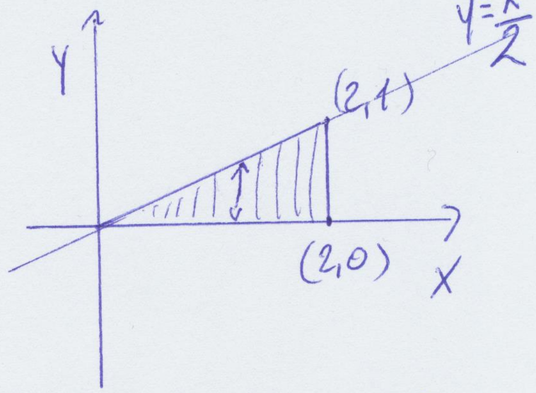
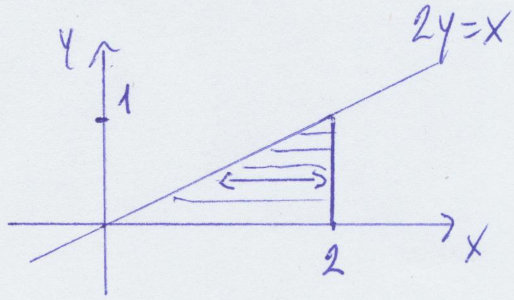
Dvojný integrál.

1 Spočítejte $\iint\limits_D (xe^y + 2)\, dxdy$, kde D je trojúhelník s vrcholy $(0,0)$, $(2,1)$ a $(2,0)$.

i) $\displaystyle\int_0^2 \int_0^{\frac{x}{2}} (xe^y + 2)\, dy\, dx$ nebo



ii) $\displaystyle\int_0^1 \int_{2y}^{2} (xe^y + 2)\, dx\, dy$

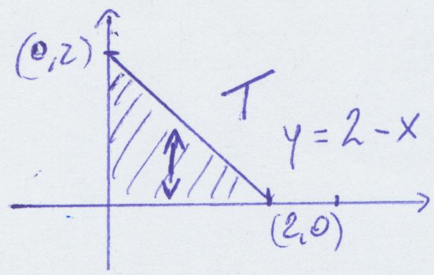


i) $\displaystyle\int_0^2 \left[xe^y + 2y\right]_{y=0}^{y=\frac{x}{2}} dx = \int_0^2 \left(xe^{\frac{x}{2}} + \cancel{\frac{2x}{2} - x}\right) dx =$

$= \begin{vmatrix} u = x & v' = e^{\frac{x}{2}} \\ u' = 1 & v = 2e^{\frac{x}{2}} \end{vmatrix} = \left[2xe^{\frac{x}{2}} - 4e^{\frac{x}{2}}\right]_0^2 =$

$= 4e - 4e - (-4) = 4$

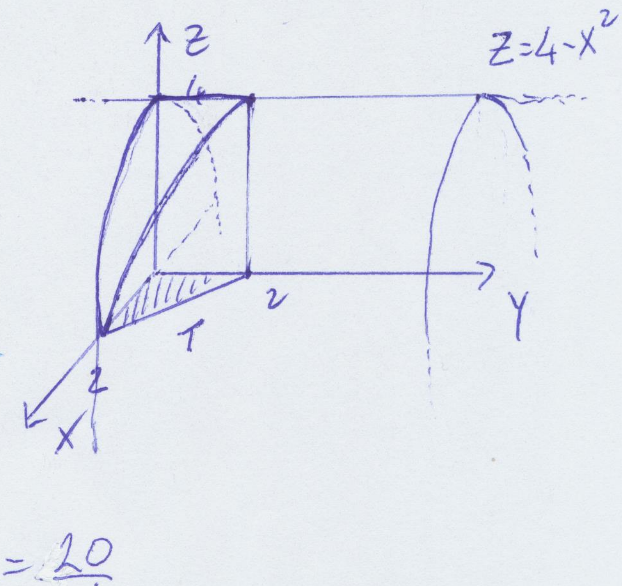
2 Určete objem tělesa pod grafem funkce $f(x,y) = 4 - x^2$ nad trojúhelníkem T s vrcholy $(0,0)$, $(2,0)$ a $(0,2)$.



$\text{Objem} = \iint\limits_T (4 - x^2)\, dA =$

$= \displaystyle\int_0^2 \int_0^{2-x} (4 - x^2)\, dy\, dx =$

$= \displaystyle\int_0^2 (4 - x^2) \left[y\right]_0^{2-x} dx =$

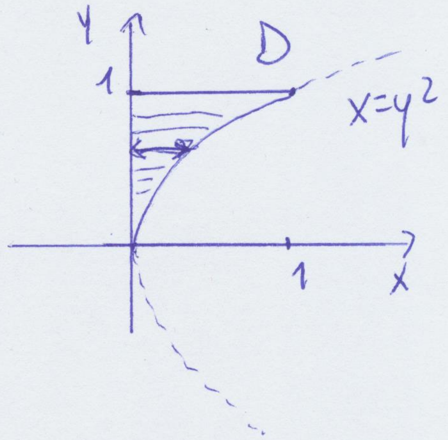


$= \displaystyle\int_0^2 (4 - x^2)(2 - x)\, dx =$

$= \displaystyle\int_0^2 (8 - 4x - 2x^2 + x^3)\, dx =$

$= \left[8x - 2x^2 - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^2 = 16 - 8 - \frac{16}{3} + 4 = \frac{20}{3}$

3 Spočítejte $\iint\limits_D 3y^3 e^{xy} dA$, kde D je omezeno křivkami $x = 0$, $y = 1$ a $x = y^2$.



$$\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy = \int_0^1 \left[3y^3 \frac{e^{xy}}{y} \right]_{x=0}^{x=y^2} dy =$$

$$= \int_0^1 \left(3y^2 e^{y^3} - 3y^2 \right) dy =$$

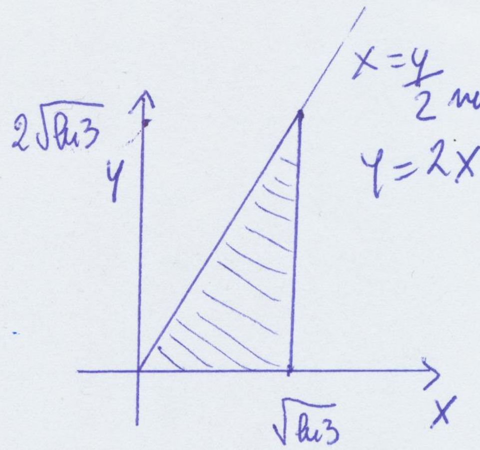
$$= \left[e^{y^3} - y^3 \right]_0^1 = e - 1 - 1 = e - 2$$

$$\int 3y^2 e^{y^3} dy = \left| \begin{array}{l} u = y^3 \\ du = 3y^2 dy \end{array} \right|$$

$$\int e^u du = e^u = e^{y^3} + C$$

4 Změňte pořadí integrace u následujících integrálů a spočítejte

$$\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\, dx dy \, .$$

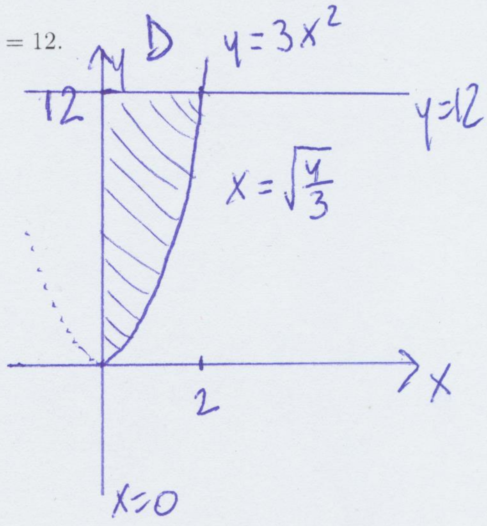


$$\int_0^{\sqrt{\ln 3}} \int_0^{2x} e^{x^2} dy\, dx =$$

$$= \int_0^{\sqrt{\ln 3}} e^{x^2} \left[y \right]_{y=0}^{y=2x} dx = \int_0^{\sqrt{\ln 3}} 2x e^{x^2} dx =$$

$$= \left[e^{x^2} \right]_0^{\sqrt{\ln 3}} = e^{\ln 3} - e^0 = 3 - 1 = 2 .$$

5 Spočítejte $\iint\limits_D x^5 \sin(y^4)\; dxdy$, kde D je omezeno křivkami $y = 3x^2$, $x = 0$ a $y = 12$.

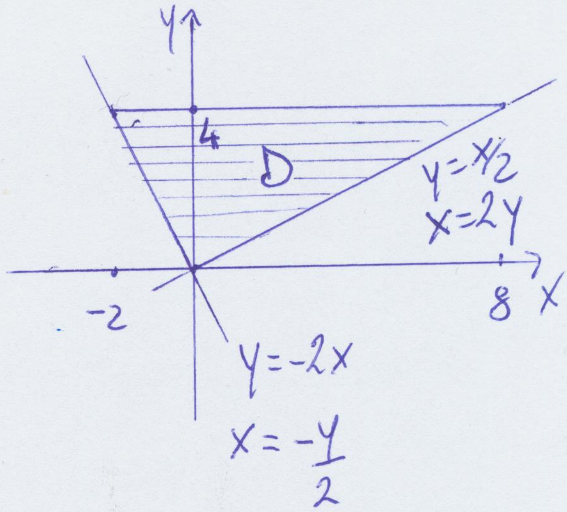


$$\int_0^{12}\int_0^{\sqrt{\frac{y}{3}}} x^5 \sin(y^4)\,dx\,dy = \int_0^{12} \sin(y^4)\left[\frac{x^6}{6}\right]_0^{\sqrt{\frac{y}{3}}} dy =$$

$$= \int_0^{12} \frac{y^3}{6\cdot 3^3}\sin(y^4)\,dy = -\left[\frac{\cos y^4}{24\cdot 3^3}\right]_0^{12} =$$

$$= -\frac{1}{648}\left(\cos(12)^4 - 1\right).$$

6 Spočítejte $\iint\limits_D e^{y^2+1}\; dxdy$, kde D je trojúhelník s vrcholy $(0,0)$, $(-2,4)$ a $(8,4)$.



$$\int_0^4 \int_{-\frac{y}{2}}^{2y} e^{y^2+1}\,dx\,dy = \int_0^4 e^{y^2+1}\left[x\right]_{-\frac{y}{2}}^{2y} dy =$$

$$= \int_0^4 e^{y^2+1}\left(2y - \frac{y}{2}\right) dy = \int_0^4 \frac{3}{2} y\, e^{y^2+1}\,dy =$$

$$= \frac{3}{4}\left[e^{y^2+1}\right]_0^4 = \frac{3}{4}\left(e^{17} - e\right).$$

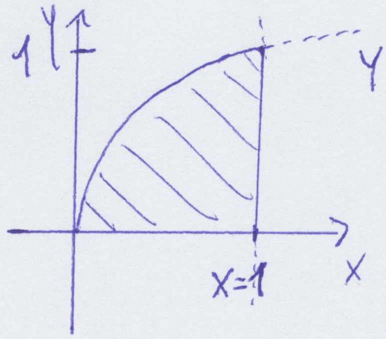
$$\int y\, e^{y^2+1}\,dy = \left| \begin{matrix} u = y^2+1 \\ du = 2y\,dy \end{matrix} \right|$$

$$\int \frac{e^u}{2}\,du = \frac{e^u}{2} = \frac{e^{y^2+1}}{2} + C$$

7 Změňte pořadí integrace u následujících integrálů:

$$i)\int_0^1\int_0^{\sqrt{x}} f(x,y)\,dydx, \qquad ii)\int_0^1\int_0^{\sqrt{x-x^2}} f(x,y)\,dydx,$$

i)

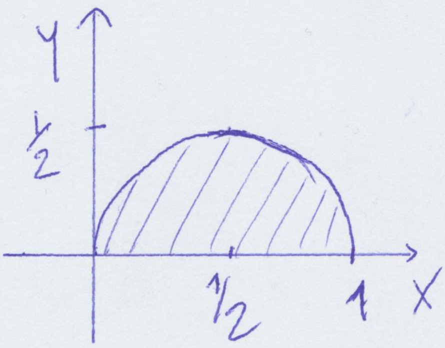


$y=\sqrt{x} \rightarrow x=y^2$

$$i) = \int_0^1\int_{y^2}^1 f(x,y)\,dx\,dy$$

ii) $y=\sqrt{x-x^2} \rightarrow \begin{cases} y>0 \\ y^2 = x-x^2 \end{cases} \rightarrow \begin{cases} y>0 \\ \left(x-\frac{1}{2}\right)^2 + y^2 = \frac{1}{4} \end{cases}$

$S=\left(\frac{1}{2},0\right) \quad r=\frac{1}{2}.$



$$\left(x-\frac{1}{2}\right)^2 = \frac{1}{4}-y^2$$

$$x-\frac{1}{2} = \pm\sqrt{\frac{1}{4}-y^2}$$

$$x = \frac{1}{2}\pm\sqrt{\frac{1}{4}-y^2}$$

$$ii) = \int_0^{1/2}\int_{\frac{1}{2}-\sqrt{\frac{1}{4}-y^2}}^{1/2+\sqrt{1/4-y^2}} f(x,y)\,dx\,dy$$