, the normal vector orthogonal to the tangent at the point $a = (x, y) \in M$ is the gradient of the function Φ. We are looking for points $a = (x, y) \in M$, at which the normal vector to M is a multiple of the orthogonal vector to the line p. Then, there exists $\lambda \in \mathbb{R}$ so that

$$\left(\frac{x}{2}, \frac{2y}{9}\right) = \text{grad}(\Phi)_{|a} = \lambda \cdot (3, 1)$$

and

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

It is not a surprise that, from the first condition, we again get the equation $y = \frac{3}{4}x$ and thus the same solutions as in the previous case.

Observation: Let's analyze what could happen if we had not guaranteed all the previously mentioned assumptions on the set M, for whose points we evaluate the distance from the line p with the method of the tangent:

- if M has tangent at every point and is bounded, but *IS NOT closed*: as such M, it is enough to take our ellipse, from which we have taken away exactly those extreme points (even if, formally, from our method we would get them, those extrema are not properly contained in the set).
- if M has tangent at every point and is closed, but *IS NOT bounded*: as such M, it is enough to take an hyperbola with asymptote p (now, no extrema can exist).
- M is bounded and closed, but *DOES NOT admit a tangent at every point*: as such M, it is enough to take a properly twisted triangle) (extrema now exist, but with the tangent method we will not find them).

(3) We may use a method that can be applied to find the distance to a general body in the plane (or in space). It is generally harder to solve the final equations, but, in our case, there will be no problem.

Let's consider the function distance (squared) of two points (x, y) and (u, v), as

$$h(x, y, u, v) = (x - u)^2 + (y - v)^2$$

and we will look for its extrema under the conditions (constraints) $\frac{x^2}{4}+\frac{y^2}{9}=1$ and $3u+v-9=0$. Since one of the conditions defines a non-bounded set (it is actually the line p), then the maximum of the function does not exist, and the method is used just to find the minimum (we will further justify this statement).

We have two constraints

$$\Phi_1(x,y,u,v)=\frac{x^2}{4}+\frac{y^2}{9}-1$$

and

$$\Phi_2(x,y,u,v)=3u+v-9$$

with gradients

$$\mathrm{grad}(\Phi_1)_{|a}=\left(\frac{x}{2},\frac{2y}{9},0,0\right)$$

$$\mathrm{grad}(\Phi_2)_{|a}=(0,0,3,1)$$

where $a=(x,y,u,v)$. Let's indicate

$$K=\{a\in\mathbb{R}^4 \mid \Phi_1(a)=0 \;\&\; \Phi_2(a)=0\}.$$

For points $a\in K$, the gradients are evidently linearly independent, and for points that are extrema of f on K there exist $\lambda,\mu\in\mathbb{R}$ so that

$$\Big(2(x-u),2(y-v),2(u-x),2(v-y)\Big)=\mathrm{grad}(h)_{|a}=\lambda\cdot\left(\frac{x}{2},\frac{2y}{9},0,0\right)+\mu\cdot(0,0,3,1)$$

and

$$\frac{x^2}{4}+\frac{y^2}{9}=1 \quad\text{and}\quad 3u+v-9=0$$

(that is, we have 6 equations in 6 variables!). Fortunately, the equations are rather simple. Subsequently, we get

$$\lambda\frac{x}{2}=2(x-u)=-3\mu$$

$$\lambda\frac{2y}{9}=2(y-v)=-\mu$$

thus again the equation $\lambda\left(\frac{x}{2}-\frac{2y}{3}\right)=0$, where the case $\lambda=0$ again has no solution. the rest, again leads toZ

$$(x_1,y_1)=\left(\frac{4}{\sqrt{5}},\frac{3}{\sqrt{5}}\right)$$

$$(x_2,y_2)=-\left(\frac{4}{\sqrt{5}},\frac{3}{\sqrt{5}}\right)$$

and with the help of the equation $x-u=3(y-v)$ we calculate the corresponding points on the line

$$(u_1,v_1)=\left(\frac{27\sqrt{5}-9}{10\sqrt{5}},\frac{9\sqrt{5}+27}{10\sqrt{5}}\right)$$

$$(u_2,v_2)=\left(\frac{27\sqrt{5}+9}{10\sqrt{5}},\frac{9\sqrt{5}-27}{10\sqrt{5}}\right).$$

For the values of the function at points $a_i=(x_i,y_i,u_i,v_i)$, we have

$$h(a_1)<h(a_2).$$

The set given by the condition K is closed, but it IS NOT bounded. On the other hand, for $a \in K$ and $\|a\| \to \infty$ also the values $h(a)$ go to infinity (because the ellipse is bounded). Thus, it is enough to take a sufficiently big ball B so that, on the set $K \cap (\mathbb{R}^4 \setminus B)$, the values of h are bigger than, for example, $h(a_2) + 1$. Moreover:

- On the closed and now bounded set $K \cap B$, the continuous function h attains its maximum and minimum.
- On the set $K \cap \partial B$, the values of the function h are bigger or equal to the value $h(a_2) + 1$ (thanks to the continuity of h and thanks to its values on $K \cap (\mathbb{R}^4 \setminus B)$).
- On the set $K \cap B^\circ$ (due to the fact that the set B° is open) then, we can (and we have actually done it already) use the usual method of finding the extrema using Lagrange multipliers. The result are the points a_1 and a_2 (that must evidently lie in $K \cap B^\circ$ due to the value of the function there: $h(a_1) < h(a_2) < h(a_2) + 1$).
- The absolute minimum of the function h on the set $K \cap B$, thus, CANNOT occur on the "border" of $K \cap \partial B$ because there the function is "too big" and it can only be at the point a_1. At the same time, also on the set $K \cap (\mathbb{R}^4 \setminus B)$ the function is "too big", and the point a_1 is really the point of absolute minimum of the function h on the original set K.

This is how a correct proof must look like, when we must justify that the found point is a point of minimum when the set given by the condition is not bounded. On the other hand, we observe that the function "at infinity grows to infinity".

And what about point a_2? In order to find out what is going on at this point, we would need a further investigation, involving higher derivatives. Intuitively, it seems that the function has a saddle there, but it would be complicated to try to prove it. The third method is useful really just to find the distance from the set (the minimum of the function h).

Exercise 5.8. *Find the distance of the parabola* $M: \ y = x^2$ *from the line* $p: \ y = x - 2$.

Solution:
We can use any of the previous methods, but we must realize that a parabola is not a bounded set (even if it is closed and has a tangent at every point). Fortunately, the function representing the distance of points on the parabola M from the line p, even now, "at infinity grows to infinity". The minimum of the function distance must be attained at a point of M, and there the tangent line must be parallel to the line p.

The angular coefficient $\alpha \in \mathbb{R}$ of the tangent at the point $a \in M$, that is the graph of the function $g(x) = x^2$, can be found using the derivative of this real function , i.e. $\alpha = \frac{d}{dx}(x^2) = 2x$. The angular coefficient of the line p is 1. Thus, from $2x = 1$ we get $x = \frac{1}{2}$ and, therefor, $y = x^2 = \frac{1}{4}$.

The distance ρ of the point $(x, y) = \left(\frac{1}{2}, \frac{1}{4}\right) \in M$ from the line $p: \ x' - y' - 2 = 0$ is, from the general formula:

$$\rho = \frac{|x - y - 2|}{\sqrt{1^2 + 1^2}} = \frac{|\frac{1}{2} - \frac{1}{4} - 2|}{\sqrt{2}} = \frac{7\sqrt{2}}{8}.$$

Exercise 5.9. *Find the smallest and the biggest value of the function* $f(x, y) = 2x - y + 1$ *with constraint* $x^2 + 2x + y^2 = 0$.

Solution:
We use the method of Lagrange multipliers for the circle $M = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = 0\}$, where $g(x,y) = x^2 + 2x + y^2 = (x+1)^2 + y^2 - 1$. For an extreme value $a = (x,y)$ in M there exists $\lambda \in \mathbb{R}$, so that

$$(2,-1) = f'_{|a} = \lambda g'_{|a} = \lambda \cdot \Big(2(x+1), 2y\Big)$$

and

$$(x+1)^2 + y^2 = 1.$$

We evaluate x and y as functions of λ and we substitute into the constraint. We get $\lambda = \pm\frac{\sqrt{5}}{2}$ and candidates for extrema are:

$$\Big(\frac{2\sqrt{5}}{5} - 1, -\frac{\sqrt{5}}{5}\Big), \quad \Big(-\frac{2\sqrt{5}}{5} - 1, \frac{\sqrt{5}}{5}\Big)$$

with values

$$f\Big(\frac{2\sqrt{5}}{5} - 1, -\frac{\sqrt{5}}{5}\Big) = \sqrt{5} - 1, \quad f\Big(-\frac{2\sqrt{5}}{5} - 1, \frac{\sqrt{5}}{5}\Big) = -\sqrt{5} - 1.$$

The set M is closed and bounded and the continuous function f at those points really attains its maximum and minimum.

Exercise 5.10. *Find the maximum and minimum value of the function* $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = x^2 - y^2 + 2xy$$

on the circle $x^2 + y^2 = 4$.

Solution:
We use the method of Lagrange multipliers for the circle $M = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = 0\}$, where $g(x,y) = x^2 + y^2 - 4$. For the extremes $a = (x,y)$ on M there exists $\lambda \in \mathbb{R}$, so that

$$(2x + 2y, -2y + 2x) = f'_{|a} = \lambda g'_{|a} = \lambda \cdot (2x, 2y)$$

and

$$x^2 + y^2 = 4.$$

These equations thus indicate that we are looking for a vector $\mathbf{a} = (x,y)^T$ so that $\|\mathbf{a}\| = 2$ and

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This system has non-trivial solutions only if the determinant of the matrix associated to the system is equal to zero, i.e. $-(1-\lambda)(1+\lambda) - 1 = 0$, thus $\lambda = \pm\sqrt{2}$. We are actually searching for the eigenvalues of the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ and its eigenvectors with norm 2.

For $\lambda = \sqrt{2}$ we get:

$$a_1 = \pm\sqrt{2+\sqrt{2}} \ \cdot \ (1, \sqrt{2} - 1)$$

and function value equal to

$$f(a_1) = 4\sqrt{2}.$$

For $\lambda = -\sqrt{2}$ we get:

$$a_2 = \pm\sqrt{2-\sqrt{2}} \ \cdot \ (-1, \sqrt{2} + 1)$$

and function value

$$f(a_2) = -4\sqrt{2}.$$

The set M is closed and bounded and the continuous function f at those points really attains its maximum and minimum.

Exercise 5.11. *On the plane $2x + y - z = 1$ find a point such that the sum of the squared values of its distance from the points $A = (1, 1, 1)$ and $B = (2, 3, 4)$ is minimal.*

Solution:
We use the method of Lagrange multipliers for the plane $M = \{(x, y, z) \in \mathbb{R}^3 \mid g(x, y, z) = 0\}$, wher $g(x, y, z) = 2x + y - z - 1$ and the function

$$f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2 + (x-2)^2 + (y-3)^2 + (z-4)^2$$

representing the sum of the square of the distance of the point (x, y, z) from the point $A = (1, 1, 1)$ and $B = (2, 3, 4)$. For an extreme $a = (x, y, z)$ on M there exists $\lambda \in \mathbb{R}$, so that

$$\Big(2(x-3), 2(y-4), 2(z-5)\Big) = f'_{|a} = \lambda g'_{|a} = \lambda \cdot (2, 1, -1)$$

and

$$2x + y - z = 1.$$

We get $\lambda = -\frac{4}{3}$ and $a = (\frac{5}{3}, \frac{10}{3}, \frac{17}{3})$ and the function has there value $f(\frac{5}{3}, \frac{10}{3}, \frac{17}{3}) = \frac{92}{3}$.

To be sure that the function attains its minimum at this point, we would need (apart from the closeness of M) its boundedness, that we don't have. We thus make the following estimate. For $U \in \mathbb{R}^3$, from the triangular inequality, we have

$$f(U) = \|U - A\|^2 + \|U - B\|^2 \geq \Big(\|U\| - \|A\|\Big)^2 + \Big(\|U\| - \|B\|\Big)^2 \to +\infty$$

for $\|U\| \to +\infty$. Thus, there exists $K > 0$ so that, for every $U \in \mathbb{R}^3$ satisfying $\|U\| \geq K$ we have $f(U) \geq f(a) + 1 = \frac{92}{3} + 1$.

Therefore:

- on the set $M_1 = M \cap \{U \in \mathbb{R}^3 \mid \|U\| \geq K\}$ the function has always value at least $f(a) + 1$.

- on the set $M_2 = M \cap \{U \in \mathbb{R}^3 \mid \|U\| \leq K\}$, that is closed and bounded, the function attains its minimum. This cannot be attached to the edge (where the value of the function is again at least $f(a) + 1$), therefore it can only be in the found point $a = (\frac{5}{3}, \frac{10}{3}, \frac{17}{3})$, that must necessary lie, due to its function value $f(a)$, on the set M_2.

We conclude that the function f actually gains in M its (unique) minimum value at the point $a = (\frac{5}{3}, \frac{10}{3}, \frac{17}{3})$.

Exercise 5.12. *Find the minimum and maximum value of the function $f(x, y) = 6 - 4x - 3y$ with constraint $x^2 + y^2 = 4y - 2x$.*

Solution:
The constraint $(x + 1)^2 + (y - 2)^2 = 5$ is the equation of a circle. We use the method of Lagrange multipliers. For an extreme $a = (x, y)$ on the circle, there exists $\lambda \in \mathbb{R}$, so that

$$(-4, -3) = f'_{|a} = \lambda \Phi'_{|a} = \lambda \cdot \Big(2(x+1), 2(y-2)\Big)$$

and

$$(x+1)^2 + (y-2)^2 = 5.$$

We express $x+1 = -\frac{2}{\lambda}$ and $y-2 = -\frac{3}{2\lambda}$ and substitute into the remaining equation. We get $\lambda = \pm\frac{\sqrt{5}}{2}$ and candidates to be extremes of the function are:

$$\Big(-\frac{4\sqrt{5}}{5} - 1, -\frac{3\sqrt{5}}{5} + 2\Big), \qquad \Big(\frac{4\sqrt{5}}{5} - 1, \frac{3\sqrt{5}}{5} + 2\Big)$$

with corresponding values

$$5\sqrt{5} + 4, \qquad -5\sqrt{5} + 4.$$

The circle is closed and bounded and the function is continuous, thus the function attains its minimum and maximum at these points.

6 Double integrals

Exercise 6.1. *Change the order of integration in the following integrals:*

(i) $\int_0^{\pi}\int_0^{\sin x} f(x,y)\ dy\ dx$

(ii) $\int_0^1\int_0^x f(x,y)\ dy\ dx + \int_1^2\int_0^{2-x} f(x,y)\ dy\ dx$

(iii) $\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\ dy\ dx$*, where* $a>0$ *is a parameter.*

Solution:
We recall **Fubini's theorem:** Let

- $D\subseteq\mathbb{R}^2$ be the region of integration (i.e. the set on which it has a meaning to consider the integral of a given function, for example the region can be enclosed by the graph of some continuous functions),
- $f : D\to\mathbb{R}$ is a measurable function (i.e. a function that can be integrated, for example, a continuous function) and
- the double integral of the absolute value of f is finite, i.e. $\iint_D |f|\ dS < \infty$ (for example if the function is bounded).

Then, the double integral $\iint_D f\ dS$ exists and the following holds:

$$\int_{\pi_2(D)}\Big(\int_{(\mathbb{R}\times\{y\})\cap D} f(x,y)\ dx\Big)\ dy = \iint_D f\ dS = \int_{\pi_1(D)}\Big(\int_{(\{x\}\times\mathbb{R})\cap D} f(x,y)\ dy\Big)\ dx,$$

where $\pi_1,\pi_2:\mathbb{R}^2\to\mathbb{R}$ are the projection on the single axis, thus $\pi_1(x,y)=x$, and $\pi_2(x,y)=y$.

Observation: The assumption that the integral of the absolute value of the function is finite, is a fundamental assumption! For example, for the function

$$f(x,y)=\frac{x^2-y^2}{(x^2+y^2)^2}$$

on $D=\langle 0,1\rangle\times\langle 0,1\rangle\setminus\{(0,0)\}$ we have

$$\int_0^1\Big(\int_0^1\frac{x^2-y^2}{(x^2+y^2)^2}\,dy\Big)\ dx=\int_0^1\left[\frac{y}{x^2+y^2}\right]_{y=0}^{y=1} dx=\int_0^1\frac{1}{x^2+1}\,dx=[\arctan(x)]_0^1=\frac{\pi}{4}$$

and

$$\int_0^1\Big(\int_0^1\frac{x^2-y^2}{(x^2+y^2)^2}\,dx\Big)\ dy=\int_0^1\left[\frac{-x}{x^2+y^2}\right]_{x=0}^{x=1} dy=\int_0^1\frac{-1}{y^2+1}\,dy=[-\arctan(y)]_0^1=-\frac{\pi}{4}$$

but

$$\int_0^1\int_0^1\left|\frac{x^2-y^2}{(x^2+y^2)^2}\right|dx\,dy=\infty.$$

In this example, we just exercise on the interchange of the order of integration, thus, we suppose that the function f satisfies the assumptions of Fubini's theorem.

(i) The region of integration is

$$D=\{(x,y)\in\mathbb{R}^2\mid 0\le x\le\pi\ \&\ 0\le y\le\sin x\}.$$

We have

$$\pi_1(D) = \langle 0, \pi \rangle$$

and

$$(\{x\} \times \mathbb{R}) \cap D = \langle 0, \sin x \rangle.$$

After the change of order of integration, we have

$$\pi_2(D) = \langle 0, 1 \rangle$$

and, from the inequality $y \leq \sin x$, we derive:

$$\arcsin y \leq \arcsin(\sin x) = \begin{cases} x & , \ x \in \langle 0, \frac{\pi}{2}) \\ \pi - x & , \ x \in \langle \frac{\pi}{2}, \pi \rangle \end{cases}$$

Attention! arcsin and sin are each inverse function of the other, only for angles in the interval $\langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$. We use the fact that $\sin x = \sin(\pi - x)$ and, for $x \in \langle \frac{\pi}{2}, \pi \rangle$, is then $\pi - x \in \langle -\frac{\pi}{2}, 0 \rangle$, thus

$$\arcsin(\sin x) = \arcsin(\sin(\pi - x)) = \pi - x.$$

Therefor, we get $\arcsin y \leq x \leq \pi - \arcsin y$, from which

$$(\mathbb{R} \times \{y\}) \cap D = \langle \arcsin y, \pi - \arcsin y \rangle.$$

The integral then looks like the following:

$$\int_0^{\pi} \int_0^{\sin x} f(x, y) \ dy \ dx = \int_0^1 \int_{\arcsin y}^{\pi - \arcsin y} f(x, y) \ dx \ dy.$$

(ii) The regions of integration are

$$D_1 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \ \& \ 0 \leq y \leq x\}$$

$$D_2 = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 2 \ \& \ 0 \leq y \leq 2 - x\}.$$

The sets D_1 and D_2 intersects only on the segment $\{1\} \times \langle 0, 1 \rangle$, that does not effect the value of the integral. The function f can be integrated on the union of the two regions $D = D_1 \cup D_2$.

Attention! This union is not an obvious thing! If the two region of integration intersects on a "more consistent" set, we would need to integrate twice the function on the intersection (a contribution from each of the regions D_i). More precisely, the following holds

$$\iint_{D_1} f \ dS \ + \ \iint_{D_2} f \ dS = \iint_{D_1 \setminus D_2} f \ dS \ + \ 2 \cdot \iint_{D_1 \cap D_2} f \ dS \ + \ \iint_{D_2 \setminus D_1} f \ dS$$

Thus $\pi_2(D) = \langle 0, 1 \rangle$ and $(\mathbb{R} \times \{y\}) \cap D = \langle y, 2 - y \rangle$. After the change of order of integration, the integral becomes:

$$\int_0^1 \int_0^x f(x, y) \ dy \ dx + \int_1^2 \int_0^{2-x} f(x, y) \ dy \ dx = \int_0^1 \int_y^{2-y} f(x, y) \ dx \ dy.$$

(iii) The region of integration is

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2a \ \& \ \sqrt{2ax - x^2} \leq y \leq \sqrt{2ax}\}.$$

The region is bounded above by the parabola $y^2 = 2ax$ and below by half of the circle $2ax - x^2 = y^2$ (or, equivalently $(x-a)^2 + y^2 = a^2$). We divide the region in three parts

$$D_1 := D \cap \{(x,y) \in \mathbb{R}^2 \mid y \geq a\}$$

$$D_2 := D \cap \{(x,y) \in \mathbb{R}^2 \mid y \leq a \;\;\&\;\; x \leq a\}$$

and

$$D_3 := D \cap \{(x,y) \in \mathbb{R}^2 \mid y \leq a \;\;\&\;\; x \geq a\}.$$

Now we express the regions D_i, using cuts parallel to the x-axis (using the curves $y^2 = 2ax$ and $(x-a)^2 + y^2 = a^2$):

D_1 :

$$a \leq y \leq 2a, \quad \frac{y^2}{2a} \leq x \leq a$$

D_2 :

$$0 \leq y \leq a, \quad \frac{y^2}{2a} \leq x \leq a - \sqrt{a^2 - y^2}$$

D_3 :

$$0 \leq y \leq a, \quad a + \sqrt{a^2 - y^2} \leq x \leq 2a$$

Thus, the result is

$$\iint_D f \, dS = \iint_{D_1} f \, dS \;+\; \iint_{D_2} f \, dS \;+\; \iint_{D_3} f \, dS =$$

$$= \int_a^{2a} \int_{\frac{y^2}{2a}}^{a} f(x,y) \, dx \, dy \;+\; \int_0^a \int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} f(x,y) \, dx \, dy \;+\; \int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y) \, dx \, dy \,.$$

Exercise 6.2. *Find the convenient order of integration and evaluate the integral:*

(i) $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \, dy \, dx$

(ii) $\int_0^8 \int_{\sqrt[3]{x}}^{2} \frac{dy \, dx}{y^4+1}$

Solution:
(i) In order to evaluate this integral, it is convenient to change the order of integration. We have

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2 \;\&\; 0 \leq y \leq 4 - x^2 \;\&\; y \neq 4\}.$$

The function $f(x,y) = \frac{xe^{2y}}{4-y}$ is not bounded on the set D.

(**Explanation:** The set D is bounded by the parabola $y = 4 - x^2$. We may, nevertheless, consider a different parabola that will lie in the interior of D, thus we consider a suitable $\lambda > 0$ so that $(x, 4 - \lambda x^2) \in D$. Then, we have

$$\lim_{\substack{(x,y)\to(0,0) \\ y=4-\lambda x^2,\; x>0}} f(x,y) = \lim_{x\to 0+} \frac{xe^{2(4-\lambda x^2)}}{\lambda x^2} = +\infty.)$$

It is not clear if $\iint\limits_D |f|\ dS < \infty$ and if, therefore, we may use Fubini's theorem on the change of order of integration. We will use a different statment:

Theorem: Let's consider

- $D \subseteq \mathbb{R}^2$, a region of integration
- $f : D \to \mathbb{R}$, a *non-negative* measurable function and
- suppose that one of the iterated integral, evaluated with a certain order of integration, is finite

Then, also the integral in the reverse order of integration is finite, and it has the same value (and the function has a double integral $\iint\limits_E f\ dS$).

Moreover, we remind the definition of the integral $\iint\limits_E f\ dS$, if the function f or the region of integration E is *unbounded*. Then, the integral is given by a (finite) value, only if it is (by definition) *absolute convergent*, i.e. if

$$\lim_{n\to\infty} \iint\limits_{E_n} |f|\ dS =: \iint\limits_E |f|\ dS < \infty$$

for a certain sequence of bounded regions $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq E_{n+1} \subseteq \cdots$ such that f on E_n is bounded and $E = \cup_n E_n$. In this case, for the integral, we may use Fubini's theorem (in an analogous way like for bounded functions on a bounded region of integration).

Thus we have

$$\pi_2(D) = \langle 0, 4)$$

and

$$(\mathbb{R} \times \{y\}) \cap D = \langle 0, \sqrt{4-y}\rangle.$$

We now get

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\ dy\ dx = \int_0^4 \Big(\int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\ dx \Big)\ dy = \int_0^4 \Big[\frac{x^2 e^{2y}}{2(4-y)} \Big]_{x=0}^{x=\sqrt{4-y}} dy =$$

$$= \int_0^4 \frac{e^{2y}}{2} dy = \Big[\frac{e^{2y}}{4} \Big]_{y=0}^{y=4} = \frac{e^8 - 1}{4}.$$

(ii) Again, it is convenient to change the order of integration. We have

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 8\ \&\ \sqrt[3]{x} \le y \le 2\},$$

thus

$$\pi_2(D) = \langle 0, 2\rangle$$

and

$$(\mathbb{R} \times \{y\}) \cap D = \langle 0, y^3\rangle.$$

We now get

$$\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy\ dx}{y^4+1} = \int_0^2 \Big(\int_0^{y^3} \frac{1}{y^4+1}\ dx \Big)\ dy = \int_0^2 \frac{y^3}{y^4+1} dy =$$

$$= \Big[\frac{\ln(y^4+1)}{4} \Big]_{y=0}^{y=2} = \frac{\ln 17}{4}.$$

Exercise 6.3. *Evaluate the integral*

$$\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2)\ dx\ dy$$

changing it to polar coordinate.

Solution:
The region of integration is

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 2\ \&\ 0 \le x \le \sqrt{4 - y^2}\} =$$
$$= \{(x, y) \in \mathbb{R}^2 \mid 0 \le x, y\ \&\ x^2 + y^2 \le 4\},$$

thus a quarter of a disk.

We use the **Theorem on substitution:** Let $U \subseteq \mathbb{R}^2$ be a region of integration, $f : U \to \mathbb{R}$ be a measurable function with a finite integral and $\Phi : U \to \mathbb{R}^2$ be a map (called *parametrization*). Moreover, suppose that

- Φ is one to one and continuously differentiable on U° (i.e. on the interior of U) and
- the set $U \setminus U^\circ$ ($\subseteq \partial U$) has *zero measure* (i.e. its contribution to the value of any integral is zero; usually it could be a curve, a segment, and so on, with zero area).

Then

$$\iint_{\Phi(U)} f\ dS = \iint_U (f \circ \Phi) \cdot |\det \Phi'|\ dS.$$

Polar coordinates are determined by the map $\Phi : \langle 0, +\infty) \times \langle 0, 2\pi) \to \mathbb{R}^2$, where $\Phi\begin{pmatrix} r \\ \varphi \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$.

We have $\Phi'_{|(r,\varphi)} = \left(\begin{smallmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{smallmatrix}\right)$ and $\det \Phi'_{|(r,\varphi)} = r$. Since D is a quarter of a disk, we can easily parametrize it $D = \Phi(U)$, where $U = \langle 0, 2\rangle \times \langle 0, \frac{\pi}{2}\rangle$. On D°, Φ is obviously one to one and continuously differentiable, and the set ∂D is formed by two segments and an arch of a circle, that are sets with zero measure. Thus, we get:

$$\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2)\ dx\ dy = \iint_{D=\Phi(U)} (x^2 + y^2)\ dS =$$

$$= \iint_U r^2 \cdot r\ dS = \int_0^{\frac{\pi}{2}} \Big(\int_0^2 r^3\ dr \Big)\ d\varphi = \int_0^{\frac{\pi}{2}} \Big[\frac{r^4}{4} \Big]_{r=0}^{r=2}\ d\varphi = \int_0^{\frac{\pi}{2}} 4\ d\varphi = 2\pi.$$

Exercise 6.4. *Evauate the integral*

$$\iint_D \sqrt{x^2 + y^2}\ dS$$

using polar coordinates, where D is bounded by the curve $r = 1 + \cos\varphi$.

Solution:
In polar coordinates, the region is given as

$$U = \{(r, \varphi) \in \mathbb{R}^2 \mid 0 \le \varphi \le 2\pi \ \&\ 0 \le r \le 1 + \cos\varphi\}$$

we set $D := \Phi(U)$ (that is in Cartesian coordinates a region bounded by a curve called *cardioid*). Using the theorem on substitution, we get

$$\iint\limits_{D=\Phi(U)} \sqrt{x^2+y^2}\ dS = \iint\limits_U r \cdot r\ dS =$$

$$= \int_0^{2\pi} \Big(\int_0^{1+\cos\varphi} r^2\ dr \Big)\ d\varphi = \int_0^{2\pi} \Big[\frac{r^3}{3} \Big]_{r=0}^{r=1+\cos\varphi}\ d\varphi = \frac{1}{3} \int_0^{2\pi} (1+\cos\varphi)^3\ d\varphi =$$

$$= \frac{1}{3} \int_0^{2\pi} (\cos^3\varphi + 3\cos^2\varphi + 3\cos\varphi + 1)\ d\varphi = \pi + \frac{2}{3}\pi = \frac{5}{3}\pi.$$

For the single integrals, we have used the relations (for $n \ge 0$):

$$\int_0^{2\pi} \cos^{2n+1}\varphi\ d\varphi = \int_{0-\frac{\pi}{2}}^{2\pi-\frac{\pi}{2}} \cos^{2n+1}\varphi\ d\varphi = \Big[\alpha = \varphi - \frac{\pi}{2}\Big] =$$

$$= \int_{-\pi}^{\pi} \cos^{2n+1}(\alpha + \frac{\pi}{2})\ d\alpha = -\int_{-\pi}^{\pi} \sin^{2n+1}\alpha\ d\alpha = 0$$

(that follows from the periodicity of the function and its property of being odd). Similarly, since

$$\int_0^{2\pi} \cos^2\varphi\ d\varphi = \int_0^{2\pi} \sin^2\varphi\ d\varphi$$

and, at the same time

$$\int_0^{2\pi} (\cos^2\varphi + \sin^2\varphi)\ d\varphi = \int_0^{2\pi} 1\ d\varphi = 2\pi$$

we have

$$\int_0^{2\pi} \cos^2\varphi\ d\varphi = \frac{2\pi}{2} = \pi.$$

Exercise 6.5. *Evaluate the integral*

$$\iint\limits_E xe^{-y} \frac{\sin y}{y}\ dS$$

for the unbounded region $E = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le \frac{y}{2}\}$.

Solution:
We repeat the definition of the integral $\iint\limits_E f\ dS$, if the function f or the region of integration E is *unbounded.* Then, the integral is defined by a (finite) value, only if it is (by definition) *absolute convergent*, i.e. if

$$\lim_{n\to\infty} \iint\limits_{E_n} |f|\ dS =: \iint\limits_E |f|\ dS < \infty$$

for a certain sequence of bounded regions $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq E_{n+1} \subseteq \cdots$ such that f on E_n is bounded and $E = \cup_n E_n$.

In our case, we have $f(x,y) = xe^{-y}\frac{\sin y}{y}$. First of all, we have to verify that the given integral exists, i.e. absolutely converges. On E we have

$$\left|xe^{-y}\frac{\sin y}{y}\right| \le xe^{-y}.$$

For this non-negative function we can use Fubini's theorem (if at the end of the integration, we get a finite value):

$$\iint_E xe^{-y}\, dS = \int_0^\infty \int_0^{\frac{y}{2}} xe^{-y}\, dx\, dy = \int_0^\infty \frac{y^2}{8}e^{-y}\, dy = \int_0^\infty \left(\frac{y^2}{8}e^{-\frac{y}{2}}\right) e^{-\frac{y}{2}}\, dy \le \int_0^\infty K\cdot e^{-\frac{y}{2}}\, dy < \infty$$

where we have used the fact that $|\frac{y^2}{8}e^{-\frac{y}{2}}| \le K$ for an appropriate constant $K > 0$ (this function is continuous and goes to zero at infinity).

The integral of $|f|$ is thus finite and we can use Fubini's theorem (as for the anagous case of a bounded function on a bounded region of integration). Thus, we can write:

$$\iint_E xe^{-y}\frac{\sin y}{y}\, dS = \int_0^\infty \int_0^{\frac{y}{2}} xe^{-y}\frac{\sin y}{y}\, dx\, dy = \frac{1}{8}\int_0^\infty y(e^{-y}\sin y)\, dy =$$

$$= \left[\begin{matrix} g(y)=y, & g'(y)=1 \\ h'(y)=e^{-y}\sin y, & h(y)=-\frac{e^{-y}}{2}(\cos y+\sin y)\end{matrix}\right] =$$

$$= \frac{1}{8}\Big[y(-\frac{e^{-y}}{2})(\cos y + \sin y)\Big]_0^\infty - \frac{1}{8}\int_0^\infty -\frac{e^{-y}}{2}(\cos y + \sin y)\, dy =$$

$$= \frac{1}{16}\int_0^\infty e^{-y}(\cos y + \sin y)\, dy = \frac{1}{16}\Big[-e^{-y}\cos y\Big]_0^\infty = \frac{1}{16}.$$

Observation: In evaluating the indefinite integral, we have used the following method (easier than several times by parts) based on complex functions:

$$\int e^{-y}\cos y\, dy + \mathbf{i}\int e^{-y}\sin y\, dy = \int e^{-y}(\cos y + \mathbf{i}\sin y)\, dy = \int e^{-y}e^{\mathbf{i}y}\, dy =$$

$$= \int e^{(\mathbf{i}-1)y}\, dy = \frac{e^{(\mathbf{i}-1)y}}{\mathbf{i}-1} + C = -\frac{\mathbf{i}+1}{2}e^{-y}(\cos y + \mathbf{i}\sin y) + C =$$

$$= \left[-\frac{e^{-y}}{2}(\cos y - \sin y)\right] + \mathbf{i}\left[-\frac{e^{-y}}{2}(\cos y + \sin y)\right] + C.$$

Comparing the real and imaginary parts, the we get:

$$\int e^{-y}\cos y\, dy = -\frac{e^{-y}}{2}(\cos y - \sin y) + C_1$$

and

$$\int e^{-y}\sin y\, dy = -\frac{e^{-y}}{2}(\cos y + \sin y) + C_2,$$

where C_1, C_2 and C are constants.

The integral can also be derived in a "heuristic" way: we guess that the primitive function must contain the functions $e^{-y}\cos y$ and $e^{-y}\sin y$. We thus try to take their derivatives:

$$\frac{d}{dy}\left(e^{-y}\cos y\right) = -e^{-y}(\sin y + \cos y)$$

$$\frac{d}{dy}\left(e^{-y}\sin y\right) = -e^{-y}(\sin y - \cos y)$$

and the integral of the given function is found as a linear combination of the equations:

$$e^{-y}\sin y = \frac{d}{dy}\left(-\frac{e^{-y}}{2}(\cos y + \sin y)\right)$$

$$e^{-y}\cos y = \frac{d}{dy}\left(-\frac{e^{-y}}{2}(\cos y - \sin y)\right).$$

Exercise 6.6. *Use the substitution $u = x + 2y$, $v = x - y$ to evaluate the integral*

$$\int_0^{\frac{2}{3}}\int_y^{2-2y} (x+2y)e^{y-x}\ dx\ dy.$$

Solution:
The region of integration is

$$E: \quad 0 \le y \le \frac{2}{3}, \qquad y \le x \le 2-2y.$$

It is a triangle with vertexes $(0,0)$, $(\frac{2}{3},\frac{2}{3})$ and $(2,0)$. The substitution Φ is linear, and we are given its inverse:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Phi^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The triangle can be expressed as the *convex hull* of its vertexes (i.e. the smallest convex set containing the given vertexes - given the points $A_1,\ldots,A_n$ the convex hull of $[A_1,\ldots,A_n]_\alpha$ is given by

$$[A_1,\ldots,A_n]_\alpha = \{\sum_{i=1}^{n}\lambda_i A_i \mid 0 \le \lambda_1,\ldots,\lambda_n \ \&\ \sum_{i-1}^{n}\lambda_i = 1\}).$$

Since a (one to one) linear map (Φ) preserves convex hulls, the set U, such that $\Phi(U) = E$, is also given as the convex hull of the vertexes

$$\Phi^{-1}(0,0) = (0,0)$$

$$\Phi^{-1}\left(\frac{2}{3},\frac{2}{3}\right) = (2,0)$$

$$\Phi^{-1}(2,0) = (2,2).$$

Thus

$$U: \quad 0 \le v \le u, \qquad 0 \le u \le 2.$$

Moreover, $(\Phi^{-1})' = \left(\begin{smallmatrix} 1 & 2 \\ 1 & -1 \end{smallmatrix}\right)$, $\det\Phi' = \frac{1}{\det(\Phi^{-1})'} = -\frac{1}{3}$. After substitution, we get

$$\int_0^{\frac{2}{3}}\int_y^{2-2y} (x+2y)e^{y-x}\ dx\ dy = \iint_{E=\Phi(U)} (x+2y)e^{y-x}\ dS = \iint_U ue^{-v}\cdot\frac{1}{3}\ dS =$$

$$= \frac{1}{3}\int_0^2\int_0^u ue^{-v}\ dv\ du = \frac{1}{3}\int_0^2 u\Big[-e^{-v}\Big]_{v=0}^{v=u}\ du = \frac{1}{3}\int_0^2 u(1-e^{-u})\ du = \frac{1}{3}\Big[\frac{u^2}{2} + (u+1)e^{-u}\Big]_0^2 =$$

$$= 1 + e^{-2}.$$

Exercise 6.7. *Evaluate the integral*

$$\iint_E \frac{y}{x} e^{xy}\ dS$$

for the region E in the first quadrant, bounded by the curves $xy = 2$, $xy = 4$, $y = 2x$ and $y = \frac{x}{2}$.

Solution:
The region of integration is

$$E: \quad 0 < x, y, \quad \frac{x}{2} \le y \le 2x. \quad \frac{2}{x} \le y \le \frac{4}{x}$$

or

$$E: \quad 0 < x, y, \quad \frac{1}{2} \le \frac{y}{x} \le 2, \quad 2 \le xy \le 4.$$

Due to the form of the region of integration and the given function, it is convenient to set new variables

$$u = \frac{y}{x} \quad \text{and} \quad v = xy.$$

Setting new variables, requires the existence of the inverse map Φ, that we use for the substitution in our. We should verify if this map Φ exists and if it is one to one - we evaluate the variables x and y in terms of u and v and we get (supposing that $x, y > 0$)

$$x = \sqrt{\frac{v}{u}} \quad \text{and} \quad y = \sqrt{uv}.$$

We thus define the map

$$\Phi : (0, +\infty)^2 \to \mathbb{R}^2, \qquad \Phi(u, v) = \Big(\sqrt{\frac{v}{u}}, \sqrt{uv}\Big)$$

whose inverse is

$$\Phi^{-1} : (0, +\infty)^2 \to \mathbb{R}^2, \qquad \Phi^{-1}(x, y) = \Big(\frac{y}{x}, xy\Big).$$

The determinant of Φ' can be easily evaluated with the use of the inverse map (where the square roots are not present):

$$(\Phi^{-1})' = \begin{pmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ y & x \end{pmatrix}, \qquad \det(\Phi^{-1})' = -2\frac{y}{x}$$

and thus

$$\det \Phi'_{|(u,v)} = \frac{1}{\det(\Phi^{-1})'}_{|\Phi(u,v)} = -\frac{1}{2u}.$$

A parametrization U of the region $E = \Phi(U)$ is

$$U: \quad \frac{1}{2} \le u \le 2, \quad 2 \le v \le 4.$$

Thus, we can write

$$\iint_{E=\Phi(U)} \frac{y}{x} e^{xy}\ dS = \iint_U ue^v \Big| -\frac{1}{2u}\Big|\ dS = \frac{1}{2}\int_2^4\int_{\frac{1}{2}}^2 e^v\ du\ dv = \frac{3}{4}(e^4 - e^2).$$

Exercise 6.8. *Find the mass and the center of mass*

(i) of the triangle with vertexes $(0,0)$, $(1,1)$, $(4,0)$, *whose density is given by the function* $\rho(x,y) = x$.

(ii) of the part of the plane bounded by the parabola $y = 9 - x^2$ *and the* x*-axis, if its density is given by the function* $\rho(x,y) = y$.

Solution:
(i) The region of integration is

$$E: \quad 0 \le y \le 1, \quad y \le x \le 4 - 3y.$$

The required integrals are the following.

Mass:

$$m = \iint_E \rho \, dS = \int_0^1 \Big(\int_y^{4-3y} x \, dx \Big) \, dy = \int_0^1 \Big[\frac{x^2}{2} \Big]_{x=y}^{x=4-3y} \, dy =$$

$$= \frac{1}{2} \int_0^1 (3y-4)^2 - y^2 \, dy = \frac{1}{2} \Big[\frac{(3y-4)^3}{9} - \frac{y^3}{3} \Big]_0^1 = \frac{10}{3}$$

x-coordinate of the center of mass:

$$T_1 = \frac{1}{m} \iint_E x\rho(x,y) \, dS = \frac{3}{10} \int_0^1 \Big(\int_y^{4-3y} x^2 \, dx \Big) \, dy = \frac{1}{10} \int_0^1 \Big[x^3 \Big]_{x=y}^{x=4-3y} \, dy =$$

$$= \frac{1}{10} \int_0^1 (4-3y)^3 - y^3 \, dy = \Big[-\frac{(4-3y)^4}{120} - \frac{y^4}{40} \Big]_0^1 = \frac{21}{10}$$

y-coordinate of the center of mass:

$$T_2 = \frac{1}{m} \iint_E y\rho(x,y) \, dS = \frac{3}{10} \int_0^1 \Big(\int_y^{4-3y} xy \, dx \Big) \, dy = \frac{3}{20} \int_0^1 \Big[x^2 y \Big]_{x=y}^{x=4-3y} \, dy =$$

$$= \frac{3}{20} \int_0^1 y(4-3y)^2 - y^3 \, dy = \frac{12}{5} \int_0^1 y^3 - 3y^2 + 2y \, dy = \frac{12}{5} \Big[\frac{y^4}{4} - y^3 + y^2 \Big]_0^1 = \frac{3}{5}.$$

(ii) The region of integration is

$$E: \quad -3 \le x \le 3, \quad 0 \le y \le 9 - x^2.$$

mass:

$$m = \iint_E \rho \, dS = \int_{-3}^{3} \Big(\int_0^{9-x^2} y \, dy \Big) \, dx = \frac{1}{2} \int_{-3}^{3} \Big[y^2 \Big]_0^{9-x^2} \, dx = \int_0^3 (x^2 - 9)^2 \, dx =$$

$$= \Big[\frac{x^5}{5} - 6x^3 + 81y \Big]_0^3 = \frac{648}{5}$$

x-coordinate of the center of mass:

$$T_1 = \frac{1}{m} \iint_E x\rho(x,y) \, dS = \frac{1}{m} \int_{-3}^{3} x \Big(\int_0^{9-x^2} y \, dy \Big) \, dx = 0$$

(it is an odd function on a set symmetric with respect to the y-axis)

y-coordinate of the center of mass:

$$T_2 = \frac{1}{m} \iint_E y\rho(x,y)\, dS = \frac{1}{m} \int_{-3}^{3} \Big(\int_0^{9-x^2} y^2\, dy \Big)\, dx = \frac{1}{3m} \int_{-3}^{3} \big[y^3 \big]_0^{9-x^2}\, dx =$$

$$= \frac{2}{3m} \int_0^3 (3^2 - x^2)^3\, dx = \frac{5}{3 \cdot 324} \Big[3^6 x - 3^4 x^3 + \frac{3^3}{5} x^5 - \frac{x^7}{7} \Big]_0^3 = \frac{36}{7}.$$

Exercise 6.9. *Evaluating the integral*

$$\int_0^1 \int_{2x}^2 (e^{y^2} + 2xy)\, dy\, dx$$

choose the convenient order of integration.

Solution:
We will change the order of integration. The region of integration is

$$E = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1 \;\&\; 2x \le y \le 2\} = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 2 \;\&\; 0 \le x \le \frac{y}{2}\}.$$

$$\int_0^1 \int_{2x}^2 (e^{y^2} + 2xy)\, dy\, dx = \int_0^2 \int_0^{\frac{y}{2}} (e^{y^2} + 2xy)\, dx\, dy = \int_0^2 \frac{y}{2} e^{y^2} + \frac{y^3}{4}\, dy = \Big[\frac{e^{y^2}}{4} + \frac{y^4}{16} \Big]_0^2 = \frac{e^4 + 3}{4}.$$

Second (much more difficult) method: We use a substitution into polar coordinates Φ

$$\Phi : \begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}, \qquad \det \Phi' = r$$

the region

$$E = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1 \;\&\; 2x \le y \le 2\}$$

with parametrization $E = \Phi(U)$ where

$$U = \{(r,\varphi) \in \mathbb{R}^2 \mid 0 \le r\cos\varphi \le 1 \;\&\; 2r\cos\varphi \le r\sin\varphi \le 2 \;\&\; 0 \le \varphi \le 2\pi\}.$$

After semplification, this can be rewritten as

$$U = \{(r,\varphi) \in \mathbb{R}^2 \mid 0 \le r \le \frac{2}{\sin\varphi} \;\&\; \arctan 2 \le \varphi \le \frac{\pi}{2}\}.$$

Then, we have

$$\iint_{E=\Phi(U)} (e^{y^2} + 2xy)\, dS = \iint_U (e^{r^2\sin^2\varphi} + 2r^2 \sin\varphi\cos\varphi) r\, dS = \int_{\arctan 2}^{\frac{\pi}{2}} \int_0^{\frac{2}{\sin\varphi}} (e^{r^2\sin^2\varphi} + 2r^2\sin\varphi\cos\varphi) r\, dr\, d\varphi =$$

$$= \int\limits_{\arctan 2}^{\frac{\pi}{2}} \left[\frac{e^{r^2 \sin^2 \varphi}}{2\sin^2\varphi} + \frac{r^4 \sin\varphi \cos\varphi}{2}\right]_{r=0}^{r=\frac{2}{\sin\varphi}} d\varphi = \int\limits_{\arctan 2}^{\frac{\pi}{2}} \frac{e^4 - 1}{2\sin^2\varphi} + \frac{8\cos\varphi}{\sin^3\varphi}\, d\varphi =$$

$$= \left[\frac{1-e^4}{2}\operatorname{cotg}(\varphi) - \frac{4}{\sin^2\varphi}\right]_{\arctan 2}^{\frac{\pi}{2}} = -4 - \frac{1-e^4}{2}\frac{1}{\tan(\arctan 2)} + 4\Big(1 + \frac{1}{\big(\tan(\arctan 2)\big)^2}\Big) = \frac{e^4+3}{4}.$$

Exercise 6.10. *Using a suitable substitution, evaluate the integral*

$$\iint\limits_E e^{y^3} \sqrt{xy - y^2}\; dS,$$

where the region of integration E is bounded by the lines $y = x$, $x = 10y$ a $y = 1$.

Solution:
The region E is a triangle with vertices $(0,0)$, $(1,1)$ and $(10,1)$ and for the expression under the square root, we thus have $xy - y^2 = y(x-y) \geq 0$. At first sight, it seems easier to integrate first with respect to x.

$$E: \quad 0 \leq y \leq 1 \quad \& \quad y \leq x \leq 10y$$

$$\iint\limits_E e^{y^3}\sqrt{xy-y^2}\; dS = \int\limits_0^1 \int\limits_y^{10y} e^{y^3}\sqrt{xy-y^2}\; dx\; dy = \int\limits_0^1 \left[e^{y^3}\frac{2}{3}\cdot\frac{(xy-y^2)^{3/2}}{y}\right]_{x=y}^{x=10y} dy =$$

$$= \int\limits_0^1 e^{y^3}\frac{2}{3}\cdot\frac{(9y^2)^{3/2}}{y}\; dy = 6\int\limits_0^1 3y^2 e^{y^3}\; dy = 6\Big[e^{y^3}\Big]_{y=0}^{y=1} = 6(e-1)\;.$$

Exercise 6.11. *Evaluate the integral*

$$\iint\limits_E x\cos(x^2+y)\; dS$$

where $E: -\sqrt{\pi} \leq x \leq 0,\; 0 \leq y \leq \pi$, using the convenient order of integration.

Solution:
The region E is a rectangle. We can integrate using either order of integration:

$$\iint\limits_E x\cos(x^2+y)\; dS = \int\limits_0^{\pi}\int\limits_{-\sqrt{\pi}}^{0} x\cos(x^2+y)\; dx\; dy = \int\limits_0^{\pi} \left[\frac{\sin(x^2+y)}{2}\right]_{x=-\sqrt{\pi}}^{x=0} dy =$$

$$= \frac{1}{2}\int\limits_0^{\pi} \sin(y) - \sin(y+\pi)\; dy = \int\limits_0^{\pi} \sin(y)\; dy = \Big[-\cos(y)\Big]_{y=0}^{y=\pi} = 2$$

or

$$\iint\limits_E x\cos(x^2+y)\; dS = \int\limits_{-\sqrt{\pi}}^{0}\int\limits_0^{\pi} x\cos(x^2+y)\; dy\; dx = \int\limits_{-\sqrt{\pi}}^{0} \Big[x\sin(x^2+y)\Big]_{y=0}^{y=\pi} dx =$$

$$= \int_{-\sqrt{\pi}}^{0} x\sin(x^2+\pi) - x\sin(x^2)\ dx = \int_{-\sqrt{\pi}}^{0} -2x\sin(x^2)\ dx = \Big[\cos(x^2)\Big]_{x=-\sqrt{\pi}}^{x=0} = 2\ .$$

Exercise 6.12. *Using a convenient method of integration, evaluate the integral*

$$\iint_E x^2 \cdot |y|\ dS,$$

where the region $E : x^2 + y^2 \leq 2$.

Solution:
The region of integration is a disk of radius $\sqrt{2}$. We will use polar coordinates:

$$x = r\cos\varphi, \quad y = r\sin\varphi$$

$$0 \leq r \leq \sqrt{2}, \quad 0 \leq \varphi \leq 2\pi$$

$$\iint_E x^2 \cdot |y|\ dS = \int_0^{\sqrt{2}} \int_0^{2\pi} r^2 \cos^2\varphi \cdot |r\sin\varphi| \cdot r\ d\varphi\ dr = \Big(\int_0^{\sqrt{2}} r^4\ dr\Big) \cdot \Big(\int_0^{2\pi} \cos^2\varphi |\sin\varphi|\ d\varphi\Big) =$$

$$= \frac{(\sqrt{2})^5}{5} \cdot \Big(2\int_0^{\pi} \cos^2\varphi \sin\varphi\ d\varphi\Big) = \frac{8\sqrt{2}}{5} \cdot \Big[-\frac{\cos^3\varphi}{3}\Big]_{\varphi=0}^{\varphi=\pi} = \frac{16\sqrt{2}}{15}\ .$$

Exercise 6.13. *Evaluate the integral*

$$\iint_E e^{\frac{x}{y}}\ dS,$$

where E *is the region in the first quadrant bounded by the curves* $x = y^2$, $x = 0$ *and* $y = 1$.

Solution:
The region of integration

$$E = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq y^2\ \&\ 0 < y \leq 1\}$$

is bounded, but the function $f(x,y) = e^{\frac{x}{y}}$ restricted on E might not be bounded - it has a problem at the point $(0,0)$. We investigate the behaviour of f on E approaching this point. Since for $(x,y) \in E$ we have $0 \leq x \leq y^2$ and $0 < y$, then $0 \leq \frac{x}{y} \leq y$ and thus

$$1 \leq e^{\frac{x}{y}} \leq e^y \to 1$$

for $(x,y) \to (0,0)$ therefore $\lim\limits_{\substack{(x,y)\to(0,0)\\(x,y)\in E}} e^{\frac{x}{y}} = 1$. The function f is thus bounded on E and continuous and the integral exists.

We evaluate the integral using Fubini's theorem

$$\iint_E e^{\frac{x}{y}}\ dS = \int_0^1 \int_0^{y^2} e^{\frac{x}{y}}\ dx\ dy = \int_0^1 \Big[ye^{\frac{x}{y}}\Big]_{x=0}^{x=y^2}\ dy = \int_0^1 ye^y - y\ dy = \Big[(y-1)e^y - \frac{y^2}{2}\Big]_0^1 = \frac{1}{2}.$$

7 Tripple integrals

Exercise 7.1. *Evaluate*

$$\iiint\limits_E y\ dV,$$

where E is bounded above by the plane $z = x + 2y$ and below by the region of the plane $z = 0$ enclosed by the curves $y = x^2$, $y = 0$, $x = 1$.

Solution:
The region of integration is

$$E: \quad 0 \leq z \leq x + 2y, \quad 0 \leq y \leq x^2, \quad 0 \leq x \leq 1.$$

Also now, it holds

Fubini's theorem (for triple integrals): Let $E \subseteq \mathbb{R}^3$ be a region of integration and let $f : E \to \mathbb{R}$ be a function with an integrable absolute value (for example, a continuous and bounded function). Then

$$\iiint\limits_E f\ dV = \iint\limits_{\pi(E)} \Big(\int\limits_{(\{x\}\times\{y\}\times\mathbb{R})\cap E} f(x, y, z)\ dz \Big)\ dS,$$

where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection, $\pi(x, y, z) = (x, y)$ and dS indicates the integration with respect to the left variables, for example x and y.

Thus, we evaluate:

$$\iiint\limits_E y\ dV = \int\limits_0^1 \int\limits_0^{x^2} \int\limits_0^{x+2y} y\ dz\ dy\ dx = \int\limits_0^1 \int\limits_0^{x^2} y(x + 2y)\ dy\ dx = \int\limits_0^1 \Big[\frac{y^2}{2}x + \frac{2y^3}{3}\Big]_{y=0}^{y=x^2}\ dx =$$

$$= \int\limits_0^1 \frac{x^5}{2} + \frac{2x^6}{3}\ dx = \frac{1}{12} + \frac{2}{21} = \frac{5}{28}.$$

Exercise 7.2. *Evaluate*

$$\iiint\limits_E xyz\ dV,$$

where E is enclosed by the surfaces $y = x^2$, $x = y^2$, $z = xy$ and $z = 0$.

Solution:
The region of integration is

$$E: \quad 0 \leq z \leq xy, \quad x^2 \leq y \leq \sqrt{x}, \quad 0 \leq x \leq 1.$$

Thus, we have:

$$\iiint\limits_E xyz\ dV = \int\limits_0^1 \int\limits_{x^2}^{\sqrt{x}} \int\limits_0^{xy} xyz\ dz\ dy\ dx = \int\limits_0^1 \int\limits_{x^2}^{\sqrt{x}} \frac{(xy)^3}{2}\ dy\ dx = \frac{1}{8} \int\limits_0^1 x^5 - x^{11}\ dx =$$

$$= \frac{1}{8}(\frac{1}{6} - \frac{1}{12}) = \frac{1}{96}.$$

Exercise 7.3. *Evaluate*

$$\iiint_E xy\ dV,$$

where E is the tetrahedron with vertexes $(0,0,0)$, $(0,1,0)$, $(1,1,0)$ and $(0,1,1)$.

Solution:
The region of integration E is the set enclosed by the planes $x=0$, $y=1$, $z=0$ and $z=y-x$. Thus, for example, we may write

$$E: \quad 0 \le z \le y-x, \quad 0 \le x \le y, \quad 0 \le y \le 1.$$

We evaluate:

$$\iiint_E xy\ dV = \int_0^1 \int_0^y \int_0^{y-x} xy\ dz\ dx\ dy = \int_0^1 \int_0^y y^2x - x^2y\ dx\ dy = \int_0^1 \left[y^2\frac{x^2}{2} - \frac{x^3}{3}y \right]_{x=0}^{x=y} dy =$$

$$= \int_0^1 \frac{y^4}{2} - \frac{y^4}{3}\ dy = \left[\frac{y^5}{30}\right]_0^1 = \frac{1}{30}.$$

Exercise 7.4. *Evaluate*

$$\iiint_E \frac{1}{\sqrt{x^2+y^2+(z-2)^2}}\ dV,$$

where $E: \quad x^2+y^2+z^2 \le 1.$

Solution:
We recall the theorem on substitution in multiple integrals:

$$\iiint_{\Phi(U)} f\ dV = \iiint_U (f \circ \Phi) \cdot |\det \Phi'|\ dV.$$

We will use *spherical coordinates*:

$$\Psi : \langle 0, +\infty) \times \langle 0, 2\pi \rangle \times \langle 0, \pi \rangle \to \mathbb{R}^3, \quad \text{where} \quad \Psi : \begin{array}{rcl} x &=& (r\sin\vartheta)\cos\varphi \\ y &=& (r\sin\vartheta)\sin\varphi \\ z &=& r\cos\vartheta \end{array}.$$

Observation: Spherical coordinates are the composition of two

$$\Psi = \Phi_2 \circ \Phi_1$$

$$\Phi_1 : \begin{array}{rcl} \tilde{r} &=& r\sin\vartheta \\ \tilde{\varphi} &=& \varphi \\ \tilde{z} &=& r\cos\vartheta \end{array}, \quad \Phi_2 : \begin{array}{rcl} x &=& \tilde{r}\cos\tilde{\varphi} \\ y &=& \tilde{r}\sin\tilde{\varphi} \\ z &=& \tilde{z} \end{array}$$

thus, for the determinant, we have

$$\det \Psi' = \det(\Phi_2)'_{|\Phi_1} \cdot \det(\Phi_1)' = \tilde{r}_{|\Phi_1} \cdot r = (r\sin\vartheta)\cdot r = r^2\sin\vartheta.$$

We choose to parametrize the ball $E = \Psi(U)$ as follows

$$U: \quad 0 \le r \le 1 \quad \& \quad 0 \le \varphi \le 2\pi \quad \& \quad 0 \le \vartheta \le \pi \ .$$

Thus, we may write

$$\iiint\limits_{E=\Psi(U)} \frac{1}{\sqrt{x^2+y^2+z^2-4z+4}}\ dV = \iiint\limits_{U} \frac{r^2 \sin\vartheta}{\sqrt{r^2-4r\cos\vartheta+4}}\ dV =$$

$$= \int_0^1 \int_0^\pi \int_0^{2\pi} \frac{r^2 \sin\vartheta}{\sqrt{r^2-4r\cos\vartheta+4}}\ d\varphi\ d\vartheta\ dr = 2\pi \int_0^1 \int_0^\pi \frac{r^2 \sin\vartheta}{\sqrt{r^2-4r\cos\vartheta+4}}\ d\vartheta\ dr =$$

$$= 2\pi \int_0^1 \Big[r\frac{\sqrt{r^2-4r\cos\vartheta+4}}{2} \Big]_{\vartheta=0}^{\vartheta=\pi} dr = \pi \int_0^1 r\Big(\sqrt{r^2+4r+4} - \sqrt{r^2-4r+4} \Big)\ dr =$$

$$= \pi \int_0^1 r\Big(|r+2| - |r-2| \Big)\ dr = \pi \int_0^1 r\Big(r+2-(2-r) \Big)\ dr = 2\pi \int_0^1 r^2\ dr = \frac{2}{3}\pi.$$

Exercise 7.5. *Evaluate the center of mass of the solid*

$$E: \quad x^2+y^2+z^2 \le R^2 \quad \& \quad z\cdot\tan(\alpha) \ge \sqrt{x^2+y^2},$$

with density $\rho = 1$, where $R > 0$ and $\alpha \in (0, \frac{\pi}{2})$ are parameters.

Solution:
The solid E is the intersection of the ball with radius R and the cone with angle at the vertex 2α, and vertex at the center of the ball. It is again convenient to use spherical coordinates

$$\Psi: \quad \begin{array}{rcl} x &=& r\sin\vartheta\cos\varphi \\ y &=& r\sin\vartheta\sin\varphi \\ z &=& r\cos\vartheta \end{array}$$

A parametrization $E = \Psi(U)$ is given by

$$U: \quad 0 \le r \le R \quad \& \quad 0 \le \varphi \le 2\pi \quad \& \quad 0 \le \vartheta \le \alpha$$

In order to evaluate the center of mass, we first need to evaluate the mass:

$$m = \iiint\limits_{E=\Psi(U)} 1\ dV = \iiint\limits_{U} r^2 \sin\vartheta\ dV = \int_0^R \int_0^\alpha \int_0^{2\pi} r^2 \sin\vartheta\ d\varphi\ d\vartheta\ dr =$$

$$= 2\pi \int_0^R \int_0^\alpha r^2 \sin\vartheta\ d\vartheta\ dr = 2\pi(1-\cos\alpha) \int_0^R r^2\ dr = \frac{2}{3}\pi R^3(1-\cos\alpha).$$

Since the solid E is a solid of rotation, symmetric with respect to the z-axis, the x-coordinate and y-coordinate of the center of mass are both equal to zero. It remains to evaluate the z-coordinate of the center of mass:

$$T_3 = \frac{1}{m} \iiint\limits_{E=\Psi(U)} z\ dV = \frac{1}{m} \iiint\limits_{U} r^3 \cos\vartheta \sin\vartheta\ dV = \frac{1}{m} \int_0^R \int_0^\alpha \int_0^{2\pi} r^3 \frac{\sin 2\vartheta}{2}\ d\varphi\ d\vartheta\ dr =$$

$$= \frac{\pi}{m} \Big(\int_0^R r^3\ dr \Big) \cdot \Big(\int_0^\alpha \sin 2\vartheta\ d\vartheta \Big) = \frac{\pi R^4}{8m}(1-\cos 2\alpha) = \frac{3R}{16} \cdot \frac{1-\cos 2\alpha}{1-\cos\alpha} = \frac{3R}{8}(1+\cos\alpha).$$

Exercise 7.6. *Find the center of mass of a homogeneous cone with height $h > 0$ and base radius $R > 0$.*

Solution:
The given cone can be described as follows

$$E: \quad 0 \le z \le H \quad \& \quad \sqrt{x^2+y^2} \le \frac{R}{h} \cdot z,$$

and usually a cone is parametrized using cylindrical coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, $z = h$

$$U: \quad 0 \le r \le R \quad \& \quad 0 \le \varphi \le 2\pi \quad \& \quad 0 \le h \le H\ .$$

This time, in our triple integral, we choose to use the order of integration corresponding to cutting the cone in horizontal circular sections that then we integrate with respect to the height. We use the known formula for area of a disk.

mass:

$$m = \iiint\limits_E 1\ dV = \int_0^h \Big(\iint\limits_{\sqrt{x^2+y^2} \le \frac{R}{h}\cdot z} 1\ dxdy \Big)\ dz = \int_0^h \pi \frac{R^2}{h^2} \cdot z^2\ dz = \frac{\pi}{3} R^2 h$$

Since the solid E is a solid of rotation, symmetric with respect to the z-axis, the x-coordinate and y-coordinate of the center of mass are both equal to zero. It remains to evaluate the z-coordinate of the center of mass:

$$T_3 = \frac{1}{m} \iiint\limits_E z\ dV = \frac{1}{m} \int_0^h \Big(\iint\limits_{\sqrt{x^2+y^2} \le \frac{R}{h}\cdot z} z\ dxdy \Big)\ dz = \frac{1}{m} \int_0^h \pi \frac{R^2}{h^2} \cdot z^3\ dz = \frac{1}{m}\frac{\pi}{4} R^2 h^2 = \frac{3}{4} h\ .$$

From our solution, it is clear that the integral depends on how the area of the horizontal section changes at different heights, thus we would obtain the same result (the center of mass is at a quarter of the height of the cone) for any cone, with any other type of base (for example a pyramid).

Exercise 7.7. *Calculate the moment of inertia of a rotational paraboloid E of height h and base radius R relative to an axis which passes through the center of mass E and is perpendicular to the axis of rotational symmetry of the paraboloid E (i.e. Emph equatorial moment).*

Solution:
The region of integration is

$$E: \quad \frac{h}{R^2}(x^2+y^2) \le z \le h.$$

We find the center of gravity of the solid of revolution E using cylindrical coordinates

$$\Phi: \begin{array}{rcl} x &=& r\cos\varphi \\ y &=& r\sin\varphi \\ z &=& z \end{array}$$

and parametrization $E = \Phi(U)$

$$U: \quad \frac{hr^2}{R^2} \le z \le h \quad \& \quad 0 \le \varphi \le 2\pi\ .$$

mass:

$$m = \iiint\limits_{E=\Phi(U)} 1\ dV = \iiint\limits_{U} r\ dV = \int_0^R \int_{\frac{hr^2}{R^2}}^{h} \int_0^{2\pi} r\ d\varphi\ dz\ dr =$$

$$= 2\pi \int_0^R r\Big(h - \frac{hr^2}{R^2}\Big)\ dr = 2\pi h \Big[\frac{r^2}{2} - \frac{r^4}{4R^2}\Big]_0^R = \frac{\pi h R^2}{2}.$$

The solid E is rotationally symmetric around the axis z, thus, it is sufficient to find the only z coordinate of the center of mass:

$$T_3 = \frac{1}{m} \iiint\limits_{E=\Phi(U)} z\ dV = \frac{1}{m} \iiint\limits_{U} zr\ dV = \frac{1}{m} \int_0^R \int_{\frac{hr^2}{R^2}}^{h} \int_0^{2\pi} zr\ d\varphi\ dz\ dr =$$

$$= \frac{4}{hR^2} \int_0^R \int_{\frac{hr^2}{R^2}}^{h} zr\ dz\ dr = \frac{2}{hR^2} \int_0^R r\Big(h^2 - \frac{h^2 r^4}{R^4}\Big)\ dr = \frac{2h}{R^2}\Big[\frac{r^2}{2} - \frac{r^6}{6R^4}\Big]_0^R = \frac{2}{3}h.$$

The moment of inertia relative to an axis perpendicular to the axis z and passing through the center of mass will not depend (due to the symmetry of the solid E respect to the axis z) on the choice of the direction of the axis. We choose for example the direction of the x-axis, i.e. the axis p will have equation $y = 0$ and $z = \frac{2}{3}h$. The moment of inertia then will be

$$M = \iiint\limits_{E} \Big(\rho_p(x,y,z)\Big)^2\ dV,$$

where $\rho_p(x,y,z) = y^2 + \Big(z - \frac{2}{3}h\Big)^2$ is the square of the distance of the point (x,y,z) from the axis p.

In the evaluation of the moment, we again use the transformation Φ:

$$M = \iiint\limits_{E=\Phi(U)} y^2 + \Big(z - \frac{2}{3}h\Big)^2\ dV = \iiint\limits_{U} r^3\sin^2\vartheta + r\Big(z - \frac{2}{3}h\Big)^2\ dV =$$

$$= \int_0^R \int_{\frac{hr^2}{R^2}}^{h} \int_0^{2\pi} r^3\sin^2\vartheta + r\Big(z - \frac{2}{3}h\Big)^2\ d\varphi\ dz\ dr = \pi \int_0^R \int_{\frac{hr^2}{R^2}}^{h} r^3 + 2r\Big(z - \frac{2}{3}h\Big)^2\ dz\ dr =$$

$$= \pi \int_0^R hr^3\Big(1 - \frac{r^2}{R^2}\Big) + \frac{2}{3}r\Big[\Big(z - \frac{2}{3}h\Big)^3\Big]_{z=\frac{hr^2}{R^2}}^{z=h}\ dr = \pi h \int_0^R r^3 - \frac{r^5}{R^2} + \frac{2}{81}rh^2 - \frac{2}{3}h^2 r\Big(\frac{r^2}{R^2} - \frac{2}{3}\Big)^3\ dr =$$

$$= \pi h\Big(\frac{R^4}{4} - \frac{R^4}{6} + \frac{R^2h^2}{81} - \Big[\frac{h^2R^2}{12}\Big(\frac{r^2}{R^2} - \frac{2}{3}\Big)^4\Big]_0^R\Big) = \frac{\pi h R^2}{12}\Big(R^2 + \frac{h^2}{3}\Big).$$

Exercise 7.8. *Evaluate*

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\ dz\ dy\ dx.$$

Solution:
The region of integration is

$$E:\quad |x|\le 2 \quad \& \quad |y|\le\sqrt{4-x^2} \quad \& \quad \sqrt{x^2+y^2}\le z\le 2$$

or

$$E:\quad x^2\le 4 \quad \& \quad x^2+y^2\le 4 \quad \& \quad \sqrt{x^2+y^2}\le z\le 2$$

and thus

$$E:\quad \sqrt{x^2+y^2}\le z\le 2,$$

that is a cone with height 2 and base radius also 2, that stands on its vertex at the origin. In order to evaluate the integral, we use *cylindrical coordinates*:

$$\Phi:\langle 0,+\infty)\times\langle 0,2\pi\rangle\times\mathbb{R}\to\mathbb{R}^3,\quad \text{where}\quad \Phi:\begin{array}{ccc} x & = & r\cos\varphi \\ y & = & r\sin\varphi \\ z & = & z\end{array}$$

that is

$$\Phi'=\begin{pmatrix}\cos\varphi & -r\sin\varphi & 0\\ \sin\varphi & r\cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\quad \text{and}\quad \det\Phi'=r\ .$$

As a parametrization of E we choose

$$U:\quad 0\le r\le z\le 2 \quad \& \quad 0\le\varphi\le 2\pi\ .$$

We may write

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\ dz\ dy\ dx=\iiint_{E=\Phi(U)}(x^2+y^2)\ dV=$$

$$=\iiint_{U}r^2\cdot r\ dV=\int_{0}^{2}\int_{0}^{z}\int_{0}^{2\pi}r^3\ d\varphi\ dr\ dz=2\pi\int_{0}^{2}\int_{0}^{z}r^3\ \ dr\ dz=$$

$$=\frac{\pi}{2}\int_{0}^{2}z^4\ \ dz=\frac{\pi}{10}\cdot 2^5=\frac{16}{5}\pi.$$

Exercise 7.9. *Evaluate the center of mass of the solid*

$$E:\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1 \quad \& \quad x,y,z\ge 0,$$

with density $\rho=1$, where $a,b,c>0$ are parameters.

Solution:
The region of integration E is one eight of a regular ellipse. We therefore parametrize the ellipse using a variation of spherical coordinates Φ:

$$\Phi: \quad \begin{array}{rcl} x/a &=& r\sin\vartheta\cos\varphi \\ y/b &=& r\sin\vartheta\sin\varphi \\ z/c &=& r\cos\vartheta \end{array},$$

that is the result of composing the classical spherical coordinates Ψ with the linear transformation $\mathcal{L}$, that deforms the single axis:

$$\Phi = \mathcal{L}\circ\Psi, \qquad \mathcal{L}(\tilde{x},\tilde{y},\tilde{z}) := (a\tilde{x}, b\tilde{y}, c\tilde{z}).$$

Thus, we have

$$\Phi' = \mathcal{L}'_{|\Phi}\circ\Psi' \quad \text{a} \quad \det\Phi' = (\det\mathcal{L}'_{|\Phi})\cdot(\det\Psi') = abc\cdot r^2\sin\vartheta,$$

since

$$\mathcal{L}' = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

The parametrization U of the region of integration $E = \Phi(U)$ is

$$U: \quad 0 \le r \le 1 \quad \& \quad 0 \le \varphi \le \frac{\pi}{2} \quad \& \quad 0 \le \vartheta \le \frac{\pi}{2}.$$

We evaluate the mass of E, using the known volume of a ball K of radius 1 and the fact that the volume of E is one eight of the volume of the ellipse F. Since $F = \mathcal{L}(K)$, we have:

$$m = \iiint\limits_E 1\ dV = \frac{1}{8}\iiint\limits_{F=\mathcal{L}(K)} 1\ dV = \frac{1}{8}\iiint\limits_K |\det\mathcal{L}'|\ dV =$$

$$= \frac{abc}{8}\iiint\limits_K 1\ dV = \frac{abc}{8}\cdot\frac{4}{3}\pi = \frac{\pi abc}{6}.$$

In order to find the center of mass $T = (T_1, T_2, T_3)$, it is enough to evaluate one of its coordinates (for example T_3), because the other coordinates can be found in an analogous way, considering a rotation of the ellipsoid and similar calculation. Thus, we have

$$T_3 = \frac{1}{m}\iiint\limits_{E=\Phi(U)} z\ dV = \frac{1}{m}\iiint\limits_U (cr\cos\vartheta)\cdot(abcr^2\sin\vartheta)\ dV =$$

$$= \frac{3c}{\pi}\int_0^1\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} r^3\sin 2\vartheta\ d\varphi\ d\vartheta\ dr = \frac{3c}{\pi}\Big(\int_0^1 r^3\ dr\Big)\cdot\Big(\int_0^{\frac{\pi}{2}}\sin 2\vartheta\ d\vartheta\Big)\cdot\Big(\int_0^{\frac{\pi}{2}} 1\ d\varphi\Big) = \frac{3}{8}c.$$

Similarly, we can evaluate $T_1 = \frac{3}{8}a$ and $T_2 = \frac{3}{8}b$.

Exercise 7.10. *Evaluate the integral*

$$\iiint\limits_E |z|\ dV,$$

where $E = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \le 1\ \&\ |z| \le 3\ \&\ y \ge 0\}$.

Solution:
The region E is half of a cylinder. We will use cylindrical coordinates:

$$\Phi(r,\varphi,z) = (r\cos\varphi, r\sin\varphi, z), \qquad \det\Phi' = r$$

and parametrization for the region $E = \Phi(U)$ as

$$U = \{(r,\varphi,z) \in \mathbb{R}^3 \mid 0 \le r \le 1 \ \&\ |z| \le 3 \ \&\ 0 \le \varphi \le \pi\}.$$

$$\iiint\limits_{E=\Phi(U)} |z|\ dV = \iiint\limits_{U} |z|r\ dV = \int_0^1 \int_0^\pi \int_{-3}^3 |z|r\ dz\ d\varphi\ dr = \Big(\int_{-3}^3 |z|\ dz\Big)\cdot\Big(\int_0^\pi 1\ d\varphi\Big)\cdot\Big(\int_0^1 r\ dr\Big) = \frac{9}{2}\pi.$$

Exercise 7.11. *Evaluate the integral*

$$\iiint\limits_E x^2\ dV,$$

where $E = \{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{x^2+y^2} \le z \le 2 \ \&\ 0 \le y \ \&\ x \le 0\}$.

Solution:
The region E is a quarter of a cone. We will use cylindrical coordinates:

$$\Phi(r,\varphi,z) = (r\cos\varphi, r\sin\varphi, z), \qquad \det\Phi' = r$$

and parametrization of the region $E = \Phi(U)$ as

$$U = \{(r,\varphi,z) \in \mathbb{R}^3 \mid 0 \le r \le z \le 2 \ \&\ \frac{\pi}{2} \le \varphi \le \pi\}.$$

$$\iiint\limits_{E=\Phi(U)} x^2\ dV = \iiint\limits_{U} r^3\cos^2\varphi\ dV = \int_{\frac{\pi}{2}}^\pi \int_0^2 \int_0^z r^3\cos^2\varphi\ dr\ dz d\varphi = \frac{1}{4}\int_{\frac{\pi}{2}}^\pi \int_0^2 z^4\cos^2\varphi\ dz d\varphi =$$

$$= \frac{1}{4}\Big(\int_0^2 z^4\ dz\Big)\cdot\Big(\int_{\frac{\pi}{2}}^\pi \cos^2\varphi\ d\varphi\Big) = \frac{8}{5}\pi.$$

8 Path integral

Exercise 8.1. *Integrate the function* $f(x,y) = \frac{x+y^2}{\sqrt{1+x^2}}$ *along the curve* Γ*:* $y = \frac{x^2}{2}$ *from the point* $A = (1, \frac{1}{2})$ *to the point* $B = (0,0)$.

Solution:
In order to evaluate the integral, we use the formula

$$\int\limits_\Gamma f\ ds = \int_a^b f(\varphi(t))\cdot\|\varphi'(t)\|\ dt,$$

where φ is a suitable parametrization of the curve Γ, i.e. a function $\varphi : \langle a, b\rangle \to \mathbb{R}^n$, that is

- continuous and piecewise differentiable on the interval $\langle a, b\rangle$,
- φ is one to one on $\langle a, b\rangle$ up to finitely many exceptions $t_1, \dots, t_n \in \langle a, b\rangle$ (the curve can intersect itself),
- $\varphi(\langle a, b\rangle) = \Gamma$.

As a parametrization, we use $\varphi(t) = \left(1-t, \frac{(1-t)^2}{2}\right)$ for $t \in \langle 0, 1\rangle$, that for sure satisfies all the assumptions. Then, we have

$$\varphi'(t) = (-1, t-1) \quad \text{a} \quad \|\varphi'(t)\| = \sqrt{1+(t-1)^2}.$$

Thus,

$$\int_\Gamma f\ ds = \int_0^1 \frac{1-t+\frac{(1-t)^4}{4}}{\sqrt{1+(1-t)^2}} \cdot \sqrt{1+(t-1)^2}\ dt = \int_0^1 1-t+\frac{(1-t)^4}{4}\ dt = \left[\begin{smallmatrix} u=1-t \\ du=-dt \end{smallmatrix}\right] =$$

$$= -\int_1^0 u + \frac{u^4}{4}\ du = \frac{1}{2} + \frac{1}{20} = \frac{11}{20}.$$

Exercise 8.2. *Evaluate the path integral*

$$\int_\Gamma (x^{\frac{4}{3}} + y^{\frac{4}{3}})\ ds,$$

where Γ *is the asteroid* $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, *with parameter* $a > 0$.

Solution:
The equation of the asteroid is similar to the one of the circle (only with different exponents). Thus, we parametrize it with a kind of polar coordinates, also with the proper exponents:

$$\varphi: \begin{array}{l} x = a \cdot \cos^3 t \\ y = a \cdot \sin^3 t \end{array}$$

for $t \in \langle 0, 2\pi\rangle$. Then, we have

$$\varphi'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \Big(-3a\cos^2 t \cdot \sin t,\ 3a\sin^2 t \cdot \cos t\Big)$$

$$\|\varphi'(t)\| = \sqrt{(3a\cos t \cdot \sin t)^2 \left(\cos^2 t + \sin^2 t\right)} = 3a\left|\cos t \cdot \sin t\right|.$$

We use the fact that both the function and the asteroid are symmetric with respect to the origin, and we evaluate the integral only on a quarter of the asteroid:

$$\int_\Gamma (x^{\frac{4}{3}} + y^{\frac{4}{3}})\ ds = \int_0^{2\pi} a^{\frac{4}{3}} \left(\cos^4 t + \sin^4 t\right) \cdot 3a\left|\cos t \cdot \sin t\right|\ dt =$$

$$= 12 \cdot a^{\frac{7}{3}} \int_0^{\frac{\pi}{2}} \left(\cos^5 t \cdot \sin t + \sin^5 t \cdot \cos t\right)\ dt = 2a^{\frac{7}{3}} \left[-\cos^6 t + \sin^6 t\right]_0^{\frac{\pi}{2}} = 4a^{\frac{7}{3}}\ .$$

Exercise 8.3. *Evaluate the path integral*

$$\int_\Gamma xy\ ds,$$

where $\Gamma : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ \ \& \ \ x, y \geq 0$ *with parameter* $a \neq b$.

Solution:
The curve is a quarter of an ellipse, thus we choose as a parametrization a "variation" of the polar coordinates

$$\varphi : \begin{array}{l} \frac{x}{a} = \cos t \\ \\ \frac{y}{b} = \sin t \end{array}$$

for $t \in \langle 0, \frac{\pi}{2} \rangle$. Then, we have

$$\varphi'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = \Big(-a \sin t,\ b \cos t \Big)$$

$$\|\varphi'(t)\| = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\ .$$

Thus:

$$\int_\Gamma xy\ ds = \int_0^{\frac{\pi}{2}} ab \cos t \sin t \cdot \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\ dt = \left[{u = a^2 \sin^2 t + b^2 \cos^2 t \atop du = 2(a^2 - b^2) \sin t \cos t\ dt} \right] =$$

$$= \frac{ab}{2(a^2 - b^2)} \int_{b^2}^{a^2} \sqrt{u}\ du = \frac{ab}{3(a^2 - b^2)} \left[u^{\frac{3}{2}} \right]_{u=b^2}^{u=a^2} = \frac{ab(a^3 - b^3)}{3(a^2 - b^2)} = \frac{ab(a^2 + ab + b^2)}{3(a + b)}\ .$$

Exercise 8.4. *A particle is moving so that its position, at time* t, *is given by* $\varphi(t) = \Big(\cos(t), \sin(t), \frac{t^2}{2} \Big)$. *Determine the length of the traveled path during the interval of time* $\langle 0, 1 \rangle$.

Solution:
The path lies on the surface of the cylinder $x^2 + y^2 = 1$ and the particle travels on a stretching spiral. The length of the path Γ is given as the integral of the constant function $f = 1$ along the curve

$$\ell(\Gamma) = \int_\Gamma 1\ ds = \int_a^b \|\varphi'(t)\|\ \ dt\ .$$

Thus, we have

$$\varphi'(t) = \big(-\sin(t), \cos(t), t \big)$$

and

$$\|\varphi'(t)\| = \sqrt{\sin^2(t) + \cos^2(t) + t^2} = \sqrt{1 + t^2},$$

then, we get

$$\ell(\Gamma) = \int_\Gamma 1\ ds = \int_0^1 \sqrt{1 + t^2}\ dt = \left[{t = \sinh(\alpha) \atop dt = \cosh(\alpha) d\alpha} \right] = \int_0^{\operatorname{arcsinh}(1)} \sqrt{1 + \sinh^2(\alpha)} \cdot \cosh(\alpha)\ d\alpha =$$

$$= \int\limits_0^{\operatorname{arcsinh}(1) = \ln(1+\sqrt{1+1^2})} \cosh^2(\alpha)\ d\alpha = \int\limits_0^{\ln(1+\sqrt{2})} \left(\frac{e^\alpha + e^{-\alpha}}{2}\right)^2 d\alpha = \frac{1}{4} \int\limits_0^{\ln(1+\sqrt{2})} 2 + e^{2\alpha} + e^{-2\alpha}\ d\alpha =$$

$$= \frac{1}{2}\ln(1+\sqrt{2}) + \frac{1}{8}\Big[e^{2\alpha} - e^{-2\alpha}\Big]_{\alpha=0}^{\alpha=\ln(1+\sqrt{2})} = \frac{1}{2}\ln(1+\sqrt{2}) + \frac{1}{8}\left((1+\sqrt{2})^2 - \frac{1}{(1+\sqrt{2})^2}\right) =$$

$$= \frac{1}{2}\ln(1+\sqrt{2}) + \frac{1}{8}\left(3 + 2\sqrt{2} - \frac{1}{3+2\sqrt{2}} \cdot \frac{3-2\sqrt{2}}{3-2\sqrt{2}}\right) = \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2})\ .$$

Observation on the substitution: Since the graph of the function $u = \sqrt{1+t^2}$ is part of the hyperbola ($u^2 - t^2 = 1$), it is convenient to use a substitution using the hyperbolic functions

$$\cosh(\alpha) = \frac{e^\alpha + e^{-\alpha}}{2} \quad \text{and} \quad \sinh(\alpha) = \frac{e^\alpha - e^{-\alpha}}{2}\ .$$

We just decomposed the function e^α into an even and an odd part, i.e. $e^\alpha = \cosh(\alpha) + \sinh(\alpha)$. Similarly, for the parametrization of the circle $u = \sqrt{1-t^2}$ we use trigonometric functions $\sin(\alpha)$ and $\cos(\alpha)$. Also their relations are similar:

$\cosh^2(\alpha) - \sinh^2(\alpha) = 1,$	$\sinh'(\alpha) = \cosh(\alpha)$
$\cosh^2(\alpha) + \sinh^2(\alpha) = \cosh(2\alpha),$	$\cosh'(\alpha) = \sinh(\alpha)$

Solving the quadratic equation, we get the form of the inverse function of $t = \sinh(\alpha)$:

$$\alpha = \operatorname{arcsinh}(t) = \ln\left(t + \sqrt{1+t^2}\right).$$

Exercise 8.5. *Evaluate the length of the spiral defined by the parametrization* $\varphi : \langle 0, 2n\pi \rangle \to \mathbb{R}^3$,

$$\varphi(t) = (t\cos(t), t\sin(t), t).$$

Solution:
The curve lies on the surface of the cone $x^2 + y^2 = z^2$. The length of the path $\mathcal{C}$ with parametrization φ then is evaluated as

$$L(\mathcal{C}) = \int\limits_a^b \|\varphi'(t)\|\ dt \ \left(= \int\limits_{\mathcal{C}} 1\ ds \right),$$

or as the integral of the constant function $f = 1$ along the curve $\mathcal{C}$. Therefore, we have

$$\varphi'(t) = \big(\cos(t) - t\sin(t), \sin(t) + t\cos(t), 1\big)$$

a

$$\|\varphi'(t)\| = \sqrt{\big(\cos(t) - t\sin(t)\big)^2 + \big(\sin(t) + t\cos(t)\big)^2 + 1^2} = \sqrt{2+t^2},$$

thus, we get

$$L(\mathcal{C}) = \int\limits_{\mathcal{C}} 1\ ds = \int\limits_0^{2n\pi} \sqrt{2+t^2}\ dt = \left[\begin{smallmatrix} t=\sqrt{2}u \\ dt=\sqrt{2}du \end{smallmatrix}\right] = 2\int\limits_0^{\sqrt{2}n\pi} \sqrt{1+u^2}\ du = \left[\begin{smallmatrix} u=\sinh(x) \\ du=\cosh(x)dx \end{smallmatrix}\right] =$$

$$= 2\int\limits_0^{\operatorname{arcsinh}(\sqrt{2}n\pi)} \cosh^2(u)\ du = \int\limits_0^{\operatorname{arcsinh}(\sqrt{2}n\pi)} 1 + \cosh(2u)\ du = \Big[u + \frac{\sinh(2u)}{2}\Big]_0^{\operatorname{arcsinh}(\sqrt{2}n\pi)} =$$

$$= \Big[u + \sinh(u)\cosh(u)\Big]_0^{\operatorname{arcsinh}(\sqrt{2}n\pi)} = \Big[u + \sinh(u)\sqrt{1+\sinh(u)^2}\Big]_0^{\operatorname{arcsinh}(\sqrt{2}n\pi)} =$$

$$= \operatorname{arcsinh}(\sqrt{2}n\pi) + \sqrt{2}n\pi\sqrt{1+2n^2\pi^2} = \ln\left(\sqrt{2}n\pi + \sqrt{1+2n^2\pi^2}\right) + \sqrt{2}n\pi\sqrt{1+2n^2\pi^2}.$$

Observation: Again, we used here the relation between the hyperbolic function $\cosh(x)$ and $\sinh(x)$:

$$\cosh^2(x) - \sinh^2(x) = 1, \qquad \sinh'(x) = \cosh(x)$$

$$\cosh^2(x) + \sinh^2(x) = \cosh(2x), \qquad \cosh'(x) = \sinh(x)$$

Summing up we get

$$\cosh^2(x) = \frac{1+\cosh(2x)}{2}$$

and taking the derivative, then

$$2\sinh(x)\cosh(x) = \sinh(2x).$$

Solving the quadratic equation, we get the form of the inverse function of $u = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$:

$$x = \operatorname{arcsinh}(u) = \ln\left(u + \sqrt{1+u^2}\right).$$

Exercise 8.6. *Find the work done by the force field $\vec{F}(x,y,z) = (y+z, z+x, x+y)$ in moving a particle along the curve Γ with parametrization $\varphi(t) = (t, t^2, t^4)$, $t \in \langle 0, 1\rangle$, and the orientation induced by this parametrization.*

Solution:
The integral is calculated using the formula

$$\int_\Gamma \vec{F}\cdot d\vec{s} = \int_a^b \vec{F}(\varphi(t))\cdot\varphi'(t)\ dt.$$

We have

$$\varphi'(t) = (1, 2t, 4t^3)$$

thus,

$$\int_\Gamma \vec{F}\cdot d\vec{s} = \int_0^1 (t^2+t^4, t^4+t, t+t^2)\cdot\begin{pmatrix}1\\2t\\4t^3\end{pmatrix}\ dt = \int_0^1 3t^2 + 5t^4 + 6t^5\ dt = \Big[t^3+t^5+t^6\Big]_0^1 = 3.$$

Exercise 8.7. *Prove that the following vector fields are conservative and find for each a potential function.*

(i) $\mathbf{F}(x,y,z) = (x^2+y,\quad y^2+x,\quad ze^z)$,

(ii) $\mathbf{F}(x,y,z) = \left(3x^2 + \frac{y}{(x+z)^2},\quad -\frac{1}{x+z},\quad \frac{y}{(x+z)^2}\right)$.

Solution:
The work done by the force $\mathbf{F}$ to move a particle in a region U (an open connected set) from point A to point B is independent from the path if and only if the vector field has a potential, i.e. there exists a function $f : U \to \mathbb{R}$ so that $\text{grad}(f) = \mathbf{F}$.

If the region U is also simply connected (that means that any closed path in U can be continuously pulled, inside U, to a point), then this happens if and only if $\text{curl}(\mathbf{F}) = 0$ on the whole U.

An example of a simply connected region is $\mathbb{R}^n$ or $\mathbb{R}^3 \setminus \{0\}$.

An example of a region that is not simply connected is $\mathbb{R}^2 \setminus \{0\}$, $\mathbb{R}^3 \setminus$ "$x -$ axis " or the torus.

In our case, the region is the whole $\mathbb{R}^3$, thus a simply connected region. The rotational vector field, called curl, is defined as

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \quad -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}, \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \Big)$$

where $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$ is a formally defined vector formed by the partial derivative operators.

(i) After evaluation, we have

$$\text{curl}(\mathbf{F}) = (0 - 0, \quad 0 - 0, \quad 1 - 1) = \mathbf{0},$$

thus, the curl is zero on the whole $\mathbb{R}^3$ and the vector field $\mathbf{F}$ has a potential.

A potential is a function $f : \mathbb{R}^3 \to \mathbb{R}$ so that

$$\frac{\partial f}{\partial x} = x^2 + y \tag{1}$$

$$\frac{\partial f}{\partial y} = y^2 + x \tag{2}$$

$$\frac{\partial f}{\partial z} = ze^z \; . \tag{3}$$

¿From the first equation, we get

$$f(x, y, z) = \int (x^2 + y) \; dx = \frac{x^3}{3} + xy + C(y, z),$$

where $C : \mathbb{R}^2 \to \mathbb{R}$ is an unknown function depending only on y and z. The form of the found function f, now can be substituted into the second equation

$$y^2 + x = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \Big(\frac{x^3}{3} + xy + C(y, z) \Big) = x + \frac{\partial C}{\partial y}$$

thus

$$\frac{\partial C}{\partial y} = y^2.$$

We get $C(y, z) = \int y^2 \; dy = \frac{y^3}{3} + D(z)$, where $D : \mathbb{R} \to \mathbb{R}$ is again an unknown function depending only on the variable z. For the moment, we have

$$f(x, y, z) = \frac{x^3}{3} + xy + \frac{y^3}{3} + D(z)$$

and, substituting into the last equation, we get

$$ze^z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\Big(\frac{x^3}{3} + xy + \frac{y^3}{3} + D(z)\Big) = \frac{\partial D}{\partial z}.$$

Thus $D(z) = \int ze^z \, dz = (z-1)e^z + K$, where $K \in \mathbb{R}$ is a constant. All together, we found as a potential

$$f(x,y,z) = \frac{x^3}{3} + xy + \frac{y^3}{3} + (z-1)e^z + K.$$

(ii) Similarly to the previous example, we have

$$\text{curl}(\mathbf{F}) = \left(\frac{1}{(x+z)^2} - \frac{1}{(x+z)^2}, \ \ \frac{2y}{(x+z)^3} - \frac{2y}{(x+z)^3}, \ \ \frac{1}{(x+z)^2} - \frac{1}{(x+z)^2}\right) = \mathbf{0}.$$

The curl is again zero on the whole $\mathbb{R}^3$ and the vector field $\mathbf{F}$ has a potential.

For the potential f, we have

$$\frac{\partial f}{\partial x} = 3x^2 + \frac{y}{(x+z)^2} \tag{4}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{x+z} \tag{5}$$

$$\frac{\partial f}{\partial z} = \frac{y}{(x+z)^2} \quad . \tag{6}$$

We start from the second equation:

$$f(x,y,z) = \int -\frac{1}{x+z} \, dy = -\frac{y}{x+z} + C(x,z),$$

where $C : \mathbb{R}^2 \to \mathbb{R}$ is an unknown function depending only on the variables x and z. The found form of the function f can now be substituted into the third equation

$$-\frac{y}{(x+z)^2} = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\Big(-\frac{y}{x+z} + C(x,z)\Big) = -\frac{y}{(x+z)^2} + \frac{\partial C}{\partial z}$$

thus

$$\frac{\partial C}{\partial z} = 0.$$

We get $C(x,z) = D(x)$, where $D : \mathbb{R} \to \mathbb{R}$ is again a function depending only on x. For the moment, we get

$$f(x,y,z) = -\frac{y}{x+z} + D(x)$$

and, substituting into the first equation, we have

$$3x^2 + \frac{y}{(x+z)^2} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\Big(-\frac{y}{x+z} + D(x)\Big) = \frac{y}{(x+z)^2} + \frac{\partial D}{\partial x}.$$

Thus $D(x) = \int 3x^2 \, dx = x^3 + K$, where $K \in \mathbb{R}$ is a constant. We finally can write a potential

$$f(x,y,z) = -\frac{y}{x+z} + x^3 + K.$$

9 Surface integral

Exercise 9.1. *Find the surface area*

(i) of the part of the plane $x + 2y + z = 4$ that lies inside the cylinder $x^2 + y^2 = 4$,

(ii) of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Solution:
(i) The surface is given by $M = \{(x, y, z) \in \mathbb{R}^3 \mid x+2y+z = 4 \ \&\ x^2+y^2 \leq 4\}$. Its area can be evaluated using the formula

$$\iint\limits_M 1\ dS = \iint\limits_U \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\|\ dS,$$

where Φ is a suitable parametrization of the surface M, i.e. a map $\Phi : U \to \mathbb{R}^3$, where $U \subseteq \mathbb{R}^2$, so that

- it is continuously differentiable and one to one on U°,
- $\Phi(U) = M$
- the matrix Φ' has rank 2 on U° (or $\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \neq 0$ on U°).

(i) Since the surface M is the graph of the function $f(x, y) = 4 - x - 2y$ with domains

$$U : \quad x^2 + y^2 \leq 4,$$

as a parametrization we simply choose

$$\Phi(x, y) = \Big(x, y, f(x, y)\Big) = (x, y, 4 - x - 2y)$$

for $(x, y) \in U$.
We have

$$\frac{\partial \Phi}{\partial x} = (1, 0, -1) \qquad \frac{\partial \Phi}{\partial y} = (0, 1, -2)$$

a

$$\frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{vmatrix} = (1, 2, 1) \qquad \left\| \frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} \right\| = \sqrt{6},$$

thus Φ satisfies all the assumptions.
Then, we can write

$$\iint\limits_M 1\ dS = \iint\limits_U \sqrt{6}\ dS = \sqrt{6} \iint\limits_U 1\ dS = \sqrt{6} \cdot 4\pi,$$

since the area of the disk U with radius 2 is 4π.

(ii) The surface is given by

$$M : \quad x^2 + y^2 = z \ \ \& \ \ z \leq 9.$$

As a parametrization we choose

$$\Phi(x, y) = (x, y, x^2 + y^2)$$

with domain

$$U : \quad x^2 + y^2 \leq 9.$$

We have

$$\frac{\partial \Phi}{\partial x} = (1, 0, 2x)$$
$$\frac{\partial \Phi}{\partial y} = (0, 1, 2y)$$

a

$$\frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} = (-2x, -2y, 1) \qquad \left\| \frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} \right\| = \sqrt{4(x^2 + y^2) + 1}.$$

Thus

$$\iint\limits_M 1 \, dS = \iint\limits_U \sqrt{4(x^2 + y^2) + 1} \, dS = \left[\begin{smallmatrix} x = r\cos\varphi \\ y = r\sin\varphi \\ (r,\varphi) \in \langle 0,3\rangle \times \langle 0, 2\pi\rangle \end{smallmatrix} \right] = \int\limits_0^{2\pi} \int\limits_0^3 r\sqrt{4r^2 + 1} \, dr \, d\varphi =$$

$$= \left(\int\limits_0^3 r\sqrt{4r^2 + 1} \, dr \right) \cdot \left(\int\limits_0^{2\pi} 1 \, d\varphi \right) = \left[\frac{(4r^2 + 1)^{\frac{3}{2}}}{12} \right]_0^3 \cdot 2\pi = \frac{\pi}{6}(37^{\frac{3}{2}} - 1).$$

Exercise 9.2. *Evaluate*

$$\iint\limits_M z \, dS,$$

where M is part of the cylinder $x^2 + y^2 = 1$ between the planes $z = 0$ and $z = x + 1$.

Solution:
The surface integral of the function $f : M \to \mathbb{R}$ is given by

$$\iint\limits_M f \, dS = \iint\limits_U f(\Phi(u, v)) \cdot \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| \, dS \ ,$$

where Φ is a suitable parametrization.

The surface is given by

$$M: \quad x^2 + y^2 = 1 \quad \& \quad 0 \le z \le x + 1.$$

Its parametrization, using cylindrical coordinates, is given by

$$\Phi(\varphi, z) = (\cos\varphi, \sin\varphi, z)$$

with domain

$$U: \quad 0 \le \varphi \le 2\pi \quad \& \quad 0 \le z \le 1 + \cos\varphi.$$

We have

$$\frac{\partial \Phi}{\partial \varphi} = (-\sin\varphi, \cos\varphi, 0)$$
$$\frac{\partial \Phi}{\partial z} = (0, 0, 1)$$

a

$$\frac{\partial \Phi}{\partial \varphi} \times \frac{\partial \Phi}{\partial z} = (\cos\varphi, \sin\varphi, 0) \qquad \left\| \frac{\partial \Phi}{\partial \varphi} \times \frac{\partial \Phi}{\partial z} \right\| = 1.$$

Thus, for the function $f(x, y, z) = z$, we have

$$\iint_M f\ dS = \iint_U z\ dS = \int_0^{2\pi} \int_0^{1+\cos\varphi} z\ dz\ d\varphi = \int_0^{2\pi} \frac{(1+\cos\varphi)^2}{2}\ d\varphi = \int_0^{2\pi} \Big(\frac{1}{2} + \cos\varphi + \frac{\cos^2\varphi}{2}\Big)\ d\varphi =$$

$$= \pi + 0 + \frac{\pi}{2} = \frac{3}{2}\pi.$$

Exercise 9.3. *Evaluate*

$$\iint_M yz\ dS,$$

where M is the surface with parametrization $x = uv$, $y = u + v$, $z = u - v$ and $u^2 + v^2 \leq 1$.

Solution:
The surface is now defined as $M = \Phi(U)$, where

$$U: \quad u^2 + v^2 \leq 1$$

and $\Phi : U \to \mathbb{R}^3$,

$$\Phi(u, v) = (uv, u + v, u - v).$$

We verify that Φ is really a well defined parametrization of M (i.e. Φ is one to one and the rank of the derivative Φ' is 2).

The injectivity of Φ follows from the fact that the second and the third component of the function (i.e. $y = u+v$ and $z = u-v$) form a regular linear map (that is one to one). The rank of the derivative can be found like in the previous example with the use of the vector product:

$$\frac{\partial\Phi}{\partial u} = (v, 1, 1)$$

$$\frac{\partial\Phi}{\partial v} = (u, 1, -1)$$

and

$$\frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v} = (-2, v+u, v-u) \qquad \left\|\frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v}\right\| = \sqrt{2(u^2+v^2)+4} \neq 0.$$

For the integral, we then have

$$\iint_M yz\ dS = \iint_U (u^2 - v^2)\sqrt{2(u^2+v^2)+4}\ dS = \left[\begin{smallmatrix} u = r\cos\varphi \\ v = r\sin\varphi \\ (r,\varphi)\in\langle 0,1\rangle\times\langle 0,2\pi\rangle \end{smallmatrix}\right] =$$

$$= \int_0^{2\pi}\int_0^1 r^3(\cos^2\varphi - \sin^2\varphi)\sqrt{2r^2+4}\ dr\ d\varphi = \Big(\int_0^3 r^3\sqrt{2r^2+4}\ dr\Big)\cdot\Big(\int_0^{2\pi} \cos 2\varphi\ d\varphi\Big) = 0,$$

because the second integral is zero.

Observation: We can try to find out how the surface M looks like. From the equations $y = u + v$ and $z = u - v$ we get $u = \frac{z+y}{2}$ and $v = \frac{y-z}{2}$. Thus $x = uv = \frac{y^2-z^2}{4}$ and $1 \geq u^2 + v^2 = \frac{z^2+y^2}{4}$. Thus, the following relations hold

$$y^2 - z^2 = 4x, \quad y^2 + z^2 \leq 4$$

that describe a part of an hyperbolic paraboloid (a saddle) inside a cylinder with axis x and radius 2.

Exercise 9.4. *Evaluate*

$$\iint\limits_M x^2z + y^2z \; dS,$$

where M is the surface of the half-sphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$.

Solution:
The surface $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 4 \;\&\; z \geq 0\}$ can be parametrized with the use of spherical coordinates as

$$\Phi(\varphi, \vartheta) = (2\sin\vartheta\cos\varphi, 2\sin\vartheta\sin\varphi, 2\cos\vartheta)$$

and domain

$$U: \quad 0 \leq \varphi \leq 2\pi \quad \& \quad 0 \leq \vartheta \leq \frac{\pi}{2}.$$

Moreover, we have

$$\frac{\partial\Phi}{\partial\varphi} = (-2\sin\vartheta\sin\varphi, 2\sin\vartheta\cos\varphi, 0)$$

$$\frac{\partial\Phi}{\partial\vartheta} = (2\cos\vartheta\cos\varphi, 2\cos\vartheta\sin\varphi, -2\sin\vartheta).$$

Before evaluating the norm of the vector product, we may notice that the given vectors are orthogonal, that is $\frac{\partial\Phi}{\partial u} \cdot \frac{\partial\Phi}{\partial v} = 0$. Then, we have

$$\left\|\frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v}\right\| = \left\|\frac{\partial\Phi}{\partial u}\right\| \cdot \left\|\frac{\partial\Phi}{\partial v}\right\| = 4|\sin\vartheta|.$$

Thus,

$$\iint\limits_M x^2z + y^2z \; dS = \iint\limits_U (8\sin^2\vartheta\cos\vartheta) \cdot 4|\sin\vartheta| \; dS = \Big(\int_0^{\frac{\pi}{2}} 32\sin^3\vartheta\cos\vartheta \; d\vartheta\Big) \cdot \Big(\int_0^{2\pi} 1 \; d\varphi\Big) =$$

$$= 2\pi\Big[8\sin^4\vartheta\Big]_0^{\frac{\pi}{2}} = 16\pi.$$

10 Green, Gauss, and Stokes theorems

Exercise 10.1. *Use Green's theorem to find the work done by the force field $\vec{F}(x, y, z) = (2xy^3, 4x^2y^2)$ on a particle moving along the path Γ, that is the boundary of the region M bounded by the curves $y = 0$, $x = 1$ and $y = x^3$ in the first quadrant. The path Γ is positively oriented (anticlockwise).*

Solution:
We have,

$$M: \quad 0 \leq x \leq 1 \quad \& \quad 0 \leq y \leq x^3.$$

Its boundary is a piecewise differentiable path, oriented like in the hypothesis of Green's theorem.
Using Green's theorem for the vector field $\vec{F} = (F_1, F_2)$, we have

$$\int\limits_{\partial M} \mathbf{F} \cdot d\mathbf{s} = \iint\limits_M \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big) \; dS,$$

where the formula on the right side can be remembered like $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix}$.

We have,

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 8xy^2 - 6xy^2 = 2xy^2$$

therefore

$$\int\limits_{\Gamma=\partial M} \mathbf{F} \cdot d\mathbf{s} = \iint\limits_M \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\ dS = \iint\limits_M 2xy^2\ dS = \int\limits_0^1 \int\limits_0^{x^3} 2xy^2\ dy\ dx = \int\limits_0^1 \frac{2}{3}x^{10}\ dx = \frac{2}{33}.$$

Exercise 10.2. *Evaluate the surface integral*

$$\iint\limits_M \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x,y,z) = (x^2, y^2, z^2)$ *and* M *is the sphere* $(x-a)^2+(y-b)^2+(z-c)^2 = R^2$ *with outward orientations and* $R > 0$ *is a parameter.*

Solution:
The flux of the vector field $\mathbf{F} : M \to \mathbb{R}^3$ through the oriented surface $M \subseteq \mathbb{R}^3$ is evaluated as

$$\iint\limits_M \mathbf{F} \cdot d\vec{S} = \iint\limits_U \mathbf{F}\big(\Phi(u,v)\big) \cdot \left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right) dS\ ,$$

where $\Phi : U \to M$ is a suitable parametrization, $U \subseteq \mathbb{R}^2$, and the orientation given by the vector field $\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}$ agrees with the one given by the parametrization of the surface M. (In case the orientation should not agree, it is enough to change the order of the vectors in the vector product, that is equivalent to changing the sign of the integral.)

We evaluate the integral using the definition (even if we could use Gauss theorem, being the surface a closed one). The surface M will be parametrized using spherical coordinates with a shift:

$$\Phi : \quad \begin{aligned} x - a &= R\sin\vartheta\cos\varphi \\ y - b &= R\sin\vartheta\sin\varphi \\ z - c &= R\cos\vartheta \end{aligned}$$

with domain

$$U : \quad 0 \le \varphi \le 2\pi \quad \& \quad 0 \le \vartheta \le \pi\ .$$

We have

$$\begin{aligned} \frac{\partial \Phi}{\partial \varphi} &= \big(\ -R\sin\vartheta\sin\varphi,\ R\sin\vartheta\cos\varphi,\ 0\ \big) \\ \frac{\partial \Phi}{\partial \vartheta} &= \big(\ R\cos\vartheta\cos\varphi,\ R\cos\vartheta\sin\varphi,\ -R\sin\vartheta\big) \end{aligned}$$

a

$$\frac{\partial \Phi}{\partial \varphi} \times \frac{\partial \Phi}{\partial \vartheta} = -R\sin\vartheta \cdot \big(R\sin\vartheta\cos\varphi, R\sin\vartheta\sin\varphi, R\cos\vartheta\big) = -R\sin\vartheta \cdot \big(x-a,\ y-b,\ z-c\big).$$

The vector $(x-a,\ y-b,\ z-c)$ is oriented from the point (a,b,c) to the point $(x,y,z)\in M$, thus the vector product has opposite orientation than the outward one required in the problem. Then, instead of $\frac{\partial\Phi}{\partial\varphi}\times\frac{\partial\Phi}{\partial\vartheta}$, in the integral we use $\frac{\partial\Phi}{\partial\vartheta}\times\frac{\partial\Phi}{\partial\varphi}$.

For the scalar product in the integral, we have

$$(\mathbf{F}\circ\Phi)\cdot\left(\frac{\partial\Phi}{\partial\vartheta}\times\frac{\partial\Phi}{\partial\varphi}\right)=$$

$$=R\sin\vartheta\cdot\Big((a+R\sin\vartheta\cos\varphi)^2,\ (b+R\sin\vartheta\sin\varphi)^2,\ (c+R\cos\vartheta)^2\Big)\cdot\begin{pmatrix}R\sin\vartheta\cos\varphi\\ R\sin\vartheta\sin\varphi\\ R\cos\vartheta\end{pmatrix}=$$

$$=R^2\sin\vartheta\cdot\Big[2R\big(a\cos^2\varphi\sin^2\vartheta+b\sin^2\varphi\sin^2\vartheta+c\cos^2\vartheta\big)+c^2\cos\vartheta+R^2\cos^3\vartheta+\cdots\Big]$$

where the dots indicate terms where it is possible to factor out either "$\sin\varphi$" or "$\cos\varphi$" with an odd power; those terms have a zero contribute in the integral. We have

$$\iint_M \mathbf{F}\cdot d\vec{S}=$$

$$=\iint_{\substack{0\le\varphi\le2\pi\\0\le\vartheta\le\pi}} R^2\sin\vartheta\cdot\Big[2R\big(a\cos^2\varphi\sin^2\vartheta+b\sin^2\varphi\sin^2\vartheta+c\cos^2\vartheta\big)+c^2\cos\vartheta+R^2\cos^3\vartheta\Big]\,d\varphi\,d\vartheta=$$

$$=\int_0^\pi R^2\sin\vartheta\cdot\Big[2R\big(\pi a\sin^2\vartheta+\pi b\sin^2\vartheta+2\pi c\cos^2\vartheta\big)+2\pi c^2\cos\vartheta+2\pi R^2\cos^3\vartheta\Big]\,d\vartheta=$$

$$=2R^3\pi(a+b)\cdot\left(\int_0^\pi\sin^3\vartheta\,d\vartheta\right)+4R^3\pi c\cdot\left(\int_0^\pi\cos^2\vartheta\sin\vartheta\,d\vartheta\right)+$$

$$+R^2\pi c^2\cdot\left(\int_0^\pi\sin(2\vartheta)\,d\vartheta\right)+2R^4\pi\cdot\left(\int_0^\pi\cos^3\vartheta\sin\vartheta\,d\vartheta\right)=$$

$$=2R^3\pi(a+b)\cdot\left(\int_0^\pi(1-\cos^2\vartheta)\sin\vartheta\,d\vartheta\right)+4R^3\pi c\cdot\Big[-\frac{\cos^3\vartheta}{3}\Big]_0^\pi+2R^4\pi\cdot\Big[-\frac{\cos^4\vartheta}{4}\Big]_0^\pi=$$

$$=\ 2R^3\pi(a+b)\cdot\left(2-\frac{2}{3}\right)+4R^3\pi c\cdot\frac{2}{3}\ =\ \frac{8\pi R^3}{3}(a+b+c)\ .$$

Exercise 10.3. *Using Stokes' theorem, evaluate*

$$\int_{\partial M}\mathbf{F}\cdot d\mathbf{s},$$

where

(i) $\mathbf{F}(x,y,z)=(y^2-z^2,z^2-x^2,x^2-y^2)$ *and* M *is the intersection of the cube* $J=\langle0,a\rangle^3$ *with the plane* $x+y+z=\frac{3a}{2}$. *The orientation is given by the order of the points* $\left(\frac{a}{2},a,0\right)$, $\left(a,0,\frac{a}{2}\right)$ *and* $\left(a,\frac{a}{2},0\right)$.

(ii) $\mathbf{F}(x,y,z)=(y^2,z^2,x^2)$ *and* M *is the triangle* $(a,0,0)$, $(0,a,0)$ *and* $(0,0,a)$. *The orientation is the one given by order of the vertexes.*

Solution:
Stokes' theorem is the extension of Green's theorem from $\mathbb{R}^2$ to $\mathbb{R}^3$:

$$\int_{\partial M} \mathbf{F} \cdot d\mathbf{s} = \iint_{M} \text{curl}(\mathbf{F}) \cdot d\mathbf{S},$$

where the orientation of the surface and the one of its boundary must agree with the rule of the right hand (posing the right hand with the fingers pointing in the positive direction of the boundary, the tum, orthogonal to the rest of the fingers, is pointing in the direction of the positive normal to the surface), (i.e. the orientation of the boundary of the surface appears anticlockwise if viewed from the "top" of the normal vector to the surface).

(i) The set M is a regular hexagon with side length $\frac{\sqrt{2}}{2}a$. Moreover, we have

$$\text{curl}(\mathbf{F}) = \big(-2y - 2z, -2x - 2z, -2x - 2y\big) \ .$$

The normal vector to the surface M is the normal vector to the plane where M lies, after we normalize it, we get $\mathbf{n} = \frac{\sqrt{3}}{3}(1,1,1)$ (the direction of the vector is the wanted one). For the evaluation of the integral, we use

- the definition of flux of a vector field through a surfce,
- the fact that points on the surface satisfy the equation $x + y + z = \frac{3a}{2}$ and
- the fact that we know the area of a regular hexagon:

$$\iint_M \text{curl}(\mathbf{F}) \cdot d\vec{S} = \iint_M \big(\text{curl}(\mathbf{F}) \cdot \mathbf{n}\big) \ dS = -\frac{4\sqrt{3}}{3} \iint_M x + y + z \ dS = -\frac{4\sqrt{3}}{3} \iint_M \frac{3a}{2} \ dS =$$

$$= -2\sqrt{3}a \iint_M 1 \ dS = -2\sqrt{3}a \cdot \frac{3\sqrt{3}}{4}a^2 = -\frac{9}{2}a^3 \ .$$

(ii) We follow a similar approach to the one used in (i). The set M is an equilateral triangle with side length $\sqrt{2}a$ and it lies on the plane $x + y + z = a$. Moreover, we have

$$\text{curl}(\mathbf{F}) = \big(-2z, -2x, -2y\big) \ .$$

The normalized normal vector to the oriented surface M is $\mathbf{n} = \frac{\sqrt{3}}{3}(1,1,1)$. We evaluate:

$$\iint_M \text{curl}(\mathbf{F}) \cdot d\vec{S} = \iint_M \big(\text{curl}(\mathbf{F}) \cdot \mathbf{n}\big) \ dS = -\frac{2\sqrt{3}}{3} \iint_M x + y + z \ dS = -\frac{2\sqrt{3}}{3} \iint_M a \ dS =$$

$$= -\frac{2\sqrt{3}a}{3} \cdot \frac{\sqrt{3}}{2}a^2 = -a^3 \ .$$

Exercise 10.4. *Using Stokes' theorem, evaluate*

$$\iint_M \text{curl}(\mathbf{F}) \cdot d\mathbf{S},$$

where $\mathbf{F}(x,y,z) = (xyz, x, e^{xy}\cos z)$ *and* M *is the half sphere* $x^2+y^2+z^2 = 1$, $z \geq 0$ *with upward orientation.*

Solution:

We have

$$M: \quad x^2+y^2+z^2=1 \ \ \& \ \ z \geq 0$$

and

$$\partial M: \ \ x^2+y^2=1 \quad \& \quad z=0.$$

The surface M has upward orientation, thus the orientation of its boundary ∂M agrees, for example, with the one given by the parametrization $\varphi(\alpha)=(\cos\alpha, \sin\alpha, 0)$, for $0 \leq \alpha \leq 2\pi$.

We have:

$$\varphi'(\alpha)=(-\sin\alpha, \cos\alpha, 0)$$

$$\iint_M \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} = \int_{\partial M} \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} (0, \cos\alpha, e^{\sin\alpha\cos\alpha}) \cdot \begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix} d\alpha = \int_0^{2\pi} \cos^2\alpha \ d\alpha = \pi.$$

Exercise 10.5. *Using Gauss' theorem evaluate the flux of the vector field* $\mathbf{F}=(3y^2z^3, 9x^2yz^2, -4xy^2)$ *through the surface of the cube* $M=\langle 0,1\rangle^3 \subseteq \mathbb{R}^3$.

Solution:

Gauss' theorem

$$\iint_{\partial M} \mathbf{F}\cdot d\mathbf{S} = \iiint_M \operatorname{div}(\mathbf{F}) \ dV$$

puts in relation the flux of a vector field $\mathbf{F}$ through the boundary ∂M of a region M in $\mathbb{R}^3$ with the triple integral over the region M. The function

$$\operatorname{div}(\mathbf{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},$$

integrated over M is a vector field called divergence of $\mathbf{F}$ and it is interpreted as a "source" of the vector field at a given point (in case it has a positive value) or a "sink" of the vector field at a given point (in case it has a negative value). With Gauss' theorem we show that the "total change of the vector field" inside the region corresponds with the flux of the vector field through its boundary.

Using Gauss' theorem, we suppose that the surface of the cube ∂M has outward orientation. We have

$$\operatorname{div}(\mathbf{F}) = 9x^2z^2$$

and

$$\iint_{\partial M} \mathbf{F}\cdot d\mathbf{S} = \iiint_M \operatorname{div}(\mathbf{F}) \ dV = \int_0^1\int_0^1\int_0^1 9x^2z^2 \ dx \ dy \ dz = \Big(\int_0^1 9x^2 \ dx\Big)\cdot\Big(\int_0^1 1 \ dy\Big)\cdot\Big(\int_0^1 z^2 \ dz\Big) = 1.$$

Exercise 10.6. *Verify Gauss' theorem for the vector field* $\mathbf{F}=(x^3, y^3, z^3)$ *and the sphere* $x^2+y^2+z^2=1$.

Solution:

We verify Gauss's theorem

$$\iint_{\partial M} \mathbf{F}\cdot d\mathbf{S} = \iiint_M \operatorname{div}(\mathbf{F}) \ dV$$

proving that both integrals have the same value.

In our case, we have

$$M: \quad x^2+y^2+z^2 \le 1$$

and

$$\partial M: \quad x^2+y^2+z^2 = 1 \ .$$

The orientation of the boundary ∂M, in this case, is given by the normal vector pointing outside.

We have

$$\operatorname{div}(\mathbf{F}) = 3(x^2+y^2+z^2)$$

and

$$\iiint\limits_M \operatorname{div}(\mathbf{F})\ dV = \iiint\limits_M 3(x^2+y^2+z^2)\ dV = \left[\begin{smallmatrix} x=r\sin\vartheta\cos\varphi \\ y=r\sin\vartheta\sin\varphi \\ z=r\cos\vartheta \\ (r,\varphi,\vartheta)\in\langle 0,1\rangle\times\langle 0,2\pi\rangle\times\langle 0,\pi\rangle \end{smallmatrix} \right] =$$

$$= \int\limits_0^\pi \int\limits_0^{2\pi} \int\limits_0^1 3r^2 \cdot |r^2 \sin\vartheta|\ dr\ d\varphi\ d\vartheta = \Big(\int\limits_0^1 3r^4\ dr \Big) \cdot \Big(\int\limits_0^{2\pi} 1\ d\varphi \Big) \cdot \Big(\int\limits_0^\pi \sin\vartheta\ d\vartheta \Big) = \frac{3}{5} \cdot 2\pi \cdot 2 = \frac{12}{5}\pi.$$

For the second integral, we choose to parametrize ∂M using spherical coordinates

$$\Phi(\varphi,\vartheta) = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta),$$

with domain

$$U: \quad 0 \le \varphi \le 2\pi \quad \& \quad 0 \le \vartheta \le \pi \ .$$

We have

$$\begin{aligned} \tfrac{\partial\Phi}{\partial\varphi} &= (\ -\sin\vartheta\sin\varphi,\ \sin\vartheta\cos\varphi,\ \ \ 0\ ) \\ \tfrac{\partial\Phi}{\partial\vartheta} &= (\ \ \cos\vartheta\cos\varphi,\ \cos\vartheta\sin\varphi,\ -\sin\vartheta\ ) \end{aligned}$$

and

$$\frac{\partial\Phi}{\partial\varphi} \times \frac{\partial\Phi}{\partial\vartheta} = -\sin\vartheta \cdot (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) = -\sin\vartheta \cdot \boldsymbol{\Phi}(\varphi,\vartheta).$$

The vector $\frac{\partial\Phi}{\partial\varphi} \times \frac{\partial\Phi}{\partial\vartheta}$ has the direction of the normal vector pointing inside. Since the surface has outward orientation, in evaluating the flux of the vector field, we will consider the opposite vector, that is $-\frac{\partial\Phi}{\partial\varphi} \times \frac{\partial\Phi}{\partial\vartheta} = \frac{\partial\Phi}{\partial\vartheta} \times \frac{\partial\Phi}{\partial\varphi}$.

Substituting, we obtain

$$\iint\limits_{\partial M} \mathbf{F} \cdot d\mathbf{S} = \iint\limits_U \mathbf{F}(\Phi(\varphi,\vartheta)) \cdot \Big(-\frac{\partial\Phi}{\partial\varphi} \times \frac{\partial\Phi}{\partial\vartheta} \Big) dS = \iint\limits_U \mathbf{F}(\Phi(\varphi,\vartheta)) \cdot \Big(\sin\vartheta \cdot \boldsymbol{\Phi}(\varphi,\vartheta) \Big)\ dS =$$

$$= \int\limits_0^\pi \int\limits_0^{2\pi} \sin\vartheta \Big(\sin^4\vartheta\cos^4\varphi + \sin^4\vartheta\sin^4\varphi + \cos^4\vartheta \Big)\ d\varphi\ d\vartheta =$$

$$= \Big(\int\limits_0^\pi \sin^5\vartheta\ d\vartheta \Big) \cdot \Big(\int\limits_0^{2\pi} \cos^4\varphi + \sin^4\varphi\ d\varphi \Big) + \Big(\int\limits_0^\pi \cos^4\vartheta\sin\vartheta\ d\vartheta \Big) \cdot \Big(\int\limits_0^{2\pi} 1\ d\varphi \Big).$$

For the first two integrals, due to shift and symmetry, we have that

$$\int\limits_0^{2\pi} \cos^4\varphi\ d\varphi = \int\limits_0^{2\pi} \sin^4\varphi\ d\varphi = 4 \int\limits_0^{\frac{\pi}{2}} \sin^4\varphi\ d\varphi$$

and

$$\int_0^{\pi} \sin^5 \vartheta \; d\vartheta = 2 \int_0^{\frac{\pi}{2}} \sin^5 \vartheta \; d\vartheta.$$

For $n \geq 2$ we evaluate the integral

$$A_n := \int_0^{\frac{\pi}{2}} \sin^n \alpha \; d\alpha = [-\cos \alpha \cdot \sin^{n-1} \alpha]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1) \sin^{n-2} \alpha \cdot \cos^2 \alpha \; d\alpha =$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} \alpha \cdot (1 - \sin^2 \alpha) \; d\alpha = (n-1)A_{n-2} - (n-1)A_n.$$

Thus, we have $A_n = \frac{n-1}{n} A_{n-2}$ a $A_2 = \frac{\pi}{4}$ a $A_0 = 1$. Carrying on the calculation, we get

$$\iint_{\partial M} \mathbf{F} \cdot d\mathbf{S} = \cdots = \left(2 \cdot \frac{4}{5} \cdot \frac{2}{3}\right) \cdot \left(8 \cdot \frac{3}{4} \cdot \frac{\pi}{4}\right) + \left(\left[-\frac{\cos^5 \vartheta}{5}\right]_0^{\pi}\right) \cdot \left(2\pi\right) = \frac{8}{5}\pi + \frac{4}{5}\pi = \frac{12}{5}\pi,$$

and Gauss' theorem is thus verified.

11 Fourier series

Exercise 11.1. *Evaluate the Fourier series of the periodic extension of $f(t) = t^2$ on $[-1, 1)$.*

Solution:

Definition Let f be a function that is T-periodic and integrable on $[0, T]$.

We define its **Fourier series** as $\frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \big[a_k \cos(k\omega t) + b_k \sin(k\omega t)\big]$, where $\omega = \frac{2\pi}{T}$ is the frequency,

$$a_k = \tfrac{2}{T} \int\limits_0^T f(t) \cos(k\omega t)\, dt, \quad \text{for} \quad k \in \mathbb{N}_0,$$

and

$$b_k = \tfrac{2}{T} \int\limits_0^T f(t) \sin(k\omega t)\, dt, \quad \text{for} \quad k \in \mathbb{N}.$$

We use the notation

$$f \sim \tfrac{a_0}{2} + \sum_{k=1}^{\infty} \big[a_k \cos(k\omega t) + b_k \sin(k\omega t)\big].$$

Observation (i) If f is odd, then $a_k = 0$ and $b_k = \frac{4}{T} \int\limits_0^{T/2} f(t) \sin(k\omega t)\, dt$.

(ii) If f is even, then $b_k = 0$ and $a_k = \frac{4}{T} \int\limits_0^{T/2} f(t) \cos(k\omega t)\, dt$.

For the given function f, we have $T = 2$, thus $\omega = \pi$. We evaluate:

$$a_0 = \tfrac{2}{2} \int\limits_0^2 f(t)\, dt = \int\limits_{-1}^1 f(t)\, dt = \int\limits_{-1}^1 t^2 dt = \frac{2}{3},$$

$$a_k = \tfrac{2}{2} \int\limits_{-1}^1 t^2 \cos(k\pi t)\, dt = \frac{4\cos(k\pi)}{\pi^2 k^2} = \frac{4(-1)^k}{\pi^2 k^2},$$

$$b_k = \tfrac{2}{2} \int\limits_{-1}^1 t^2 \sin(k\pi t)\, dt = 0.$$

Thus

$$f \sim \tfrac{1}{3} + \sum_{k=1}^{\infty} \tfrac{4(-1)^k}{\pi^2 k^2} \cos(k\pi t).$$

Observation: Jordan criterion states the following:

Let f be a T-periodic function that is piecewise continuous on some interval I of length T, assume that it has a derivative f' piecewise continuous on I.

Let $f \sim \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \big[a_k \cos(k\omega t) + b_k \sin(k\omega t)\big]$. Then for every $t \in \mathbb{R}$ we have

$$\lim_{N\to\infty} \Big(\tfrac{a_0}{2} + \sum_{k=1}^{N} \big[a_k \cos(k\omega t) + b_k \sin(k\omega t)\big] \Big) = \tfrac{1}{2}[f(t^-) + f(t^+)].$$

If moreover f is continuous on $\mathbb{R}$, then $\frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \big[a_k \cos(k\omega t) + b_k \sin(k\omega t)\big]$ is uniformly convergent to f.

In the given example, we have evaluated that, for every $t \in [-1, 1]$, $t^2 = \frac{1}{3} + \sum\limits_{k=1}^{\infty} \frac{4(-1)^k}{\pi^2 k^2} \cos(k\pi t)$.

If we use this equality for $t = 0$, we get the sum of a particular numerical series:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}.$$

While, for $t = 1$, we get

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Exercise 11.2. *Consider the function*

$$f(t) = \begin{cases} t & , \ t \in [0,1) \\ 0 & , \ t \in [1,2) \end{cases} .$$

Evaluate (a) the Fourier series, (b) the sine Fourier series and (c) the cosine Fourier series of the appropriate periodic extension of f.

Solution:

Definition Let f be a function defined and continuous on $[0, L)$. We define its **s**ine series as the Fourier series of its odd periodic extension. We define its **c**osine series as the Fourier series of its even periodic extension.

Observation The sine Fourier series of f can be obtained as a Fourier series with $a_k = 0$, $b_k = \frac{2}{L}\int_0^L f(t)\sin(k\omega t)\,dt$ and $\omega = \frac{\pi}{L}$.

The cosine Fourier series of f can be obtained as a Fourier series with $b_k = 0$, $a_k = \frac{2}{L}\int_0^L f(t)\cos(k\omega t)\,dt$ and $\omega = \frac{\pi}{L}$.

Remark: The sum of the sine series is a $T = 2L$-periodic extension of f into an odd function. The sum of the cosine series is a $T = 2L$-periodic extension of f into an even function. Both sums must be also modified using the Jordan criterion.

In our case

$$f(t) = \begin{cases} t & , \ t \in [0,1) \\ 0 & , \ t \in [1,2) \end{cases}$$

(a) For the Fourier series of f, we have: $T = 2$, $\omega = \pi$.

$$a_0 = \tfrac{2}{2}\int_0^2 f(t)\,dt = \int_0^1 dt = 1.$$

$$a_k = \tfrac{2}{2}\int_0^1 \cos(k\pi t)\,dt = \Big[\frac{1}{k\pi}\sin(k\pi t)\Big]_0^1 = 0.$$

$$b_k = \tfrac{2}{2}\int_0^1 \sin(k\pi t)\,dt = \Big[-\frac{1}{k\pi}\cos(k\pi t)\Big]_0^1 = \frac{1}{k\pi}[1-\cos(k\pi)] = \frac{1}{k\pi}[1-(-1)^k] = \begin{cases} 0 & , \ k \text{ even} \\ \frac{2}{(2k+1)\pi} & , \ k \text{ odd} \end{cases}$$

Thus

$$f \sim \tfrac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{k\pi}[1-(-1)^k]\sin(k\pi t) = \tfrac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi}\sin((2k+1)\pi t).$$

(b) For the sine Fourier series of f, we have: $L = 2$, $T = 4$, $\omega = \frac{\pi}{2}$, $a_k = 0$.

$$b_k = \tfrac{2}{2}\int_0^1 \sin\big(k\frac{\pi}{2}t\big)\,dt = \Big[-\frac{2}{k\pi}\cos\big(k\frac{\pi}{2}t\big)\Big]_0^1 = \frac{2}{k\pi}[\cos(k\pi) - \cos\big(k\frac{\pi}{2}\big)].$$

Thus the sine Fourier series of the given function f is

$$\sum_{k=1}^{\infty} \frac{2}{k\pi}[(-1)^k - \cos\big(k\frac{\pi}{2}\big)]\sin\big(k\frac{\pi}{2}t\big).$$

(c) For the cosine Fourier series of f, we have $L = 2$, $T = 4$, $\omega = \frac{\pi}{2}$, $b_k = 0$.

$$a_0 = \frac{2}{2}\int_0^1 dt = 1.$$

$$a_k = \frac{2}{2}\int_0^1 \cos\big(k\frac{\pi}{2}t\big)\,dt = \big[\frac{2}{k\pi}\sin\big(k\frac{\pi}{2}t\big)\big]_0^1 = \frac{2}{k\pi}\sin\big(k\frac{\pi}{2}\big).$$

Thus the cosine Fourier series of the given function f is

$$\frac{1}{2} + \sum_{k=1}^{\infty}\frac{2}{k\pi}\sin\big(k\frac{\pi}{2}\big)\cos\big(k\frac{\pi}{2}t\big).$$

Here $a_{2k} = 0$, $a_{2k+1} = (-1)^{k+1}\frac{2}{(2k+1)\pi}$, so $\frac{1}{2} + \sum\limits_{k=0}^{\infty}(-1)^{k+1}\frac{2}{(2k+1)\pi}\cos\big((2k+1)\frac{\pi}{2}t\big)$.

Exercise 11.3. *Evaluate the Fourier series of the periodic extension of $f(t) = \sin t$, $0 \le t < \frac{\pi}{2}$. Specify to which function the Fourier series converges.*

Solution:
The period of the function is $T = \frac{\pi}{2}$, thus $\omega = 4$. The given function is neither even nor odd. The coefficients of the Fourier series are evaluated as follows

$$a_0 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \sin t\,dt = \frac{4}{\pi}[-\cos t]_0^{\frac{\pi}{2}}\,dt = \frac{4}{\pi}.$$

$$a_k = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \sin t\cos(4kt)\,dt = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\big(\sin(4k+1)t - \sin(4k-1)t\big)\,dt =$$

$$= \frac{2}{\pi}\left[-\frac{\cos(4k+1)t}{4k+1} + \frac{\cos(4k-1)t}{4k-1}\right]_0^{\frac{\pi}{2}} = \frac{2}{\pi}\left(\frac{1}{4k+1} - \frac{1}{4k-1}\right) = \frac{-4}{\pi(16k^2-1)}.$$

$$b_k = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \sin t\sin(4kt)\,dt = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\big(-\cos(4k+1)t + \cos(4k-1)t\big)\,dt =$$

$$= \frac{2}{\pi}\left[-\frac{\sin(4k+1)t}{4k+1} + \frac{\cos(4k-1)t}{4k-1}\right]_0^{\frac{\pi}{2}} = \frac{2}{\pi}\left(-\frac{1}{4k+1} - \frac{1}{4k-1}\right) = \frac{-16}{\pi(16k^2-1)}.$$

Thus

$$f \sim \frac{2}{\pi} - \sum_{k=1}^{\infty}\frac{4}{\pi(16k^2-1)}(\cos 4kt + 4k\sin 4kt),\ t \in \mathbb{R}.$$

The periodic extension of the given function is not continuous at points $t = \frac{k\pi}{2}$, $k \in \mathbb{Z}$. At this points the Fourier series converges to $\frac{1}{2}[f(t^-) + f(t^+)] = \frac{1}{2}$. At any other point, the Fourier series converges to the periodic extension of the given function.

Exercise 11.4. *Evaluate the Fourier series of the periodic extension of* $f(t) = |t|$, $-1 \le t < 1$.

Solution:
The period of the function is $T = 2$, thus $\omega = \frac{2\pi}{2}$. The extension of the given function is even, thus $b_k = 0$. The other coefficients of the Fourier series are evaluated as follows:

$$a_0 = 2\frac{2}{2}\int_0^1 t\,dt = 2\left[\frac{t^2}{2}\right]_0^1 dt = 1;$$

$$a_k = 2\int_0^1 t\cos k\pi t\,dt = 2\left[t\frac{\sin k\pi t}{k\pi} + \frac{\cos k\pi t}{k^2\pi^2}\right]_0^1 = \frac{2}{k^2\pi^2}(\cos k\pi - 1).$$

We now observe that, for every even k, $\cos k\pi = 1$, thus $a_k = 0$; while for every odd k, $k = 2n+1$, $\cos(2n+1)\pi = -1$, thus $a_{2n+1} = \frac{-4}{\pi^2(2n+1)^2}$. We conclude that the Fourier series can be written as follows

$$f = \frac{1}{2} - \sum_{n=0}^{\infty}\frac{4}{\pi^2(2n+1)^2}\cos(2n+1)\pi t,\ t \in \mathbb{R}.$$

Since the periodic extension of the given function is continuous, the Fourier series converges uniformly to it on $\mathbb{R}$, that is why we have written a sign of equality between f and the Fourier series.

Exercise 11.5. *Evaluate the sine Fourier series of the appropriate periodic extension of* $f(t) = \sin t$, $0 \le t < \frac{\pi}{2}$. *Specify to which function the Fourier series converges.*

Solution:
In evaluating the sine Fourier series of a function defined on an interval $[0, L)$, we must start from extending the given function in an odd way to the interval $[-L, L)$. In our case $L = \frac{\pi}{2}$, and extending the function $\sin t$ in an odd way to the interval $[-\frac{\pi}{2}, \frac{\pi}{2})$, we get again the sine function. Thus the sine Fourier series of the function $f(t) = \sin t$, $0 \le t < \frac{\pi}{2}$ is equivalent to the Fourier series of the function $f(t) = \sin t$, $-\frac{\pi}{2} \le t < \frac{\pi}{2}$. All coefficients a_k are zero due to the symmetry, for the b_k we get

$$b_k = 2\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin t \sin(2kt)\,dt = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\big(-\cos(2k+1)t + \cos(2k-1)t\big)\,dt =$$

$$= \frac{2}{\pi}\left[-\frac{\sin(2k+1)t}{2k+1} + \frac{\cos(2k-1)t}{2k-1}\right]_0^{\frac{\pi}{2}} = \frac{2}{\pi}\left(-\frac{(-1)^{k+1}}{2k+1} + \frac{(-1)^{k+1}}{2k-1}\right) = \frac{8k(-1)^{k+1}}{\pi(4k^2-1)}.$$

We get

$$f \sim \sum_{k=1}^{\infty}\frac{8k(-1)^{k+1}}{\pi(4k^2-1)}\sin 2kt,\ t \in \mathbb{R}.$$

The odd periodic extension of the given function is not continuous at points $t = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$. At this points the sine Fourier series converges to $\frac{1}{2}[f(t^-) + f(t^+)] = 0$. At any other point, the sine Fourier series converges to the odd periodic extension of the given function.

Exercise 11.6. *Consider the function*

$$f(t) = \begin{cases} 1 & ,\ t \in [0, 1) \\ -2 & ,\ t \in [1, 2) \end{cases}$$

Evaluate the Fourier series of the appropriate periodic extension of f.

Solution:
For the Fourier series, we have $T = 2$, thus $\omega = \pi$. We evaluate now the coefficients:

$$a_0 = \frac{2}{2}\Big(\int_0^1 1\,dt + \int_1^2 -2\,dt\Big) = -1;$$

$$a_k = \frac{2}{2}\Big(\int_0^1 \cos k\pi t\,dt - 2\int_1^2 \cos k\pi t\,dt\Big) = 0;$$

$$b_k = \frac{2}{2}\Big(\int_0^1 \sin k\pi t\,dt - 2\int_1^2 \sin k\pi t\,dt\Big) = \frac{3}{k\pi}[1-(-1)^k] = \begin{cases} 0 & , \ k \text{ even}, \\ \frac{6}{k\pi} & , \ k \text{ odd}. \end{cases}$$

Thus, using $k = 2n+1$ to indicate all odd numbers like in Exercise 4:

$$f \sim -\frac{1}{2} + \sum_{k=1}^{\infty} \frac{3}{k\pi}[1-(-1)^k]\sin k\pi t = -\frac{1}{2} + \sum_{n=0}^{\infty} \frac{6}{(2n+1)\pi}\sin(2n+1)\pi t,\ t \in \mathbb{R}.$$

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