Dvojný integrál 2 (polární souřadnice).

$x = \rho \cos\varphi$

$y = \rho \sin\varphi$

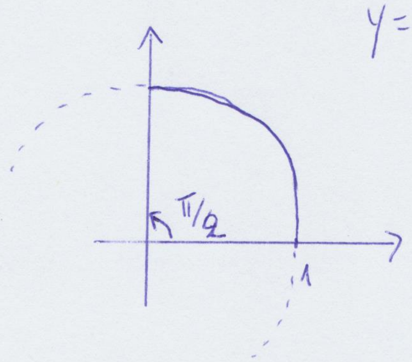
nebo

$x = r\cos\theta$

$y = r\sin\theta$

1 Použitím polárních souřadnic spočítejte integrály $\int\limits_0^1 \int\limits_0^{\sqrt{1-x^2}} \arctan \frac{y}{x}\, dy\, dx.$

$y = \sqrt{1-x^2}; \quad y^2 = 1 - x^2, \; x^2 + y^2 = 1 \rightarrow \rho = 1$

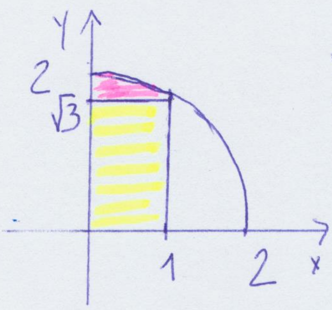


$$\int_0^{\pi/2}\int_0^1 \rho \arctan\left(\frac{\rho \sin\varphi}{\rho\cos\varphi}\right) d\rho\, d\varphi =$$

$$= \int_0^{\pi/2}\int_0^1 \rho\cdot\varphi \; d\rho\, d\varphi = \int_0^{\pi/2} \varphi\, d\varphi \cdot \int_0^1 \rho\, d\rho =$$

$$= \left[\frac{\varphi^2}{2}\right]_0^{\pi/2} \cdot \left[\frac{\rho^2}{2}\right]_0^1 = \frac{1}{2}\cdot\frac{\pi^2}{4}\cdot\frac{1}{2} = \frac{\pi^2}{16}$$

2 Přepište následující integrál $\int_0^1 \int_0^{\sqrt{4-x^2}} f\, dy\, dx$ v opačném pořadí integrace, a v polárních souřadnicích v pořadí $d\varrho\, d\varphi$.

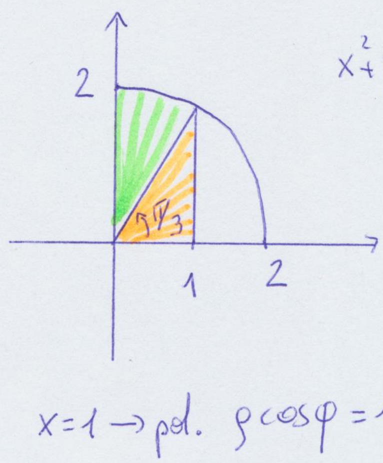


$y = \sqrt{4-x^2}$

$\begin{cases} x^2 + y^2 = 4 \\ y > 0,\; x > 0 \end{cases}$

$x = \sqrt{4-y^2}$

$$\int_0^{\sqrt{3}}\int_0^1 f(x,y)\, dx\, dy + \int_{\sqrt{3}}^2 \int_0^{\sqrt{4-y^2}} f(x,y)\, dx\, dy$$



$x^2 + y^2 = 4$

$\downarrow$ pol.

$\rho = 2$

$$\int_0^{\pi/3}\int_0^{\frac{1}{\cos\varphi}} f(\rho\cos\varphi, \rho\sin\varphi)\, \rho\, d\rho\, d\varphi +$$

$$+ \int_{\pi/3}^{\pi/2}\int_0^2 f(\rho\cos\varphi, \rho\sin\varphi)\, \rho\, d\rho\, d\varphi$$

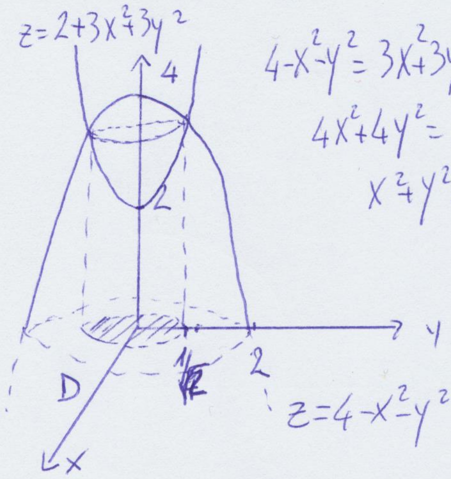
$x = 1 \rightarrow$ pol. $\rho\cos\varphi = 1$

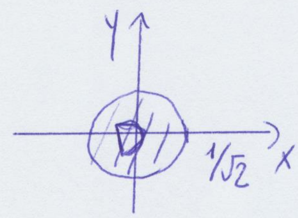
$\rho = \frac{1}{\cos\varphi}$

$\rho =$ Jakobián

3 Určete objem tělesa pod grafem funkce $f(x,y) = 4 - x^2 - y^2$ nad grafem funkce $f(x,y) = 3x^2 + 3y^2 + 2$.

$$4-x^2-y^2 = 3x^2+3y^2+2$$
$$4x^2+4y^2=2$$
$$x^2+y^2=\tfrac{1}{2}$$





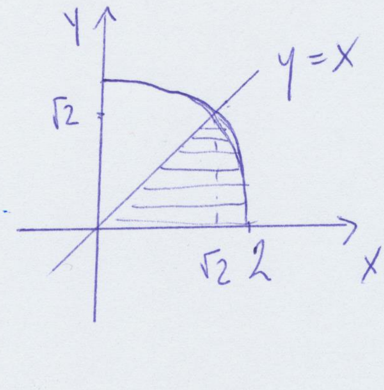
$$\text{Objem} = \iint_D (4-x^2-y^2)-(2+3x^2+3y^2)\, dA =$$

$$= \text{pol.} \int_0^{2\pi}\int_0^{\frac{1}{\sqrt2}} (2-4\rho^2)\rho \, d\rho\, d\varphi =$$

$$= \int_0^{2\pi} d\varphi \cdot \left[\rho^2 - \rho^4\right]_0^{\frac{1}{\sqrt2}} =$$

$$= 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\pi}{2}$$

4 Použitím polárních souřadnic spočítejte integrály $\int\limits_0^{\sqrt2}\int\limits_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\,dx\,dy$.



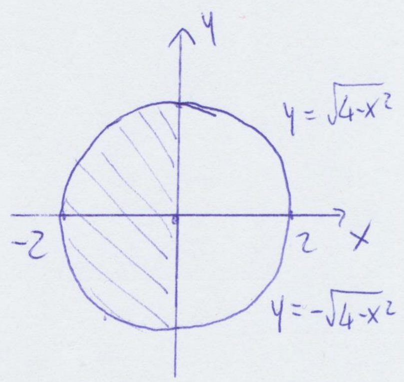
$$\int_0^{\pi/4}\int_0^2 \frac{1}{1+\rho^2}\rho\, d\rho\, d\varphi =$$

$$= \int_0^{\pi/4} d\varphi \cdot \int_0^2 \frac{1}{2}\, \frac{2\rho}{1+\rho^2}\, d\rho =$$

$$= \frac{\pi}{4}\cdot\frac{1}{2}\left[\ln(1+\rho^2)\right]_0^2 = \frac{\pi}{8}\ln 5.$$

5 Použitím polárních souřadnic spočítejte integrály $\int_{-2}^{0}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{x^2-y^2}{\sqrt{x^2+y^2}}\,dy\,dx$.

$$\int_{\pi/2}^{3/2\pi}\int_0^2 \frac{\varrho^2\cos^2\varphi-\varrho^2\sin^2\varphi}{\cancel{\varrho}}\,\cancel{\varrho}\,d\varrho\,d\varphi =$$

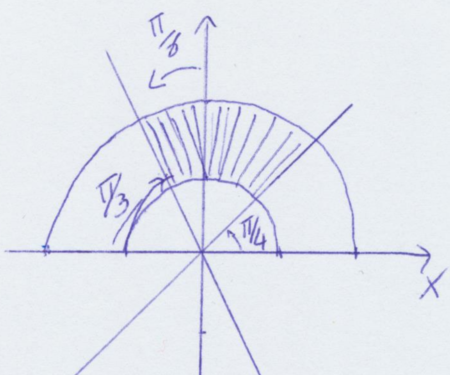


$$=\int_{\pi/2}^{3/2\pi}(\cos^2\varphi-\sin^2\varphi)\,d\varphi\cdot\int_0^2\varrho^2\,d\varrho=$$

$$=\int_{\pi/2}^{3/2\pi}\cos 2\varphi\,d\varphi\cdot\left[\frac{\varrho^3}{3}\right]_0^2=\left[\frac{\sin 2\varphi}{2}\right]_{\pi/2}^{3/2\pi}\cdot\frac{8}{3}=0$$

6 Spočítejte $\iint_D \frac{y}{\sqrt{x^2+y^2}}\,dA$, kde D je omezeno křivkami $x^2+y^2=1$, $x^2+y^2=4$, $y=-\sqrt{3}x$, $y=x$ a $y\geq 0$.

$$\int_{\pi/4}^{\frac{2}{3}\pi}\int_1^2\frac{\varrho\sin\varphi}{\varrho}\,\varrho\,d\varrho\,d\varphi=$$

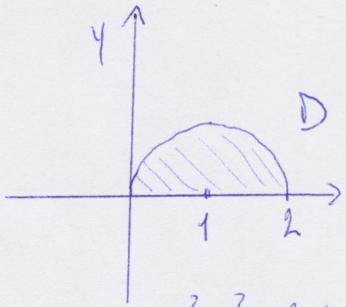


$$=\int_{\pi/4}^{\frac{2}{3}\pi}\sin\varphi\,d\varphi\cdot\int_1^2\varrho\,d\varrho=$$

$$=\left[-\cos\varphi\right]_{\pi/4}^{\frac{2}{3}\pi}\left[\frac{\varrho^2}{2}\right]_1^2=\left[\frac{1}{2}+\frac{\sqrt{2}}{2}\right]\cdot\left[2-\frac{1}{2}\right]=\frac{3}{4}\left(1+\sqrt{2}\right)$$

7 Spočítejte $\iint\limits_D xy\sqrt{x^2+y^2}\, dA$, kde $D = \{(x, y) : y \geq 0,\ x^2 - 2x + y^2 \leq 0\}\}$.

$(x-1)^2+y^2 \leq 1$



$x^2+y^2 = 2x$

$\downarrow$ pol.

$\varrho^2 = 2\varrho\cos\varphi$

$\varrho = 2\cos\varphi$

$$\int_0^{\pi/2}\int_0^{2\cos\varphi} \varrho\cos\varphi\, \varrho\sin\varphi\, \varrho \cdot \varrho \, d\varrho\, d\varphi =$$

$$= \int_0^{\pi/2} \left[\frac{\varrho^5}{5}\right]_0^{2\cos\varphi} \cos\varphi \sin\varphi \, d\varphi =$$

$$= \int_0^{\pi/2} \frac{32}{5} \cos^6\varphi \sin\varphi \, d\varphi = \frac{32}{5}\left[-\frac{\cos^7\varphi}{7}\right]_0^{\pi/2} =$$

$$= \frac{32}{35}.$$

8 S použitím substituce spočítejte $\int\limits_0^1 \int\limits_0^{1-x} \sqrt{x+y}(y-2x)^2\, dA$.

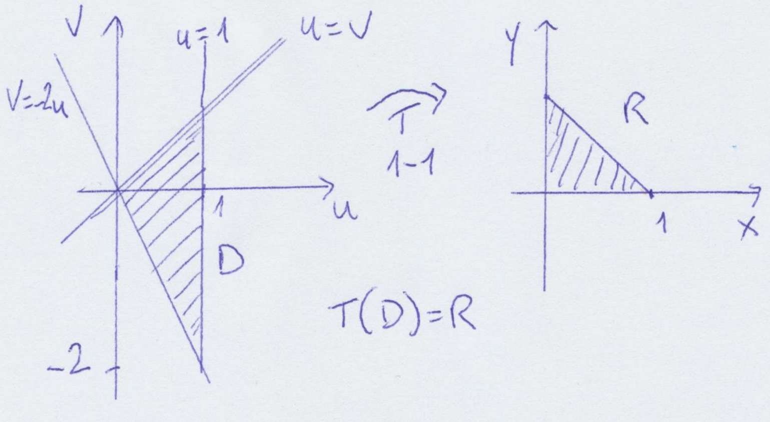
$dA = dy\,dx$

$x+y = u$

$y - 2x = v$

$T : \mathbb{R}^2 \to \mathbb{R}^2 \quad T(u,v) = \left(\frac{u-v}{3},\ \frac{2u+v}{3}\right)$

$$J_T = \begin{pmatrix} 1/3 & -1/3 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$



$$|\det J_T| = \frac{1}{3}$$

$x = 0 \to \frac{u-v}{3} = 0 \to v = u$

$y = 0 \to \frac{2u+v}{3} = 0 \to v = -2u$

$y = 1-x \to x+y = 1 \to u = 1$

$$\int_0^1\int_{-2u}^{u} \sqrt{u}\cdot v^2 \cdot \frac{1}{3}\, dv\, du = \frac{1}{3}\int_0^1 \left[\frac{v^3}{3}\right]_{-2u}^{u} \sqrt{u}\, du = \frac{1}{3}\int_0^1 \left(\frac{u^3}{3} + \frac{8u^3}{3}\right)\sqrt{u}\, du$$

$$= \frac{1}{3}\int_0^1 3\, u^{7/2}\, du = \left[u^{9/2}\cdot\frac{2}{9}\right]_0^1 = \frac{2}{9}.$$