Definitions and claims—functions, integrals

Definition. Let f be a function defined on an interval I.
We say that a function F is **an antiderivative** of f on I, if F is continuous on I, differentiable on the interior of I^O and $F' = f$ on I^O.

Theorem. Let F be an antiderivative of f on an interval I.
1) $\forall c \in \mathbb{R}$: $G(x) = F(x) + c$ is an antiderivative of f on I.
2) Let G be a different antiderivative of f on I. Then $\exists c \in \mathbb{R}$: $G = F + c$ on I.

Fact.
Let F be a function, that has on I a derivative. Then this derivative satisfies the intermediate value property on I.

Corollary. Let f be a function on an interval I.
If this function does not satisfy the intermediate value property on I, then an antiderivative of f on I cannot exist.

Theorem.
If a function is continuous on an interval, then it has antiderivative on this interval.

Definition. Let f be a function that has antiderivative on an interval I.
We define **the indefinite integral** of f on I as the set of all possible antiderivatives of f.

Notation: $\int f(x)\,dx = \{F;\ F \text{ is an antiderivative function of } f \text{ on } I\}$.

If we have one antiderivative F, then traditionally we write

$$\int f(x)\,dx = F(x) + C,\ x \in I.$$

Fact. (**table of integrals**)

$\int x^a dx = \frac{x^{a+1}}{a+1} + C$, $a \neq -1$, $x \in D_a$ where D_a depends on a.

$\int \frac{1}{x} dx = \ln|x| + C$, $x \neq 0$.

$\int e^x dx = e^x + C$, $x \in \mathbb{R}$;

$\int a^x dx = \frac{a^x}{\ln(a)} + C$, $x \in \mathbb{R}$.

$\int \sin(x)\,dx = -\cos(x) + C$, $x \in \mathbb{R}$;

$\int \cos(x)\,dx = \sin(x) + C$, $x \in \mathbb{R}$;

$\int \frac{dx}{\cos^2(x)} = \tan(x) + C$, $x \neq \frac{\pi}{2} + to\pi$;

$\int \frac{dx}{\sin^2(x)} = -\cot(x) + C$, $x \neq to\pi$.

$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$, $x \in (-1,1)$;

$\int \frac{dx}{x^2+1} = \arctan(x) + C$, $x \in \mathbb{R}$.

$\int \sinh(x)\,dx = \cosh(x) + C$, $x \in \mathbb{R}$;

$\int \cosh(x)\,dx = \sinh(x) + C$, $x \in \mathbb{R}$;

$\int \frac{dx}{\cosh^2(x)} = \tanh(x) + C$, $x \in \mathbb{R}$;

$\int \frac{dx}{\sinh^2(x)} = -\coth(x) + C$, $x \neq 0$.

$\int \frac{dx}{\sqrt{x^2+1}} = \operatorname{argsinh}(x) + C$, $x \in \mathbb{R}$;

$\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{argcosh}(x) + C$, $x \geq 1$;

$\int \frac{dx}{x^2-1} = \operatorname{argtanh}(x) + C$, $x \in (-1,1)$;

$\int \frac{dx}{x^2-1} = \operatorname{argcoth}(x) + C$, $|x| > 1$.

Theorem. (linearity)
Let F be an antiderivative of f on an interval I and G be an antiderivative of g on I, let $\alpha, \beta \in \mathbb{R}$. Then $\alpha F + \beta G$ is an antiderivative of $\alpha f + \beta g$ on I.

Notation: $\int (\alpha f + \beta g)(x)\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$.

Theorem. (direct substitution)
Let F be an antiderivative of f on an interval I. Let φ be a function $J \mapsto I$, that is continuous on J and has derivative on J^O. Then $F(\varphi)$ is an antiderivative of $f(\varphi) \cdot \varphi'$ on J.

Notation: $\int f(\varphi(t))\varphi'(t)\,dt = F(\varphi(t)) + C$, $t \in J$.

Theorem. (indirect substitution)
Let f be a function on interval I. Let φ be a continuous bijection $J \mapsto I$, that is differentiable on J^O. If G is an antiderivative of $f(\varphi)\varphi'$ on J, then $G(\varphi_{-1})$ is an antiderivative of f on I.

Notation: $\int f(x)\,dx = G(\varphi_{-1}(x)) + C$, $x \in I$.

Notation for calculations of substitution:
direct:

$$\int f(\varphi(t))\varphi'(t)\,dt = \left|\begin{matrix} x = \varphi(t) \\ dx = \varphi'(t)\,dt \end{matrix}\right| = \int f(x)\,dx = F(x) + C = F(\varphi(t)) + C.$$

indirect:

$$\int f(x)\,dx = \left|\begin{matrix} x = \varphi(t) \\ dx = \varphi'(t)\,dt \end{matrix}\right| = \int f(\varphi(t))\varphi'(t)\,dt = G(t) + C = G(\varphi_{-1}(x)) + C.$$

Theorem. (integration by parts or per partes)
Let f, g be differentiable functions on interval I. If there exists an antiderivative F of $f(x)g'(x)$ on I, then $G(x) = f(x)g(x) - F(x)$ is an antiderivative of $f'(x)g(x)$ on I.

Notation: $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$

Often used notations:

$$\int f'g\,dx = fg - \int fg'\,dx,$$

$$\int u'v\,dx = uv - \int uv'\,dx,$$

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

Fact. Let p be a polynomial of degree n.
1) p has exactly n complex roots (counting their multiplicity).
2) If c is a root of the polynomial p, then there exists a polynomial q such that degree(q) = degree$(p) - 1$ and $p(x) = (x - c)q(x)$. If p and c are real, then also q is real.
3) If c is a complex root of the polynomial p, also its complex conjugate c^* is a root of the polynomial p. Moreover $(x - c)(x - c^*) = x^2 - 2\,\mathrm{Re}(c)x - |c|^2 = x^2 + \alpha x + \beta$, where α, β are real.

Corollary. Let $p(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial of degree n.
Then $p(x) = a_n(x - c_1) \cdot (x - c_2) \cdot \ldots \cdot (x - c_n)$, kde c_i are roots of the polynomial p.
Then , also, $p(x) = a_n(x - d_1)^{n_1} \cdot (x - d_2)^{n_2} \cdot \ldots \cdot (x - d_N)^{n_N}$, where d_i are distinct roots of p with multiplicity n_i, $\sum n_i = n$.
Then,

$$p(x) = a_n(x - d_1)^{n_1} \cdot (x - d_2)^{n_2} \cdot \ldots \cdot (x - d_N)^{n_N} \cdot (x^2 + \alpha_1 x + \beta_1)^{m_1} \cdot \ldots \cdot (x^2 + \alpha_M x + \beta_M)^{m_M},$$

where d_i are distinct real roots of p with multiplicity n_i and $(x^2 + \alpha_j x + \beta_j)$ are distinct irreducible second degree factors, each corresponding to a couple of complex conjugated roots c_j, c_j^* with multiplicity m_j, and $\sum n_i + 2\sum m_j = n$.

Definition. A **Rational Function** is a function that can be written in the form $\frac{p}{q}$, where p, q are polynomials.

Fact.
Let p, q be polynomials. Then there exist uniquely determined polynomials a, r such that

$$p(x) = a(x) \cdot q(x) + r(x) \qquad \text{and} \qquad \text{degree}(r) < \text{degree}(q).$$

Theorem. (separation of roots) Let p, q be polynomials and degree(p) < degree(q).
1) If d is a root of q with multiplicity k, tj. $q(x) = (x - d)^k \tilde{q}(x)$ and $\tilde{q}(d) \neq 0$, then $\exists A \in I\!R$ $\exists$ polynomial $\tilde{p}$ such that degree$(\tilde{p})$ < degree$(q) - 1$:

$$\tfrac{p(x)}{q(x)} = \tfrac{A}{(x-d)^k} + \tfrac{\tilde{p}(x)}{(x-d)^{k-1}\tilde{q}(x)}.$$

2) If $(x^2 + \alpha x + \beta)$ is an irreducible factor of q with multiplicity k, i.e. $(x^2 + \alpha x + \beta)$ cannot be written as the product of two real linear factors, $q(x) = (x^2 + \alpha x + \beta)^k \tilde{q}(x)$ and $\tilde{q}$ is not divisible by $(x^2 + \alpha x + \beta)$), then $\exists B, C \in I\!R$ $\exists$ polynomial $\tilde{p}$ such that degree$(\tilde{p})$ < degree$(q) - 2$:

$$\frac{p(x)}{q(x)} = \frac{Bx + C}{(x^2 + \alpha x + \beta)^k} + \frac{\tilde{p}(x)}{(x^2 + \alpha x + \beta)^{k-1}\tilde{q}(x)}.$$

The constant A, or B, C are uniquely determined.
Conclusion:

Theorem. (decomposition into **partial fractions**)
Let p, q be polynomials such that degree(p) < degree(q). Let

$$q(x) = a_n(x - d_1)^{n_1} \cdot (x - d_2)^{n_2} \cdot \ldots \cdot (x - d_N)^{n_N} \cdot (x^2 + \alpha_1 x + \beta_1)^{m_1} \cdot \ldots \cdot (x^2 + \alpha_M x + \beta_M)^{m_M},$$

where d_i are distinct roots of q with multiplicity n_i a $(x^2 + \alpha_j x + \beta_j)$ are distinct irreducible quadratic factors, each corresponding to a couple of complex conjugated roots c_j, c_j^* with multiplicity m_j, and $\sum n_i + 2\sum m_j = n$.
Then $\exists A_{1,1}, \ldots A_{1,n_1}, A_{2,1}, \ldots, A_{N,n_N} \in I\!R$ $\exists B_{1,1}, C_{1,1}, \ldots, B_{1,m_1}, C_{1,m_1}, B_{2,1}, \ldots, C_{M,m_M} \in I\!R$:

$$\begin{aligned}\frac{p(x)}{q(x)} = \frac{A_{1,1}}{x-d_1} + \frac{A_{1,2}}{(x-d_1)^2} + \ldots + \frac{A_{1,n_1}}{(x-d_1)^{n_1}}\\ + \frac{A_{2,1}}{x-d_2} + \frac{A_{2,2}}{(x-d_2)^2} + \ldots + \frac{A_{2,n_2}}{(x-d_2)^{n_2}} + \ldots + \frac{A_{N,n_N}}{(x-d_N)^{n_N}}\\ + \frac{B_{1,1}x+C_{1,1}}{x^2+\alpha_1 x+\beta_1} + \frac{B_{1,2}x+C_{1,2}}{(x^2+\alpha_1 x+\beta_1)^2} + \ldots + \frac{B_{1,m_1}x+C_{1,m_1}}{(x^2+\alpha_1 x+\beta_1)^{m_1}}\\ + \frac{B_{2,1}x+C_{2,1}}{x^2+\alpha_2 x+\beta_2} + \frac{B_{2,2}x+C_{2,2}}{(x^2+\alpha_2 x+\beta_2)^2} + \ldots + \frac{B_{2,m_2}x+C_{2,m_2}}{(x^2+\alpha_2 x+\beta_2)^{m_2}} + \ldots + \frac{B_{M,m_M}x+C_{M,m_M}}{(x^2+\alpha_M x+\beta_M)^{m_M}}.\end{aligned}$$

Definitions and claims—definite integral.

Definition. Let $[a,b]$ be an interval.
A **partition** of the interval is choice of points $D=\{x_0,x_1,\ldots,x_n\}$ such that $a=x_0<x_1<\ldots<x_n=b$.
Definition. Let f be a bounded function on $[a,b]$.
Given a partition D of this interval we define
Upper sum $\overline{S}(f,D)=\sum\limits_{to=1}^{n}\sup\limits_{\langle x_{to-1},x_to\rangle}(f)\cdot(x_to-x_{to-1})$,
Lower sum $\underline{S}(f,D)=\sum\limits_{to=1}^{n}\inf\limits_{\langle x_{to-1},x_to\rangle}(f)\cdot(x_to-x_{to-1})$.
Lemma. Let D be a partition of the interval $[a,b]$, let $\tilde{D}$ be a refinement of it, i.e. $\tilde{D}=D\cup\{x\}$ for a certain $x\in\langle a,b\rangle\setminus D$. Let f be bounded on $[a,b]$. Then

$$\underline{S}(f,D)\le\underline{S}(f,\tilde{D})\le\overline{S}(f,\tilde{D})\le\overline{S}(f,D).$$

Definition. Let f be a bounded function on $[a,b]$.
For partitions D_1,D_2 of the interval $[a,b]$:

$$\inf_{[a,b]}(f)\cdot(b-a)\le\underline{S}(f,D_1)\le\overline{S}(f,D_2)\le\sup_{[a,b]}(f)\cdot(b-a).$$

Conclusions:
Fact. Let f be a bounded function on $[a,b]$.
Then $\exists\ \underline{S}=\sup\{\underline{S}(f,D);\ D \text{ partition of } [a,b]\}\in I\!R$,
$\exists\ \overline{S}=\inf\{\overline{S}(f,D);\ D \text{ partition of } [a,b]\}\in I\!R$ a $\underline{S}\le\overline{S}$.
Definition. Let f be a bounded function on $[a,b]$.
We say that f is **Riemann integrable** on $[a,b]$, if $\underline{S}=\overline{S}$.
Then we define the **(Riemann) definite integral** f from a to b as $\int\limits_a^b f(x)\,dx=\underline{S}$.

Theorem. Let f be a bounded function on $[a,b]$.
f is Riemann integrable on $[a,b]$ $\iff$ $\exists\ D_1,D_2,\ldots$ partitions of $[a,b]$: $\lim\limits_{n\to\infty}\big(\overline{S}(f,D_n)-\underline{S}(f,D_n)\big)=0$.
Then also $\int\limits_a^b f(x)\,dx=\lim\limits_{n\to\infty}\big(\overline{S}(f,D_n)\big)=\lim\limits_{n\to\infty}\big(\underline{S}(f,D_n)\big)$.
Numerical methods for calculation of the definite integral.
Rectangular rule: $\int_a^b f(x)\,dx\approx(b-a)\,f\left(\frac{a+b}{2}\right)$.
Trapezoidal rule: $\int_a^b f(x)\,dx\approx(b-a)\,\frac{f(a)+f(b)}{2}$.
Simpson's rule: $\int_a^b f(x)\,dx\approx\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$.

Theorem.
1) If f is monotone on $[a,b]$, then it is Riemann integrable on $[a,b]$.
2) If is f continuous on $[a,b]$, then it is Riemann integrable on $[a,b]$.

Theorem.
1) (linearity) Let f,g be Riemann integrable on $\langle a,b\rangle$. Then $\forall\alpha,\beta\in I\!R$: $\alpha f+\beta g$ is Riemann integrable on $[a,b]$ and

$$\int\limits_a^b(\alpha f+\beta g)(x)\,dx=\alpha\int\limits_a^b f(x)\,dx+\beta\int\limits_a^b g(x)\,dx.$$

2) (comparison) Let f,g be Riemann integrable on $\langle a,b\rangle$. If $f\le g$ on $[a,b]$, then

$$\int\limits_a^b f(x)\,dx\le\int\limits_a^b g(x)\,dx.$$

3) Let f be Riemann integrable on $[a,b]$. Then also $|f|$ is there Riemann integrable and

$$\left|\int\limits_a^b f(x)\,dx\right|\le\int\limits_a^b|f(x)|\,dx.$$

4) Let f be Riemann integrable on $[a,b]$. Then $\forall$ closed interval $J\subset[a,b]$: f is Riemann integrable on J.
5) Let $a<b<c$, f a function on $[a,c]$.
f is Riemann integrable on $[a,c]$ $\iff$ f is Riemann integrable on $[a,b]$ a f is Riemann integrable on $[b,c]$.

Then also $\int\limits_a^c f(x)\,dx = \int\limits_a^b f(x)\,dx + \int\limits_b^c f(x)\,dx$.

Fact.

1) Let f be Riemann integrable on $[a,b]$. If $\exists m, M \in I\!R$: $m \le f(x) \le M$ on $[a,b]$, then $m(b-a) \le \int\limits_a^b f(x)\,dx \le M(b-a)$.

2) Let f be Riemann integrable on $[-a,a]$.

If f is odd, then $\int\limits_{-a}^a f(x)\,dx = 0$. If f is even, then $\int\limits_{-a}^a f(x)\,dx = 2\int\limits_0^a f(x)\,dx$.

Definition. Let f be a bounded function on $[a,b]$. We define

$$\int\limits_b^a f(x)\,dx = -\int\limits_a^b f(x)\,dx \qquad \text{and} \qquad \int\limits_a^a f(x)\,dx = 0.$$

Theorem. Let f be a bounded Riemann integrable function on $I = [a,b]$.

Chosen a $c \in I$, we define $F(x) = \int\limits_c^x f(t)\,dt$ for $x \in I$.

1) F is continuous on I.

2) If f is continuous at $a \in I^O$, then F is differentiable at a and it holds that $F'(a) = f(a)$.

Theorem. (Fundamental Theorems of Calculus)

Let f be a continuous function on $I = [a,b]$.

1) Chosen $c \in I$ and define $F(x) = \int\limits_c^x f(t)\,dt$ for $x \in I$. Then F is an antiderivative function of f on I.

2) Let F be any antiderivative function of f on I. Then

$$\int\limits_a^b f(x)\,dx = F(b) - F(a) \qquad \text{(Newton-Leibniz formula).}$$

Notation: $F(b) - F(a) = \big[F(x)\big]_a^b$.

Definitions and claims—application of integrals.

Fact.

Let f, g be continuous functions on $[a,b]$ and $f \le g$ on $[a,b]$. **Area** of the region of the plane between the graphs of f and g on $[a,b]$ is

$$A = \int\limits_a^b [g(x) - f(x)]\,dx.$$

Fact.

Let f have a continuous derivative on $[a,b]$. Then the **arc length** of the portion of graph of f on $[a,b]$ is

$$L = \int\limits_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

Fact.

Let f be continuous on $[a,b]$.

1) Let $c \in I\!R$ such that $c \le \inf\limits_{[a,b]}(f)$. **The volume of the solid of revolution** obtained rotating the graph of f on $[a,b]$ **around the horizontal axis** $y = c$ is

$$V = \int\limits_a^b \pi\big(f(x) - c\big)^2 dx \qquad \text{(slice method).}$$

Let f have also continuous derivative on $[a,b]$. **The surface area** of this solid of revolution is

$$S = \int\limits_a^b 2\pi\big(f(x) - c\big)\sqrt{1 + [f'(x)]^2}dx.$$

2) Let $f \ge 0$ on $[a,b]$, let $d \in I\!R \setminus [a,b]$. **The volume of the solid of revolution** obtained rotating the graph of f on $[a,b]$ **around the vertical axis** $x = d$ is

$$V = \int\limits_a^b 2\pi(x - d)f(x)\,dx \qquad \text{(peel method).}$$

Definition. Let f be Riemann integrable on $[a,b]$.

We define **average** or **mean value** of f on $[a,b]$ as $\operatorname*{Ave}\limits_{[a,b]}(f) = \frac{1}{b-a}\int\limits_a^b f(x)\,dx$.

Theorem. (Mean Value Theorem for the integral)

Let f be continuous on $[a,b]$. Then $\exists c \in [a,b]$: $f(c) = \operatorname*{Ave}\limits_{[a,b]}(f)$.

Theorem. (continuity with respect to the limits of integration)

Let f be Riemann integrable on $[a,b]$. Then

$$\int\limits_a^b f(x)\,dx = \lim_{B\to b^-}\Bigl(\int\limits_a^B f(x)\,dx\Bigr), \qquad \int\limits_a^b f(x)\,dx = \lim_{A\to a^+}\Bigl(\int\limits_A^b f(x)\,dx\Bigr).$$

Definitions and claims—functions–improper integral.

Definition.

1) Let $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{\infty\}$ such that $a < b$. Let f be a function on $[a, b)$ such that $\forall B \in (a, b)$: f is Riemann integrable on $[a, B]$. Then we define the **improper Riemann integral** of f from a to b as

$$\int\limits_a^b f(x)\,dx = \lim_{B\to b^-}\Bigl(\int\limits_a^B f(x)\,dx\Bigr), \text{ if this limit exists.}$$

2) Let $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R}$ such that $a < b$. Let f be a function on $(a, b]$ such that $\forall A \in (a, b)$: f is Riemann integrable on $[A, b]$. Then we define the **improper Riemann integral** of f from a to b as

$$\int\limits_a^b f(x)\,dx = \lim_{A\to a^+}\Bigl(\int\limits_A^b f(x)\,dx\Bigr), \text{ if this limit exists.}$$

Terminology: If one of those limits exists, we say that the corresponding integral exists, otherwise that it does not exist. If one of the limits converges, we say that the corresponding **integral converges**, otherwise the integral **diverges**.

Fact.

$$\int\limits_1^\infty \frac{1}{x^p}\,dx \begin{cases} \text{diverges}, & p \le 1; \\ = \frac{1}{p-1}, & p > 1. \end{cases} \qquad \int\limits_0^1 \frac{1}{x^p}\,dx \begin{cases} \text{diverges}, & p \ge 1; \\ = \frac{1}{1-p}, & p < 1. \end{cases}$$

Definition.

Let f be a function on the set $[a, b] \setminus P$, where $P = \{x_0 < x_1 < \ldots < x_n\}$ is a partition of $[a, b]$. Suppose that $\forall k = 1, \ldots, n$ the improper integral $\int\limits_{x_{k-1}}^{x_k} f(x)\,dx$ exists. Then we define the **improper Riemann integral** of f from a to b as $\int\limits_a^b f(x)\,dx = \sum\limits_{to=1}^{n} \int\limits_{x_{to-1}}^{x_to} f(x)\,dx$, if the sum makes sense.

In this case we say that the integral $\int\limits_a^b f(x)\,dx$ exists. If it is a real number, we say that the integral $\int\limits_a^b f(x)\,dx$ converges and in this case every integral $\int\limits_{x_{to-1}}^{x_to} f(x)\,dx$ must converge.

Theorem. (test for "bad right bound")

Let $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{\infty\}$, $a < b$. Let f, g be functions on $[a, b)$ such that $\forall B \in (a, b)$: f, g are Riemann integrable on $[a, B]$.

1) (comparison test) Let $0 \le f \le g$ on $[a, b)$.

If $\int\limits_a^b f(x)\,dx$ diverges, then $\int\limits_a^b g(x)\,dx$ diverges.

If $\int\limits_a^b g(x)\,dx$ converges, then $\int\limits_a^b f(x)\,dx$ converges.

2) (limit comparison test) Let $f, g > 0$ on $[a, b)$, let $\lim\limits_{x\to b^-}\bigl(\frac{f(x)}{g(x)}\bigr)$ converges to a positive number. Then $\int\limits_a^b f(x)\,dx$ converges $\iff$ $\int\limits_a^b g(x)\,dx$ converges.

Equivalently, $\int\limits_a^b f(x)\,dx$ diverges $\iff$ $\int\limits_a^b g(x)\,dx$ diverges.

Theorem. (test for "bad left bound")

Let $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R}$, $a < b$. Let f, g be functions on $(a, b]$ such that $\forall A \in (a, b)$: f, g are Riemann integrable on $[A, b]$.

1) (comparison test) Let $0 \le f \le g$ on $(a, b]$.

If $\int\limits_a^b f(x)\,dx$ diverges, then $\int\limits_a^b g(x)\,dx$ diverges.

If $\int\limits_a^b g(x)\,dx$ converges, then $\int\limits_a^b f(x)\,dx$ converges.

2) (limit comparison test) Let $f, g \ge 0$ on $(a, b]$, let $\lim\limits_{x\to a^+}\bigl(\frac{f(x)}{g(x)}\bigr)$ converges to a positive number.

Then $\int\limits_a^b f(x)\,dx$ converges $\iff$ $\int\limits_a^b g(x)\,dx$ converges.

Equivalently, $\int\limits_a^b f(x)\,dx$ diverges $\iff$ $\int\limits_a^b g(x)\,dx$ diverges.