Definitions and claims—functions (derivative)

Definition. Let f be defined on a neighbourhood of a.
We say, that f is **differentiable** at a, if $\lim\limits_{x\to a}\bigl(\frac{f(x)-f(a)}{x-a}\bigr)$ converges.
Then we define the **derivative** of f at a as $f'(a)=\lim\limits_{x\to a}\bigl(\frac{f(x)-f(a)}{x-a}\bigr)$.
Leibnitz's symbols: $f'(a)=\frac{\mathrm{d}f}{\mathrm{d}x}(a)=\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_a=\frac{\mathrm{d}}{\mathrm{d}x}f(a)=\frac{\mathrm{d}}{\mathrm{d}x}f\Big|_a$.
Alternative equivalent form: $f'(a)=\lim\limits_{h\to 0}\bigl(\frac{f(a+h)-f(a)}{h}\bigr)$.

Fact.
If a function f has a derivative at a point a, then at a there is a tangent line to the graph of the function [with slope $k_T=f'(a)$] and a perpendicular line [with slope $k_N=-\frac{1}{k_T}$ if $k_T\neq 0$].

Application: Looking for a solution of $f(x)=0$ with Newton's method: $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$.

Theorem.
If a function f is differentiable at a, then f is continuous at a.

Definition.
We say, that a function f is differentiable on an open set G, if $\forall a\in G\ \exists f'(a)$.

Fact. (**dictionary**)
$[c]'=0$, $x\in\mathbb{R}$.
$[x^a]'=a\cdot x^{a-1}$, $x\in\mathbb{R}$ for $a\in\mathbb{N}$, $x\in\mathbb{R}\setminus\{0\}$ for $a\in\mathbb{Z}$, $x\in\mathbb{R}^+$ for $a\in\mathbb{R}$.

$[e^x]'=e^x$, $x\in\mathbb{R}$;	$[a^x]'=\ln(a)a^x$, $x\in\mathbb{R}$.
$[\ln(x)]'=\frac{1}{x}$, $x>0$;	$[\log_a(x)]'=\frac{1}{\ln(a)}\frac{1}{x}$, $x>0$.
$[\sin(x)]'=\cos(x)$, $x\in\mathbb{R}$;	$[\cos(x)]'=-\sin(x)$, $x\in\mathbb{R}$;
$[\mathrm{tg}(x)]'=\frac{1}{\cos^2(x)}$, $x\neq\frac{\pi}{2}+k\pi$;	$[\mathrm{cotg}(x)]'=\frac{-1}{\sin^2(x)}$, $x\neq k\pi$.
$[\arcsin(x)]'=\frac{1}{\sqrt{1-x^2}}$, $x\in(-1,1)$;	$[\arccos(x)]'=\frac{-1}{\sqrt{1-x^2}}$, $x\in(-1,1)$;
$[\mathrm{arctg}(x)]'=\frac{1}{x^2+1}$, $x\in\mathbb{R}$;	$[\mathrm{arccotg}(x)]'=\frac{-1}{x^2+1}$, $x\in\mathbb{R}$.
$[\sinh(x)]'=\cosh(x)$, $x\in\mathbb{R}$;	$[\cosh(x)]'=\sinh(x)$, $x\in\mathbb{R}$;
$[\mathrm{tgh}(x)]'=\frac{1}{\cosh^2(x)}$, $x\in\mathbb{R}$;	$[\mathrm{cotgh}(x)]'=\frac{-1}{\sinh^2(x)}$, $x\neq 0$.
$[\mathrm{argsinh}(x)]'=\frac{1}{\sqrt{x^2+1}}$, $x\in\mathbb{R}$;	$[\mathrm{argcosh}(x)]'=\frac{1}{\sqrt{x^2-1}}$, $x\in(1,\infty)$;
$[\mathrm{argtgh}(x)]'=\frac{1}{1-x^2}$, $x\in(-1,1)$;	$[\mathrm{argcotgh}(x)]'=\frac{1}{1-x^2}$, $x\in(-\infty,-1)\cup(1,\infty)$.

Other useful formulas: $[Ax+B]'=A$, $\bigl[\sqrt{x}\bigr]'=\frac{1}{2\sqrt{x}}$.

Theorem. Let f,g be differentiable at a.
Then, functions $f+g$, $f-g$, $f\cdot g$, $\frac{f}{g}$ (pokud $g(a)\neq 0$) are also differentiable there and their derivatives are given by the formulas:
$(f+g)'(a)=f'(a)+g'(a)$ $(f-g)'(a)=f'(a)-g'(a)$
$(f\cdot g)'(a)=f'(a)g(a)+f(a)g'(a)$ $\bigl(\frac{f}{g}\bigr)'(a)=\frac{f'(a)g(a)-f(a)g'(a)}{g^2(a)}$

Theorem. Let f be differentiable at a and g be differentiable at $b=f(a)$.
Then $g\circ f=g(f)$ is differentiable at a and it holds $(g\circ f)(a)=g'(f(a))\cdot f'(a)$.

Theorem. (**grammar**)
$[\alpha f\pm\beta g]'=\alpha f'\pm\beta g'$ for $\alpha,\beta\in\mathbb{R}$ (linearity)
$[f\cdot g]'=f'g+fg'$ (product rule)
$\bigl[\frac{f}{g}\bigr]'=\frac{f'g-fg'}{g^2}$ (quotient rule)
$[g(f)]'=g'(f)\cdot f'$ (chain rule)

Theorem. (inverse function)
Let f be continuous and strictly monotone on a neighbourhood of a, let $b=f(a)$. If f is differentiable at a, then also its inverse function f_{-1} is differentiable at b and

$$[f_{-1}]'(b)=\frac{1}{f'(a)}=\frac{1}{f'(f_{-1}(b))}.$$

Definition.
Let f be defined on an open left neighbourhood of a.
We say, that f is **differentiable** at a **from the left**, if $\lim\limits_{x\to a^-}\bigl(\frac{f(x)-f(a)}{x-a}\bigr)$ converges.
Then we define the **left derivative** f at a as $f'_-(a)=\lim\limits_{x\to a^-}\bigl(\frac{f(x)-f(a)}{x-a}\bigr)$.
Let f be defined on an open right neighbourhood of a.
We say, that f is **differentiable** at a **from the right**, if $\lim\limits_{x\to a^+}\bigl(\frac{f(x)-f(a)}{x-a}\bigr)$ converges.
Then we define the **right derivative** f at a as $f'_+(a)=\lim\limits_{x\to a^+}\bigl(\frac{f(x)-f(a)}{x-a}\bigr)$.

Theorem.

$\exists f'_-(a) \implies f$ is continuous at a from the left. $\exists f'_+(a) \implies f$ is continuous at a from the right.

Theorem.
f is differentiable at $a \iff \exists f'_-(a), f'_+(a)$ and they are equal. Then $f(a) = f'_-(a) = f'_+(a)$.

Definition. Let $n \in I\!N_0$.
We define the n-th derivative of f as $f^{(0)} = f$, $f^{(1)} = f'$ and $f^{(n+1)} = [f^{(n)}]'$.
We also call it the derivative of order n.

Definitions and claims—functions (derivative and properties of functions)

Theorem. (Rolle's Theorem)
Let f be continuous on an interval $\langle a,b\rangle$ and differentiable on (a,b). If $f(a) = f(b)$, then $\exists c \in (a,b)$: $f'(c) = 0$.

Theorem. (Mean value Theorem, Lagrange's Theorem)
Let f be continuous on an interval $\langle a,b\rangle$ and differentiable on (a,b). Then $\exists c \in (a,b)$: $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Corollary.
Let f be continuous on an interval $\langle a,b\rangle$ and differentiable on (a,b). If $f' = 0$ on (a,b), then f is constant on $\langle a,b\rangle$.

Corollary.
Let f,g be continuous on an intervalu $\langle a,b\rangle$ and differentiable on (a,b). If $\exists c \in \langle a,b\rangle$: $f(c) = g(c)$ and $f' = g'$ on (a,b), then $f = g$ on $\langle a,b\rangle$.

Theorem.
Let f be continuous at a from the left and differentiable on a left annular neighbourhood of a. Then $f'_-(a) = \lim\limits_{x\to a^-}\big(f'(x)\big)$, if the limit converges.
Let f be continuous at a from the right and differentiable on a right annular neighbourhood of a. Then $f'_+(a) = \lim\limits_{x\to a^+}\big(f'(x)\big)$, if the limit converges.

Theorem. (l'Hôpital's rule)
Let f,g be differentiable on an annular neighbourhood of $a \in I\!R^*$.
Suppose that either $\lim\limits_{x\to a}\big(f(x)\big) = \lim\limits_{x\to a}\big(g(x)\big) = 0$ or $\lim\limits_{x\to a}\big(|g(x)|\big) = \infty$.
If $\exists \lim\limits_{x\to a}\big(\frac{f'(x)}{g'(x)}\big)$, then $\lim\limits_{x\to a}\big(\frac{f(x)}{g(x)}\big) = \lim\limits_{x\to a}\big(\frac{f'(x)}{g'(x)}\big)$.

Definitions and claims—functions (Taylor polynom)

Definition. Suppose that f has at a n-th derivative.
We define the **Taylor polynom of degree n with center at a** as

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

We define the rest $R_n(x) = f(x) - T_n(x)$ for $x \in D(f)$.

Theorem. Suppose that f has at a n-th derivative.
Then T_n is the only polynom of degree n that , at the point a, has same derivatives as f up to order n.

Theorem. (Taylor's Theorem)
Suppose that f has at a n-th derivative, let x be such that f is defined on a closed interval I from a to x.
If f has continuous n-th derivative on I and there exists $f^{(n+1)}$ on the interior of I^O, then
$R_n(x) = \frac{1}{n!}\int\limits_a^x f^{(n+1)}(t)(x-t)^n\,dt.$
Then also $\exists\ c \in I^O$: $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$.
Observation: The first is called the integral formula of the rest, the second Lagrange's formula.

Definitions and claims—functions (graph)

Definition. Let f be a function defined on an interval I.
We say that f is **increasing** on I, if $\forall x < y \in I$: $f(x) < f(y)$.
We say that f is **not-decreasing** on I, if $\forall x < y \in I$: $f(x) \le f(y)$.
We say that f is **decreasing** on I, if $\forall x < y \in I$: $f(x) > f(y)$.
We say that f is **not-increasing** on I, if $\forall x < y \in I$: $f(x) \ge f(y)$.
We say that f is **monotone** on I, if f has on I one of the previous four properties.
We say that f is **strictly monotone** on I, if f is increasing on I or f is decreasing on I.

Theorem. Let f be continuous on an interval I and differentiable on its interior I^O.
If $f' > 0$ on I^O, then f is increasing on I.
If $f' \ge 0$ on I^O, then f is not-decreasing on I.
If $f' < 0$ on I^O, then f is decreasing on I.
If $f' \le 0$ on I^O, then f is not-increasing on I.
If f is not-decreasing on I, then $f' \ge 0$ on I^O.
If f is not-increasing on I, then $f' \le 0$ on I^O.

Theorem. If f, at a point a, changes from decreasing to increasing (or the opposite way), then $f'(a)=0$ or $f'(a)$ does not exist.
Definition. Let f be a function defined on a neighbourhood of c.
c is a **critical point**, if $f'(c)=0$ or $f'(c)$ does not exist.

Definition. Let f be a function defined on a neighbourhood of a.
We say that f has a **local maximum** at a, resp. that $f(a)=b$ is a local maximum, if $\exists$ a neighbourhood $U=U(a)$ such that $\forall x\in U$ we have $f(x)\le f(a)$.
We say that f has a **local minimum** at a , resp. that $f(a)=b$ is a local minimum, if $\exists$ a neighbourhood $U=U(a)$ such that $\forall x\in U$ we have $f(x)\ge f(a)$.
We say that f has, at the point a, a **local extreme**, if it has there a local maximum or a local minimum.
We say that f has, at the point a, a **sharp local extreme**, if a is a local extreme and $\exists$ annular neighbourhood $P=P(a)$ such that $\forall x\in P$ we have $f(x)\ne f(a)$.
Theorem.
If a function f has a local extreme at a point a , then a is a critical point.
Theorem. Let c be a critical point.
If $f''(c)>0$, then $f(c)$ is a local minimum.
If $f''(c)<0$, then $f(c)$ is a local maximum.
Theorem.
If a point c is a global extreme (absolute maximum or absolute minimum) for a function f on intervalu I, then c is either a local extreme or a boundary point of I.

Definition. Let f be a function defined on an interval I.
We say that a function f on the interval I is **convex**, or **concave down** if $\forall x<y<z\in I$: $\frac{f(y)-f(x)}{y-x}\le\frac{f(z)-f(y)}{z-y}$.
We say that a function f on the intervalu I is **concave**, or **concave up** if $\forall x<y<z\in I$: $\frac{f(y)-f(x)}{y-x}\ge\frac{f(z)-f(y)}{z-y}$.
We say that $a\in D(f)$ is a **point of inflexion**, if at a f changes concavity (from up to down or viceversa).
Theorem.
If f at a point a changes its concavity, then we must have $f''(a)=0$ or $f''(a)$ does not exist.
Theorem. Let f be a function with continuous first derivative f' on the interval I, let f'' exists on the interior of I^O.
If $f''\ge 0$ on I^O, then f is convex on I.
If $f''\le 0$ on I^O, then f is concave on I.

Definition.
Let f be a function defined at least on a one-side annular neighbourhood of $a\in\mathbb{R}$.
We say that the line $x=a$ is **a vertical asymptote** for f, or that f has a vertical asymptote at a, if
$\lim\limits_{x\to a^+}\big(f(x)\big)=\pm\infty$ nebo $\lim\limits_{x\to a^-}\big(f(x)\big)=\pm\infty$.
Definition.
Let f be a function defined on a neighbourhood of ∞.
We say that the line $y=B$ is a **horizontal asymptote** of f at ∞, if $\lim\limits_{x\to\infty}\big(f(x)\big)=B$.

Let f be a function defined on a neighbourhood of $-\infty$.
We say that the line $y=B$ is a **horizontal asymptote** of f at $-\infty$, if $\lim\limits_{x\to-\infty}\big(f(x)\big)=B$.
Definition.
Let f be defined on a neighbourhood of ∞.
We say that the line $y=Ax+B$, $A\ne 0$, is an **oblique asymptote** of f at ∞, if $\lim\limits_{x\to\infty}\big(f(x)-(Ax+B)\big)=0$.
Let f be defined on a neighbourhood of $-\infty$.
We say that the line $y=Ax+B$, $A\ne 0$, is an **oblique asymptote** of f at $-\infty$, if $\lim\limits_{x\to-\infty}\big(f(x)-(Ax+B)\big)=0$.
Theorem.
Let f be defined on a neighbourhood of ∞.
A line $y=Ax+B$ is an oblique asymptote of f at ∞ $\iff$ $0\ne A=\lim\limits_{x\to\infty}\big(\frac{f(x)}{x}\big)$ and $B=\lim\limits_{x\to\infty}\big(f(x)-Ax\big)$.
Let f is definovna on neighbourhood $-\infty$.
A line $y=Ax+B$ is an oblique asymptote of f at $-\infty$ $\iff$ $0\ne A=\lim\limits_{x\to-\infty}\big(\frac{f(x)}{x}\big)$ and $B=\lim\limits_{x\to-\infty}\big(f(x)-Ax\big)$.