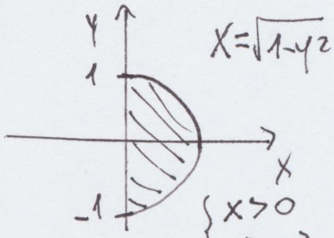
Trojný integrál 2 (Sférické a cylindrické souřadnice).

1 Zapište integrál pomocí cylindrických souřadnic a pak ho spočítejte:

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xy^2z \; dz\; dx\; dy.$$

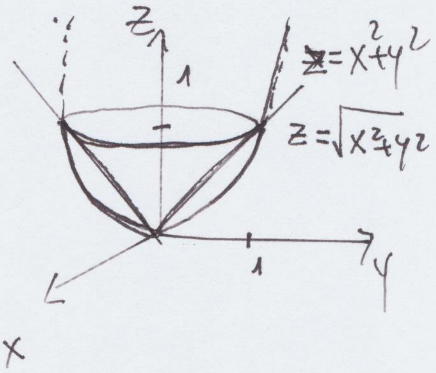


$\begin{cases} x>0 \\ x^2+y^2=1 \end{cases}$

$\phi \begin{cases} x=\rho\cos\varphi \\ y=\rho\sin\varphi \\ z=z \end{cases}$

$$\int_{-\pi/2}^{\pi/2}\int_0^1\int_{\rho^2}^{\rho} \rho\cos\varphi\, \rho^2\sin^2\varphi\, z\, \rho\; dz\, d\rho\, d\varphi =$$

Jakobián

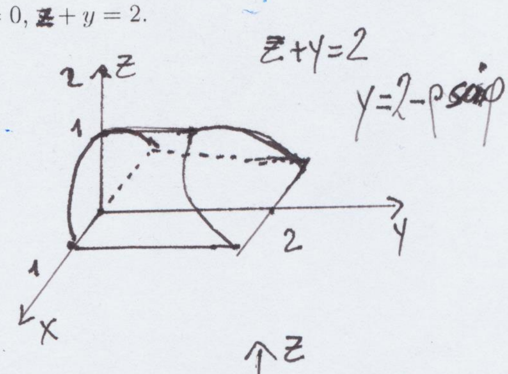


$$=\int_{-\pi/2}^{\pi/2}\int_0^1 \cos\varphi\sin^2\varphi\, \rho^4\left[\frac{z^2}{2}\right]_{\rho^2}^{\rho} d\rho\, d\varphi =$$

$$=\int_{-\pi/2}^{\pi/2}\cos\varphi\sin^2\varphi\, d\varphi\cdot\int_0^1 \frac{\rho^6-\rho^8}{2}\, d\rho =$$

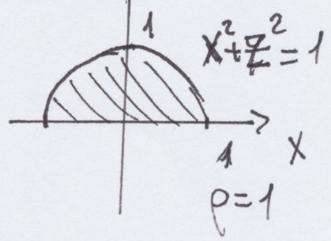
$$=\left[\frac{\sin^3\varphi}{3}\right]_{-\pi/2}^{\pi/2}\cdot\frac{1}{2}\left[\frac{\rho^7}{7}-\frac{\rho^9}{9}\right]_0^1=\frac{2}{3}\cdot\frac{1}{2}\cdot\left(\frac{1}{7}-\frac{1}{9}\right)$$

2 Nalezněte objem tělesa M ohraničeného plochami $x^2+z^2=1$, $y=0$, $z=0$, $\cancel{z}+y=2$.



$\phi \begin{cases} x=\rho\cos\varphi \\ y=y \\ z=\rho\sin\varphi \end{cases}$ Jakobián $=\rho$

$$V(M)=\iiint_M 1\, dV=\int_0^{\pi}\int_0^1\int_0^{2-\rho\sin\varphi}\rho\; dy\, d\rho\, d\varphi =$$



$$=\int_0^{\pi}\int_0^1 \rho\,[y]_0^{2-\rho\sin\varphi}\, d\rho\, d\varphi=\int_0^{\pi}\int_0^1 \rho(2-\rho\sin\varphi)\, d\rho\, d\varphi =$$

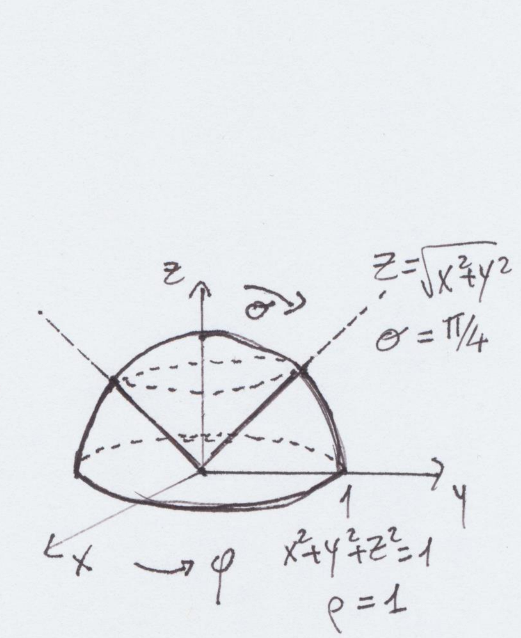
$$=\int_0^{\pi}\int_0^1 2\rho\, d\rho\, d\varphi+\int_0^{\pi}\int_0^1 -\rho^2\sin\varphi\, d\rho\, d\varphi =$$

$$=\pi\left[\rho^2\right]_0^1+\left[\frac{\rho^3}{3}\right]_0^1\left[\cos\varphi\right]_0^{\pi}=\pi-\frac{2}{3}$$

3 Vypočtěte

$$\iiint_E x^2 + y^2 + z^2 \, dV,$$

kde $E = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1, \ 0 \leq z \leq \sqrt{x^2 + y^2}\}$.



sférické souř.

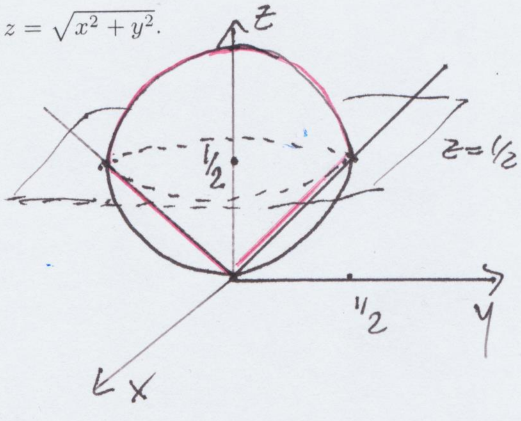
$$\phi: \ x = \rho \sin\theta \cos\varphi \qquad y = \rho \sin\theta \sin\varphi \qquad z = \rho \cos\theta \qquad J = \rho^2 \sin\theta$$

$$\int_0^{2\pi}\int_0^1\int_{\pi/4}^{\pi/2} \rho^2 \rho^2 \sin\theta \, d\theta \, d\rho \, d\varphi =$$

$$= \int_0^{2\pi} d\varphi \cdot \int_0^1 \rho^4 d\rho \cdot \int_{\pi/4}^{\pi/2} \sin\theta \, d\theta =$$

$$= 2\pi \left[\frac{\rho^5}{5}\right]_0^1 \left[-\cos\theta\right]_{\pi/4}^{\pi/2} = \frac{\sqrt{2}}{5}\pi$$

4 Nalezněte objem tělesa M ohraničeného plochami $z = x^2 + y^2 + z^2$, a $z = \sqrt{x^2 + y^2}$.



$$z = x^2+y^2+z^2 \longrightarrow \rho^2 = \rho\cos\theta$$

$$(z - 1/2)^2 + x^2 + y^2 = 1/4 \qquad \text{(sferické)}$$

$$z = \sqrt{x^2+y^2} \longrightarrow \theta = \pi/4$$

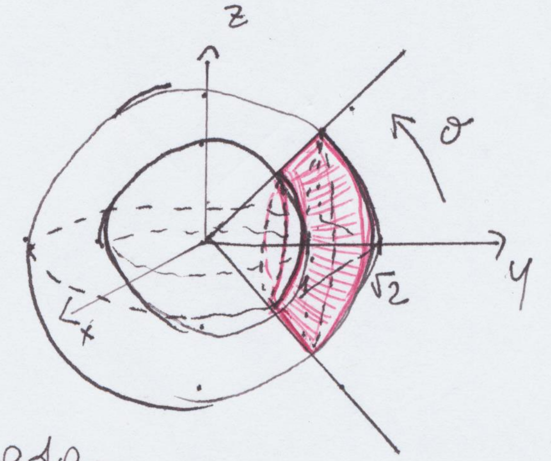
$$V(M) = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\theta} \rho^2 \sin\theta \, d\rho \, d\theta \, d\varphi =$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\cos^3\theta}{3} \sin\theta \, d\theta \, d\varphi = \frac{2\pi}{3}\left[-\frac{\cos^4\theta}{4}\right]_0^{\pi/4} = \pi/8$$

5 Vypočtěte

$$\iiint_E \frac{x^2}{x^2+z^2}\,dV,$$

kde $E = \{(x,y,z) : 1 \le x^2+y^2+z^2 \le 2,\ y>0,\ x^2-y^2+z^2<0\}$.



"sferické"

$x = \rho \sin\theta \cos\varphi$

$y = \rho \cos\theta$ Jakobián $= \rho^2 \sin\theta$

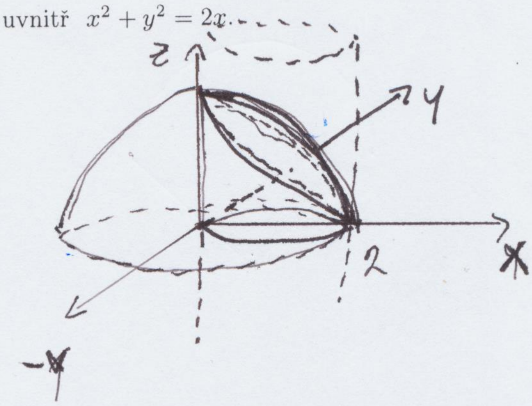
$z = \rho \sin\theta \sin\varphi$

$$\int_0^{2\pi}\int_1^{\sqrt{2}}\int_0^{\pi/4} \frac{\rho^2 \cancel{\sin^2\theta}\cos^2\varphi}{\rho^2\cancel{\sin^2\theta}}\,\rho^2\sin\theta\, d\theta\, d\rho\, d\varphi =$$

$$= \int_0^{2\pi}\cos^2\varphi\, d\varphi \cdot \int_1^{\sqrt{2}} \rho^2\, d\rho \int_0^{\pi/4}\sin\theta\, d\theta = \qquad \left[\cos^2\varphi = \frac{1}{2} + \frac{\cos 2\varphi}{2}\right]$$

$$= 2\pi\cdot\frac{1}{2}\left[\frac{\sqrt{2}^3-1}{3}\right]\cdot\left[-\frac{\sqrt{2}}{2}+1\right] = \frac{5\sqrt{2}-6}{6}\,\pi.$$

6 Nalezněte objem tělesa M ohraničeného plochou $z=\sqrt{4-x^2-y^2}$, uvnitř $x^2+y^2=2x$.



$$z = \sqrt{4-x^2-y^2} \Rightarrow \begin{cases} x^2+y^2+z^2=4 \\ z \ge 0\end{cases} \xrightarrow{\text{cylind.}} z = \sqrt{4-\rho^2}$$

$$x^2+y^2 = 2x \Rightarrow (x-1)^2+y^2 = 1 \xrightarrow{\text{cyl.}} \rho = 2\cos\varphi$$

$$V(M) = 2\cdot\int_0^{\pi/2}\int_0^{2\cos\varphi}\int_0^{\sqrt{4-\rho^2}} \rho\, dz\, d\rho\, d\varphi =$$

$$= 2\cdot\int_0^{\pi/2}\int_0^{2\cos\varphi} \rho\sqrt{4-\rho^2}\, d\rho\, d\varphi =$$

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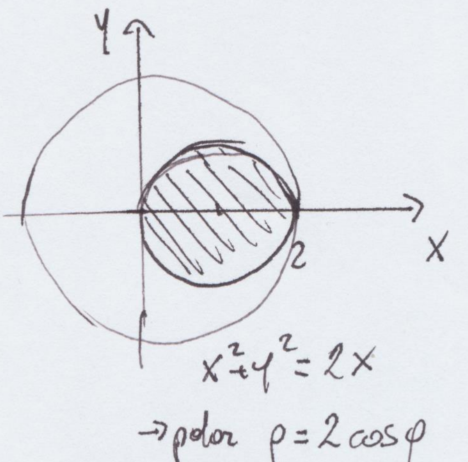
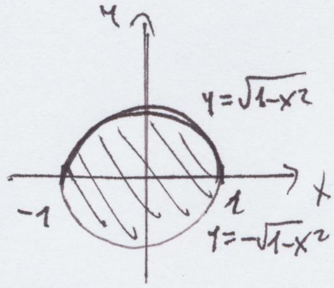


$x^2+y^2 = 2x$

$\rightarrow$ polar $\rho = 2\cos\varphi$

$$\left[\int \rho\cdot(4-\rho^2)^{1/2}\, d\rho = \frac{(4-\rho^2)^{3/2}}{-2}\cdot\frac{2}{3}\right]$$

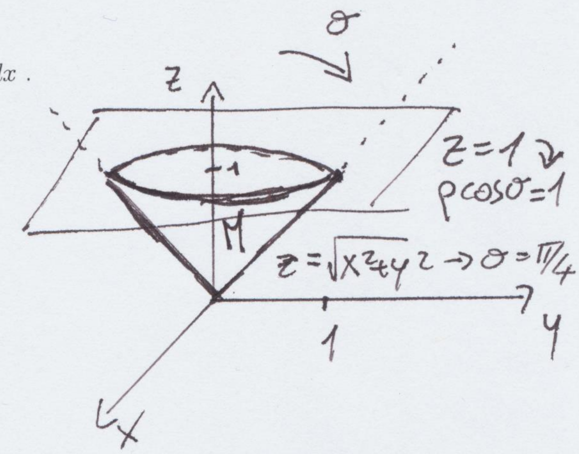
7 Zapište integrál pomocí sférických souřadnic a pak ho spočítejte:



$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} dz\, dy\, dx\,.$$

$$\| $$

$$\iiint_M dV = V(M)$$



$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\frac{1}{\cos\theta}} \rho^2 \sin\theta \, d\rho\, d\theta\, d\varphi =$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\left[\frac{\rho^3}{3}\right]_0^{\frac{1}{\cos\theta}} \sin\theta\, d\theta\, d\varphi = \int_0^{2\pi} d\varphi \cdot \int_0^{\pi/4} \frac{1}{3}\,\frac{\sin\theta}{\cos^3\theta}\, d\theta =$$

$$= 2\pi \cdot \frac{1}{3}\left[\frac{1}{2}\,\frac{1}{\cos^2\theta}\right]_0^{\pi/4} = \pi/3\left(\sqrt{2}^2-1\right) = \pi/3$$

$$\dots = 2\cdot\int_0^{\pi/2} -\left[\frac{(4-\rho^2)^{3/2}}{3}\right]_0^{2\cos\varphi} d\varphi = \frac{2}{3}\int_0^{\pi/2}\left[-\left(4-4\cos^2\varphi\right)^{3/2} + 8\right] d\varphi =$$

$\sqrt{\sin^2\varphi} = |\sin\varphi|$!

$\varphi \in [0, \pi/2]$, $\sin\varphi \geq 0$!

$$= \frac{2}{3}\int_0^{\pi/2} 8\cdot\left(1-\sin^3\varphi\right) d\varphi =$$

$$= \frac{16}{3}\int_0^{\pi/2}\left[1-\sin\varphi\left(1-\cos^2\varphi\right)\right] d\varphi = \frac{16}{3}\int_0^{\pi/2}\left[1-\sin\varphi+\sin\varphi\cos^2\varphi\right] d\varphi =$$

$$= \frac{16}{3}\left[\varphi+\cos\varphi-\frac{\cos^3\varphi}{3}\right]_0^{\pi/2} = \frac{16}{3}\left(\pi/2-\frac{2}{3}\right)$$