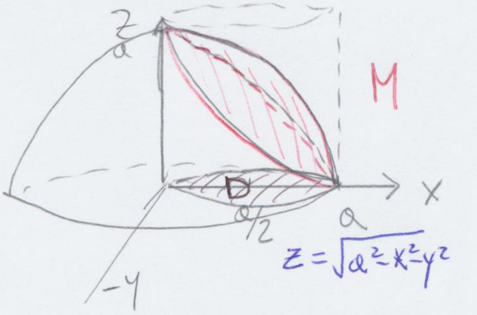
Plošný integrál.

Připomenutí: Integrál z funkce $f : M \to \mathbb{R}$ je určený jako

$$\iint_M f\, dS = \iint_U f(\Phi(u,v)) \cdot \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| \, dA,$$

kde $\Phi : U \to M$ je vhodná parametrizace.

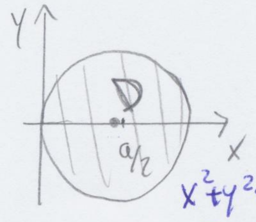
1 Stanovte obsah části kulové plochy o rovnici $x^2 + y^2 + z^2 = a^2$ (kde $a > 0$ je parametr), kterou z ní vytíná válcová plocha určená podmínkami $x^2 + y^2 = ax$ a $z \geq 0$.



$M: \phi(x,y) = (x, y, \sqrt{a^2-x^2-y^2}) \quad (x,y) \in D$

$\phi_x = \left(1, 0, \frac{-x}{\sqrt{a^2-x^2-y^2}}\right) \qquad \phi_x \times \phi_y = \left(\frac{x}{\sqrt{a^2-x^2-y^2}}, \frac{y}{\sqrt{a^2-x^2-y^2}}, 1\right)$

$\phi_y = \left(0, 1, -\frac{y}{\sqrt{a^2-x^2-y^2}}\right) \qquad \|\phi_x \times \phi_y\| = \sqrt{\frac{x^2+y^2}{(\sqrt{a^2-x^2-y^2})^2}+1} = \frac{a}{\sqrt{a^2-x^2-y^2}}$



$\text{Obsah}(M) = \iint_M dS = \iint_D \frac{a}{\sqrt{a^2-x^2-y^2}}\, dA = \text{polar. sour.} \qquad x = \rho\cos\theta,\ y = \rho\sin\theta$

$x^2+y^2 \le ax$

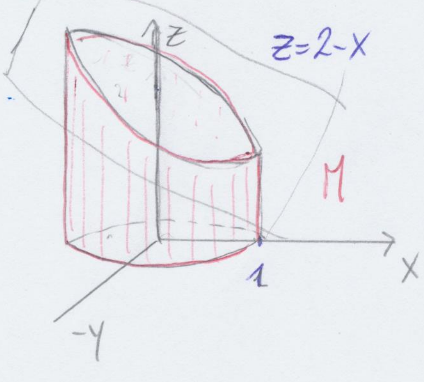
$\rho^2 \le a\rho\cos\theta$

$\rho \le a\cos\theta$

$= \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} \frac{a}{\sqrt{a^2-\rho^2}} \rho\, d\rho\, d\theta = \int_{-\pi/2}^{\pi/2} -a\left[\sqrt{a^2-\rho^2}\right]_0^{a\cos\theta} d\theta =$

$= \int_{-\pi/2}^{\pi/2} \left(-a^2\sqrt{1-\cos^2\theta} + a^2\right) d\theta = 2\int_0^{\pi/2} -a^2\sin\theta\, d\theta + \int_{-\pi/2}^{\pi/2} a^2\, d\theta = 2a^2\left[\cos\theta\right]_0^{\pi/2} + \pi a^2 = (\pi - 2)a^2.$

2 Spočítejte $\iint_M z\, dS$, kde M je částí válce $x^2 + y^2 = 1$ mezi rovinami $z = 0$ a $z = 2 - x$.



$M: \phi(\theta, z) = (\cos\theta, \sin\theta, z), \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le 2-\cos\theta$

$\phi_\theta = (-\sin\theta, \cos\theta, 0) \qquad \phi_\theta \times \phi_z = (\cos\theta, \sin\theta, 0)$

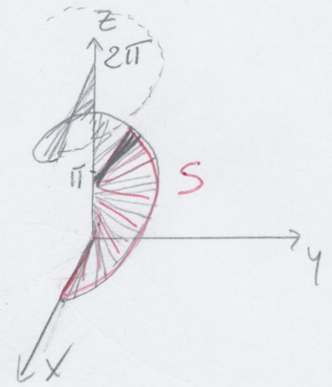
$\phi_z = (0, 0, 1) \qquad \|\phi_\theta \times \phi_r\| = 1$

$\iint_M z\, dS = \int_0^{2\pi}\int_0^{2-\cos\theta} z\, dz\, d\theta =$

$= \int_0^{2\pi} \left[\frac{z^2}{2}\right]_0^{2-\cos\theta} d\theta = \int_0^{2\pi} \frac{(2-\cos\theta)^2}{2} d\theta = \frac{1}{2}\int_0^{2\pi} \left(4 - \underbrace{4\cos\theta}_{0} + \cos^2\theta\right) d\theta =$

$= \frac{1}{2}\int_0^{2\pi} \left(4 + \frac{1}{2} + \underbrace{\frac{\cos 2\theta}{2}}_{0}\right) d\theta = \frac{1}{2}\cdot\frac{9}{2}\cdot 2\pi = \frac{9}{2}\pi$

3 Jaký je celkový náboj na šroubové ploše S dané parametrizací $\Phi(a,b) = (a\cos b,\ a\sin b,\ b)$, $0 \le a \le 1$, $0 \le b \le \pi$, je-li plošná hustota rozložení náboje $\varrho(x,y,z) = \sqrt{1+x^2+y^2}$?



$$\iint_S \varrho(x,y,z)\, dS$$

$$S: \phi(a,b) = (a\cos b, a\sin b, b) \qquad \begin{array}{l} 0 \le a \le 1 \\ 0 \le b \le \pi \end{array}$$

$$\phi_a = (\cos b, \sin b, 0)$$

$$\phi_b = (-a\sin b, a\cos b, 1)$$

$$\phi_a \times \phi_b = (\sin b, -\cos b, a)$$

$$\|\phi_a \times \phi_b\| = \sqrt{1+a^2}$$

$$\int_0^1\int_0^\pi \sqrt{1+a^2}\,\sqrt{1+a^2}\, db\, da = \int_0^1\int_0^\pi (1+a^2)\, db\, da =$$

$$= \pi \left[a + \frac{a^3}{3} \right]_0^1 = \frac{4}{3}\pi$$

4 Spočítejte $\iint\limits_M x^2\, dS$, kde M je povrch koule $x^2+y^2+z^2 = 4$. M parametrizujeme pomocí sfer. souř.

$$\phi(\varphi,\sigma) = (2\sin\sigma\cos\varphi, 2\sin\sigma\sin\varphi, 2\cos\sigma) \qquad 0 \le \varphi \le 2\pi, 0 \le \sigma \le \pi \; \} U$$

$$\phi_\varphi = (-2\sin\sigma\sin\varphi, 2\sin\sigma\cos\varphi, 0) \qquad \phi_\varphi \times \phi_\sigma = (-4\sin^2\sigma\cos\varphi, 4\sin^2\sigma\sin\varphi, -4\cos\sigma\sin\sigma)$$

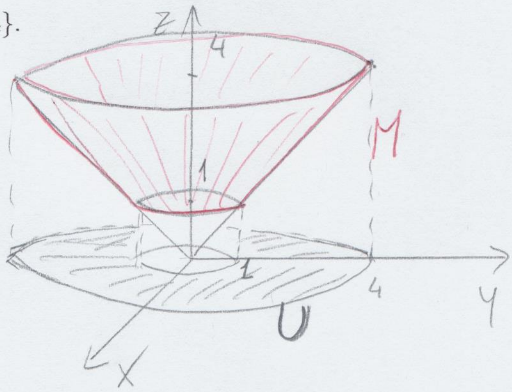
$$\phi_\sigma = (2\cos\sigma\cos\varphi, 2\cos\sigma\sin\varphi, -2\sin\sigma) \qquad \|\phi_\varphi \times \phi_\sigma\| = \|\phi_\varphi\| \cdot \|\phi_\sigma\| = 4\sin\sigma$$

$$\iint_M x^2\, dS = \int_0^{2\pi}\int_0^\pi 4\sin^2\sigma\cos^2\varphi \cdot 4\sin\sigma\, d\sigma\, d\varphi = 16\int_0^{2\pi}\cos^2\varphi\, d\varphi \cdot \int_0^\pi \sin^3\sigma\, d\sigma =$$

$$= 16\int_0^{2\pi}\left(\frac{1}{2} + \frac{\cos 2\varphi}{2}\right) d\varphi \cdot \int_0^\pi \sin\sigma(1-\cos^2\sigma)\, d\sigma =$$

$$= 16 \cdot 2\pi \cdot \frac{1}{2} \cdot \left[-\cos\sigma + \frac{\cos^3\sigma}{3}\right]_0^\pi = 16\pi\left(1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right) = \frac{64}{3}\pi.$$

5 Spočítejte $\iint\limits_M (10-z)\,dS$, kde $M=\{(x,y,z): z=\sqrt{x^2+y^2}, 1\le z\le 4\}$.



$M: \phi(x,y) = (x, y, \sqrt{x^2+y^2}) \quad (x,y)\in U: 1\le x^2+y^2\le 16$

$\phi_x = \left(1, 0, \frac{x}{\sqrt{x^2+y^2}}\right) \quad \phi_x\times\phi_y = \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\right)$

$\phi_y = \left(0, 1, \frac{y}{\sqrt{x^2+y^2}}\right) \quad \|\phi_x\times\phi_y\| = \sqrt{2}$

$\iint\limits_M (10-z)\,dS = \iint\limits_U \left(10-\sqrt{x^2+y^2}\right)\sqrt{2}\,dA =$ pola-souř. $\quad x = r\cos\theta,\ y = r\sin\theta \quad 1\le r\le 4,\ 0\le\theta\le 2\pi$

$= \int_0^{2\pi}\int_1^4 (10-r)\sqrt{2}\, r\,dr\,d\theta =$

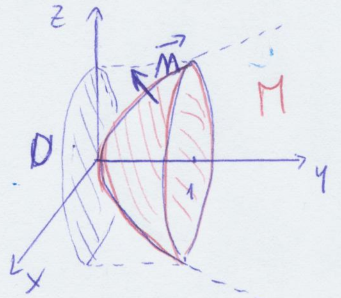
$= 2\pi\cdot\sqrt{2}\left[10\frac{r^2}{2} - \frac{r^3}{3}\right]_1^4 = 2\sqrt{2}\pi\left(5\cdot 16 - \frac{4\cdot 16}{3} - 5 + \frac{1}{3}\right) = \frac{324}{3}\sqrt{2}\pi$

Připomenutí: Tok vektorového pole $\vec{F}: M\to\mathbb{R}^3$ orientovanou plochou $M\subseteq\mathbb{R}^3$ se spočítá jako

$$\iint\limits_M \vec{F}\cdot d\vec{S} = \iint\limits_U \vec{F}(\Phi(u,v))\cdot\left(\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}\right) dA,$$

kde $\Phi: U\to M$ je opět vhodná parametrizace, $U\subseteq\mathbb{R}^2$, a orientace daná vektorovým polem $\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}$ souhlasí se zadanou parametrizací plochy M. (Pokud by orientace nesouhlasila, stačí jen změnit pořadí ve vektorovém součinu, tj. změnit znaménko integrálu.)

6 Spočítejte $\iint\limits_M \vec{F}\cdot d\vec{S}$ kde $\vec{F}(x,y,z) = (0, y, -z)$ a M je část paraboloidu $\cancel{x}\ y = x^2+z^2$ pro $x\le 1$ s orientací danou vektorovým polem směřujícím do leva ($y\le 0$).



$M: \phi(x,z) = (x, x^2+z^2, z) \quad (x,z)\in D: x^2+z^2\le 1$

$\phi_x = (1, 2x, 0) \quad \phi_x\times\phi_z = \langle 2x, -1, 2z\rangle$

správ. orient.

$\phi_z = (0, 2z, 1)$

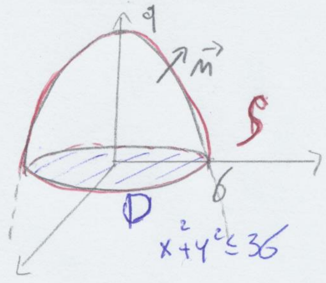
$\iint_M \vec{F}\cdot d\vec{S} = \iint\limits_D \langle 0, x^2+z^2, -z\rangle\cdot\langle 2x, -1, 2z\rangle\,dA = \iint\limits_D \left((x^2+z^2)(-1) - 2z^2\right) dA =$ pol. $\quad \begin{pmatrix} x = r\cos\theta \\ z = r\sin\theta \end{pmatrix}$

$= \int_0^{2\pi}\int_0^1 (-r^2 - 2r^2\sin^2\theta)\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 -r^2\,dr\,d\theta + \int_0^{2\pi}\int_0^1 -2r^3\sin^2\theta\,dr\,d\theta$

$\sin^2\theta = \frac{1}{2} - \frac{\cos 2\theta}{2}$

$= 2\pi\left[-\frac{r^4}{4}\right]_0^1 + \left[-\frac{2r^4}{4}\right]_0^1 \frac{1}{2}\, 2\pi =$ $-\pi$

7 Kapalina s hustotou 1 protéká s rychlostí danou polem $\vec{v}(x,y,z) = (y,\ 1,\ z)$. Určete průtok kapaliny směrem vzhůru plochou S, která je částí paraboloidu $z = 9 - \frac{(x^2+y^2)}{4}$ pro $x^2 + y^2 \leq 36$ (neboli určete $\iint\limits_S \vec{v} \cdot d\vec{S}$).



$$S : \phi(x,y) = \left(x, y, 9 - \frac{x^2+y^2}{4}\right) \quad (x,y) \in D$$

$$\phi_x = \left(1, 0, -\frac{x}{2}\right)$$

$$\phi_y = \left(0, 1, -\frac{y}{2}\right)$$

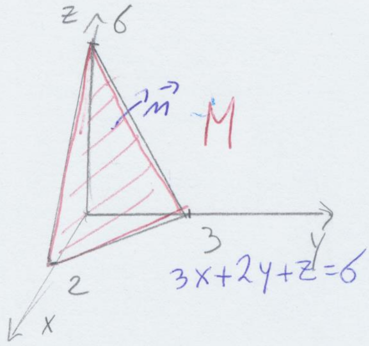
$$\phi_x \times \phi_y = \left(\frac{x}{2}, \frac{y}{2}, 1\right) \nearrow \text{sp. směr!}$$

$$\iint_S \vec{v} \cdot d\vec{S} = \iint_D (y,1,z)\big|_\phi \cdot (\phi_x \times \phi_y)\, dA = \iint_D \left(y, 1, 9 - \frac{x^2+y^2}{4}\right) \cdot \left(\frac{x}{2}, \frac{y}{2}, 1\right) dA =$$

$$= \iint_D \left(\frac{xy}{2} + \frac{y}{2} + 9 - \frac{x^2+y^2}{4}\right) dA = \text{pol.} \int_0^{2\pi} \int_0^6 \left(\frac{\rho^3}{2} \underbrace{\sin\varphi\cos\varphi}_{0} + \frac{\rho^2}{2} \underbrace{\sin\varphi}_{0} + 9\rho - \frac{\rho^3}{4}\right) d\rho\, d\varphi$$

$$= 2\pi \left[9\frac{\rho^2}{2} - \frac{\rho^4}{16}\right]_0^6 = 2\pi \left[9 \cdot \frac{36}{2} - \frac{6^4}{16}\right].$$

8 Spočítejte $\iint\limits_M \vec{F} \cdot d\vec{S}$, kde $\vec{F}(x,y,z) = (x, xy, xz)$, a M je část roviny $3x + 2y + z = 6$, která leží v prvním oktantu a je orientovaná směrem vzhůru.

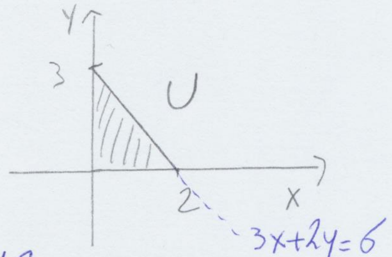


$$M : \phi(x,y) = (x, y, 6 - 3x - 2y) \quad (x,y) \in U$$

$$\phi_x = (1, 0, -3)$$

$$\phi_y = (0, 1, -2)$$

$$\phi_x \times \phi_y = (3, 2, 1) \nearrow \text{sp. směr!}$$



$$\iint_M \vec{F} \cdot dS = \iint_U (x, xy, x(6-3x-2y)) \cdot (3,2,1)\, dA =$$

$$= \int_0^2 \int_0^{\frac{6-3x}{2}} (3x + \cancel{2xy} + 6x - 3x^2 - \cancel{2xy})\, dy\, dx =$$

$$= \int_0^2 (9x - 3x^2)\left(\frac{6-3x}{2}\right) dx = \left[27\frac{x^2}{2} - \frac{45x^3}{6} + \frac{9}{6}x^4\right]_0^2 = 12.$$