Definitions and claims—Real line

Symbols (popular sets of numbers).
$\mathbb{N}$ natural numbers 1, 2, 3, 4, . . . $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, 4, \dots\}$
$\mathbb{Z}$ integers 0, 1, -1, 2, -2, 3, -3, . . .
$\mathbb{Q}$ rational numbers
$\mathbb{R}$ real numbers, rational and irrational (i.e. $\sqrt{2}$, e, π)

Definition. **Proper intervals:** Let $a \le b \in \mathbb{R}$.
$(a, b) = \{x \in \mathbb{R}; a < x < b\}$ (open interval)
$\langle a, b) = \{x \in \mathbb{R}; a \le x < b\}$ (half-closed interval)
$(a, b\rangle = \{x \in \mathbb{R}; a < x \le b\}$ (half-closed interval)
$\langle a, b\rangle = \{x \in \mathbb{R}; a \le x \le b\}$ (closed interval)
Observation: If $a = b$, then we have **degenerate interval** $(a, a) = \langle a, a) = (a, a\rangle = \emptyset$ a $\langle a, a\rangle = \{a\}$.
Improper intervals: Let $a, b \in \mathbb{R}$.
$(a, \infty) = \{x \in \mathbb{R}; a < x\}$ (open interval)
$(-\infty, b) = \{x \in \mathbb{R}; x < b\}$ (open interval)
$\langle a, \infty) = \{x \in \mathbb{R}; a \le x\}$ (half-closed interval)
$(-\infty, b\rangle = \{x \in \mathbb{R}; x \le b\}$ (half-closed interval).

Definition. Let M be a subset of $\mathbb{R}$.
A number $K \in \mathbb{R}$ is an **upper bound** of the set M, if $\forall a \in M$: $a \le K$.
A number $k \in \mathbb{R}$ is a **lower bound** for the set M, if $\forall a \in M$: $a \ge k$.
We say that M is **bounded above**, if it has an upper bound.
We say that M is **bounded below**, if it has a lower bound.
We say that M is **bounded**, if it is bounded above and bounded below.
If M is bounded above, we define its **supremum** $\sup(M)$ as the smallest upper bound, otherwise $\sup(M) = \infty$.
If M bounded below, we define its **infimum** $\inf(M)$ as the biggest lower bound, otherwise $\inf(M) = -\infty$.
We say that $x \in \mathbb{R}$ is the **maximum** for the set M, indicated with $\max(M)$, if $x \in M$ and $\forall a \in M$: $a \le x$.
We say that $x \in \mathbb{R}$ is the **minimum** of the set M, indicated with $\min(M)$, if $x \in M$ and $\forall a \in M$: $a \ge x$.

Theorem.
Every subset of real numbers has supremum and infimum.
Fact.
$\max(M)$ exists $\iff$ $\sup(M) \in M$, then $\max(M) = \sup(M)$.
$\min(M)$ exists $\iff$ $\inf(M) \in M$, then $\min(M) = \inf(M)$.

Definition.
Extended real axis $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$. Terminology: proper numbers $x \in \mathbb{R}$, improper numbers $\pm\infty$.
Order: $\forall x \in \mathbb{R}$: $-\infty < x < \infty$.
Neighbourhoods: $U_\varepsilon(\infty) = P_\varepsilon(\infty) = \left(\frac{1}{\varepsilon}, \infty\right)$, $U_\varepsilon(-\infty) = P_\varepsilon(-\infty) = \left(-\infty, -\frac{1}{\varepsilon}\right)$.
Operations: $\infty + \infty = \infty$, $\infty - (-\infty) = \infty$, $(-\infty) + (-\infty) = -\infty$, $(-\infty) - \infty = -\infty$; $\infty \cdot \infty = \infty$, $\infty^\infty = \infty$.
Undetermined: $\infty - \infty$, $\frac{\infty}{\infty}$.
$\infty + a = \infty - a = \infty$ for $a \in \mathbb{R}$, $-\infty + a = -\infty - a = -\infty$ for $a \in \mathbb{R}$.
$\frac{a}{\infty} = 0$ for $a \in \mathbb{R}$, $\frac{\infty}{a} = \infty$ for $a > 0$, $\frac{\infty}{a} = -\infty$ for $a < 0$.
$a \cdot \infty = \infty$ for $a > 0$, $a \cdot (-\infty) = -\infty$ for $a > 0$, $a \cdot \infty = -\infty$ for $a < 0$, $a \cdot (-\infty) = \infty$ for $a < 0$.
Undetermined: $\frac{a}{0}$, $\frac{\infty}{0}$, $0 \cdot \infty$.
$\infty^a = \infty$ for $a > 0$, $\infty^a = 0$ for $a < 0$; $a^\infty = \infty$ for $a > 1$, $a^\infty = 0$ for $|a| < 1$.
Undetermined: ∞^0, 1^∞.

Definitions and claims—functions (introduction, elementary properties)

Definition.
A **real function of a real variable** is a map $f : D \mapsto \mathbb{R}$, where D is a subset of $\mathbb{R}$.
From now on, we will just say real function or even just function.
Definition. Let f be a function.
The **domain** of f is the set $D(f) = \{x \in \mathbb{R};\ f(x) \text{ is defined}\}$.
The **range** of f is the set $R(f) = \{f(x);\ x \in D(f)\}$.
The **graph** of f is the set $G(f) = \{(x, f(x)),\ x \in D(f)\}$.
Definition. (comparison) Let f, g be functions.
We say that $f = g$, if $D(f) = D(g) = D$ and $\forall x \in D$: $f(x) = g(x)$.
Let $M \ne \emptyset$ be a subset of $D(f) \cap D(g)$.
We say that $f = g$ on M, if $\forall x \in M$: $f(x) = g(x)$.
We say that $f \le g$ on M, if $\forall x \in M$: $f(x) \le g(x)$.
We say that $f < g$ on M, if $\forall x \in M$: $f(x) < g(x)$.

We say that $f \geq g$ on M, if $\forall x \in M$: $f(x) \geq g(x)$.
We say that $f > g$ on M, if $\forall x \in M$: $f(x) > g(x)$.

Definition. (operations) Let f, g be functions so that $M = D(f) \cap D(g) \neq \emptyset$.
We define their **sum** $f + g$ with the formula $(f+g)(x) = f(x) + g(x)$ for $x \in M$.
We define their **difference** $f - g$ with the formula $(f-g)(x) = f(x) - g(x)$ for $x \in M$.
We define their **product** $f \cdot g$ with the formula $(f \cdot g)(x) = f(x) \cdot g(x)$ for $x \in M$.
We define their **quotient** $\frac{f}{g}$ with the formula $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ for $x \in M$, $g(x) \neq 0$.
We define their **general power** f^g with the formula $(f^g)(x) = e^{\ln[f(x)]g(x)}$ for $x \in D(f^g)$, where

$$D(f^g) = D(g) \cap \{x \in D(f);\ f(x) > 0\}.$$

Definition. (composed function) Let f, g be functions such that $R(f) \cap D(g) \neq \emptyset$.
We define their **composition** $g(f) = g \circ f$ as $(g \circ f)(x) = g\big(f(x)\big)$ for $x \in D(g \circ f)$, where

$$D(g \circ f) = \{x \in D(f);\ f(x) \in D(g)\}.$$

Definition. Let f be a function.
We say that f is **bounded above**, if $\exists K \in \mathbb{R}\ \forall x \in D(f)$: $f(x) \leq K$.
We say that f is **bounded below**, if $\exists k \in \mathbb{R}\ \forall x \in D(f)$: $f(x) \geq k$.
We say that f is **bounded**, if it is bounded above and bounded below.
Observation: If such a number K exists, we call it an **upper bound**. Similarly, we call the number k a **lower bound**.

Definition. (symmetry)
We say that a subset M of real numbers is symmetric, if $\forall x \in M$: $(-x) \in M$.
Let f be a function. We say that it is **even**, if $D(f)$ is a symmetric set and $\forall x \in D(f)$: $f(-x) = f(x)$.
We say that it is **odd**, if $D(f)$ is a symmetric set and $\forall x \in D(f)$: $f(-x) = -f(x)$.

Definition. Let f be a function, $T > 0$.
We say that T is a **period** for f, or that f is T-**periodic**, if $\forall x \in D(f)$ such that $x + T \in D(f)$, we have $f(x+T) = f(x)$.

Definition. Let f be a function.
We say that f is **injective** or **1-1** (one to one), if $\forall x_1, x_2 \in D(f)$: $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.

Definition. Let f, g be functions.
We say that g is the **inverse function** of f, indicated with $g = f_{-1}$, if

$$\forall x \in D(f)\colon\ g\big(f(x)\big) = x \text{ and } \forall y \in R(f)\colon\ f\big(g(y)\big) = y.$$

Fact. Let f be a function.
f has an inverse function $\iff$ f is one to one.
Then f_{-1} is uniquely determined and $D(f_{-1}) = R(f)$ a $R(f_{-1}) = D(f)$.

Definitions and claims—functions (limits)

Definition. Let $a \in \mathbb{R}$, $\varepsilon > 0$. We define neighbourhoods of a:

$U_\varepsilon(a) = \{x \in \mathbb{R};\ \lvert x-a\rvert < \varepsilon\} = (a-\varepsilon, a+\varepsilon)$	**ε-neighbourhood of a**
$P_\varepsilon(a) = \{x \in \mathbb{R};\ 0 < \lvert x-a\rvert < \varepsilon\} = (a-\varepsilon, a) \cup (a, a+\varepsilon)$	**annular ε-neighbourhood of a**
$U_\varepsilon^+(a) = \{x \in \mathbb{R};\ a \leq x < a+\varepsilon\} = \langle a, a+\varepsilon)$	**right ε-neighbourhood of a**
$P_\varepsilon^+(a) = \{x \in \mathbb{R};\ a < x < a+\varepsilon\} = (a, a+\varepsilon)$	**right annular ε-neighbourhood of a**
$U_\varepsilon^-(a) = \{x \in \mathbb{R};\ a-\varepsilon < x \leq a\} = (a-\varepsilon, a\rangle$	**left ε-neighbourhood of a**
$P_\varepsilon^-(a) = \{x \in \mathbb{R};\ a-\varepsilon < x < a\} = (a-\varepsilon, a)$	**left annular ε-neighbourhood of a**

A neighbourhood of the point a it means an ε-neighbourhood of a for some concrete (and arbitrary) $\varepsilon > 0$, we denote it $U(a)$. Similarly, an annular neighbourhood $P(a)$, a left neighbourhood $U^-(a)$, etc. When we say "suppose there exists an arbitrary neighbourhood $U(a)$ of the point a", we mean that we want an arbitrary $U_\varepsilon(a)$, where the precise value of ε is not important, it is enough that a value exists. When we say "for every neighbourhood $U(a)$ of the point a we have that", then that means that it must work for neighbourhoods $U_\varepsilon(a)$ for every possible $\varepsilon > 0$.

Definition. Let f be a function defined on an annular neighbourhood of $a \in \mathbb{R}^*$, let $L \in \mathbb{R}^*$.
We say that "L is the **limit** of the function f for x approaching a", or that "f goes to L for x going to a", if $\forall$ neighbourhood $U = U(L)$ $\exists$ an annular neighbourhood $P = P(a)$, such that $\forall x \in P$: $f(x) \in U$.
We also say that "f has limit L at a", or that "f goes to L at a".
If such L exists, we say that the **limit** of f at a **exists**, we write it "$\lim\limits_{x \to a}\big(f(x)\big) = L$" or "$f(x) \to L$ for $x \to a$".
Otherwise, we say that the **limit does not exist**.
If the limit exists and $L = \pm\infty$, we talk of an **improper limit**.
If the limita exists and $L \in \mathbb{R}$, we call it a **proper limit**, we also say that "f **converges** to L at a" or that "$\lim\limits_{x \to a}\big(f(x)\big)$ **converges**". Otherwise, we say that "$\lim\limits_{x \to a}\big(f(x)\big)$ **diverges**".
Observation: We have defined limits as proper or improper. Also the point at which we discuss the limit, can be proper for $a \in \mathbb{R}$ or improper for $a = \pm\infty$.

Observation: The definition for proper limit at a proper point can also be written as follows:
$\lim\limits_{x\to a}\bigl(f(x)\bigr)=L \iff \forall\varepsilon>0\ \exists\delta>0$ such that $\forall x$: $\bigl[0<|x-a|<\delta \implies |f(x)-L|<\varepsilon\bigr]$.
Let's show another definition, for example if ∞ is the limit of f at $-\infty$:
$\lim\limits_{x\to-\infty}\bigl(f(x)\bigr)=\infty \iff \forall K\in I\!R\ \exists m\in I\!R$ such that $\forall x$: $\bigl[x<m \implies f(x)>K\bigr]$.

Definition. (one-side limits)
Let f be a function defined on a left annular neighbourhood of $a\in I\!R\cup\{\infty\}$, let $L\in I\!R^*$.
We say that "L is the **limit** of the function f for x going to a **from the left**", if $\forall$ neighbourhood $U=U(L)$ $\exists$ a left annular neighbourhood $P=P^-(a)$, such that $\forall x\in P$: $f(x)\in U$.
We also say that "f goes to L for x going to a from the left", or "f has limit L at a from the left", or that "f goes to L at a from the left". We write it, $\lim\limits_{x\to a^-}\bigl(f(x)\bigr)=L$, in short $f(a^-)=L$.
Let f be a function defined on a right annular neighbourhood of $a\in I\!R\cup\{-\infty\}$, let $L\in I\!R^*$.
We say that "L is the **limit** of the function f for x going to a **from the right**", if $\forall$ neighbourhood $U=U(L)$ $\exists$ a right annular neighbourhood $P=P^+(a)$, such that $\forall x\in P$: $f(x)\in U$.
We also say that "f goes to L for x going to a from the right", or "f has limit L at a from the right", or that "f goes to L at a from the right". We write it, $\lim\limits_{x\to a^+}\bigl(f(x)\bigr)=L$, in short $f(a^+)=L$.

Theorem.
$\lim\limits_{x\to a}\bigl(f(x)\bigr)=L \iff f(a^-)=L=f(a^+)$.

Theorem.
At a given point a, if a limit for f exists, then it is unique.

Theorem.
At a given point a, if f has a proper limit, then f is bounded on some annular neighbourhood of a.

Theorem. (limits and operations) Let $f\to A$ and $g\to B$ for $x\to a$. Then
$(f+g)\to(A+B)$ for $x\to a$, $(f-g)\to(A-B)$ for $x\to a$,
$(f\cdot g)\to(A\cdot B)$ for $x\to a$, $\bigl(\frac{f}{g}\bigr)\to\frac{A}{B}$ for $x\to a$,
$f^g\to A^B$ for $x\to a$, if the right hand sides are defined.
For calculations, we may write:
$\lim\limits_{x\to a}\bigl[(f\pm g)(x)\bigr]=\lim\limits_{x\to a}\bigl(f(x)\bigr)\pm\lim\limits_{x\to a}\bigl(g(x)\bigr)$,
$\lim\limits_{x\to a}\bigl[(f\cdot g)(x)\bigr]=\lim\limits_{x\to a}\bigl(f(x)\bigr)\cdot\lim\limits_{x\to a}\bigl(g(x)\bigr)$,
$\lim\limits_{x\to a}\bigl[\bigl(\frac{f}{g}\bigr)(x)\bigr]=\frac{\lim\limits_{x\to a}(f(x))}{\lim\limits_{x\to a}(g(x))}$,
$\lim\limits_{x\to a}\bigl[(f^g)(x)\bigr]=\lim\limits_{x\to a}\bigl(f(x)\bigr)^{\lim\limits_{x\to a}(g(x))}$, if the resulting right hand sides are defined.

Theorem. Let $\lim\limits_{x\to a}\bigl(f(x)\bigr)=b$, nech $\lim\limits_{y\to b}\bigl(g(y)\bigr)=L$.
If $g(b)=L$ or $\exists$ an annular neighbourhood $P=P(a)$ such that $\forall x\in P$: $f(x)\neq b$, then

$$\lim_{x\to a}\bigl((g\circ f)(x)\bigr)=\lim_{x\to a}\bigl((g(f(x))\bigr)=L.$$

Observation: If g is continuous (see definition below) at a, then the assumptions hold.
For calculations, we use: $\lim\limits_{x\to a}\bigl[g\bigl(f(x)\bigr)\bigr]=g\bigl(\lim\limits_{x\to a}\bigl[f(x)\bigr]\bigr)$.

Algebra for evaluating limits with infinity:
$\infty+\infty=\infty$, $\infty-(-\infty)=\infty$, $(-\infty)+(-\infty)=-\infty$, $(-\infty)-\infty=-\infty$; $\infty\cdot\infty=\infty$, $\infty^\infty=\infty$.
$\infty+a=\infty-a=\infty$ for $a\in I\!R$, $-\infty+a=-\infty-a=-\infty$ for $a\in I\!R$.
$\frac{a}{\infty}=0$ for $a\in I\!R$, $\frac{\infty}{a}=\infty$ for $a>0$, $\frac{\infty}{a}=-\infty$ for $a<0$.
$\frac{1}{0^+}=\infty$, $\frac{1}{0^-}=-\infty$.
$a\cdot\infty=\infty$ for $a>0$, $a\cdot(-\infty)=-\infty$ for $a>0$, $a\cdot\infty=-\infty$ for $a<0$, $a\cdot(-\infty)=\infty$ for $a<0$.
$\infty^a=\infty$ for $a>0$, $\infty^a=0$ for $a<0$; $a^\infty=\infty$ for $a>1$, $a^\infty=0$ for $|a|<1$.
Undetermined forms: $\infty-\infty$, $\frac{\infty}{\infty}$, $\frac{a}{0}$, $\frac{\infty}{0}$, $0\cdot\infty$, 0^0, ∞^0, 1^∞.
Extension for functions:
$\ln(0^+)=-\infty$, $\ln(\infty)=\infty$, $e^{-\infty}=0$, $e^\infty=\infty$, $\operatorname{arctg}(-\infty)=-\frac{\pi}{2}$, $\operatorname{arctg}(\infty)=\frac{\pi}{2}$.

Definition. Let $\lim\limits_{x\to a}\bigl(f(x)\bigr)=L$.
We denote this result as L^+, if $\exists$ an annular neighbourhood $P=P(a)$ such that $\forall x\in P$: $f(x)>L$.
We denote this result as L^-, if $\exists$ an annular neighbourhood $P=P(a)$ such that $\forall x\in P$: $f(x)<L$.

Theorem. (limits and comparison) Let $f\to A$ and $g\to B$ for $x\to a$.
1) If $\exists$ an annular neighbourhood $P=P(a)$ such that $f\leq g$ on P, then $A\leq B$.
2) If $A<B$, then $\exists$ an annular neighbourhood $P=P(a)$ such that $f<g$ on P.

Theorem.
Let f,g be functions defined on an annular neighbourhood P of $a\in I\!R^*$ such that $f\leq g$ on P. If $f\to\infty$ at a, then necessarily $g\to\infty$ at a. If $g\to-\infty$ at a, then necessarily $f\to-\infty$ at a.

Theorem. (Squeeze Theorem)

Let f, g, h be functions defined on an annular neighbourhood of $a \in I\!\!R^*$ such that $f \le g \le h$ on P. If $f \to L$ and $h \to L$ at a, then necessarily $g \to L$ at a.
Corollary.
Let f, g be functions defined on an annular neighbourhood P of $a \in I\!\!R^*$ such that $|f| \le g$ on P. If $g \to 0$ at a, then necessarily $f \to 0$ at a.

Fact.
Let f be bounded on an annular neighbourhood of a.
If $g \to 0$ at a, then $f \cdot g \to 0$ at a.
If $g \to \infty$ at a, then $\frac{f}{g} \to 0$ at a.
If $g \to \pm\infty$ at a, then $f + g \to \pm\infty$ at a.

Definitions and claims—functions (continuity)

Definition. Let f be a function defined on a neighbourhood of $a \in I\!\!R$.
We say that f is **continuous** at a, if $\forall$ neighbourhood $U = U(f(a))$ $\exists$ a neighbourhood $V = V(a)$ such that $\forall x \in V$: $f(x) \in U$.
Epsilon-delta version:
f is **continuous** at a, if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that $\forall x$: $\big[|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon\big]$.
Definition. (one-side continuity)
Let f be a function defined on a left neighbourhood of $a \in I\!\!R$. We say that f is **continuous from the left** at a, if $\forall$ neighbourhood $U = U(f(a))$ $\exists$ a left neighbourhood $V = V^-(a)$ such that $\forall x \in V$: $f(x) \in U$.
Let f be a function defined on a right neighbourhood of $a \in I\!\!R$. We say that f is **continuous from the right** at a, if $\forall$ neighbourhood $U = U(f(a))$ $\exists$ a right neighbourhood $V = V^+(a)$ such that $\forall x \in V$: $f(x) \in U$.
Theorem.
A function is continuous at a point $\iff$ is there continuous from the right and from the left.
Theorem.
A function f is continuous at a $\iff$ $\lim\limits_{x\to a}\big(f(x)\big)$ exists and is equal to $f(a)$.
Similarly for one-side continuity.
Theorem. (continuity and functions operations)
Let f, g be functions continuous at a. Then, also the following functions are continuous at a: $f \pm g$, $f \cdot g$, $\frac{f}{g}$ (if $g(a) \neq 0$), f^g (if $f(a) > 0$).
Let f be continuous at a, let g be continuous at $b = f(a)$. Then $g \circ f = g(f)$ is continuous at a.
Definition. Let f be a function defined on a non-degenerate interval I.
We say that it is continuous on the interval I, if the following hold:
— f is continuous on all internal points of I,
— if I contains its left boundary, then f is continuous at that point from the right,
— if I contains its right boundary, then f is continuous at that point from the left.
Definition. Let f be a function, whose $D(f)$ is a union of non-degenerate intervals.
We say that f is **continuous**, if it is continuous on all intervals that compose its domain $D(f)$.
Theorem. Given f, g continuous, then $f + g$, $f - g$, $f \cdot g$, $\frac{f}{g}$, f^g, $g \circ f$ are continuous.
Theorem.
All elementary functions are continuous.

Definition. Let f be defined on a neighbourhood of a.
A point a is **a point of discontinuity**, if f is not continuous at a.
Definition. (classification of discontinuity)
Let f be defined on a neighbourhood of a.
We say that f has at a a **removable discontinuity**, if $\lim\limits_{x\to a}\big(f(x)\big)$ converges, but the limit is not equal to $f(a)$.
We say that f has at a a **jump discontinuity**, if both one-side limits $\lim\limits_{x\to a^-}\big(f(x)\big)$ and $\lim\limits_{x\to a^+}\big(f(x)\big)$ converge, but they are not equal.
We say that f has at a a **discontinuity of third type**, if at least one of the one-side limits $\lim\limits_{x\to a^-}\big(f(x)\big)$ or $\lim\limits_{x\to a^+}\big(f(x)\big)$ does not converge.
Definition. Let f be a function defined on a set M.
We say that f satisfy the **intermediate value property** on M, if $\forall a, b \in f(M)\ \forall c \in (a, b)\ \exists x \in M$: $f(x) = c$.
Verbally, if f obtains on M any two values, then there shall obtain even all values in between.
Theorem. (Intermediate Value Theorem) Let f be continuous on an interval I.
Then f satisfies the intermediate value property on I.
Corollary. Let f be a function defined on an interval $\langle a, b\rangle$.
If $f(a)$ and $f(b)$ have different sign and f is continuous on $\langle a, b\rangle$, then f must obtain the zero value at least at one point on $\langle a, b\rangle$.

Theorem.
A continuous function defined on a closed interval is bounded.

Definition. Let f be a function defined on a nonempty set M.
If f is bounded above on M, we define its **supremum** on M, indicated by $\sup\limits_M(f)$, as the least upper bound. Otherwise we define $\sup\limits_M(f)=\infty$.
If f is bounded below on M, we define its **infimum** on M, indicated by $\inf\limits_M(f)$, as the greatest lower bound. Otherwise we define $\inf\limits_M(f)=-\infty$.
We define the **maximum** of f on M as the number $m=\max\limits_M(f)$ that satisfies these two conditions: $\forall x\in M$: $f(x)\le m$ and $\exists c\in M$: $f(c)=m$. If such maximum exists, we say that f **obtains its maximum** on M.
We define the **minimum** of f on M as the number $m=\min\limits_M(f)$ that satisfies these two conditions: $\forall x\in M$: $f(x)\ge m$ and $\exists c\in M$: $f(c)=m$. If such minimum exists, we say that f **obtains its minimum** on M.
Observation: Let $f(M)$ be the image of the set M under f. Then $\sup\limits_M(f)=\sup\big(f(M)\big)$, similarly for the other three definitions.

Fact.
Every function has supremum and infimum on any nonempty subsets of $D(f)$.

Theorem. (Extreme Value Theorem)
A function continuous on a closed bounded interval obtains there its minimum and maximum.

Theorem. Let f be a function continuous on an interval I.
f is on I one to one $\iff$ f is on I strictly monotone (increasing or decreasing [precise definitions are given later]). Then also its inverse function f_{-1} is continuous.