## Definitions and claims—sequence

**Definition.**
A **sequence** is a function $a: I\!N \mapsto I\!R$. We indicate it "$\{a(n)\}_{n=1}^{\infty}$", more often "$\{a_n\}_{n=1}^{\infty}$", or also "$a_n$, $n = 1, 2, 3, \dots$".
It is a indexed countable set of real numbers $\{a_1, a_2, a_3, \dots\}$. It is also possible to numerate it starting not from the number one, for example, $\{a_0, a_1, a_2, \dots\} = \{a_n\}_{n=0}^{\infty}$, $\{a_5, a_6, a_7, \dots\} = \{a_n\}_{n=5}^{\infty}$ and so on.

**arithmetic sequence** : $a_n = q \cdot n + a$ for some fixed $a, q \in I\!R$.
**geometric sequence**: $a_n = a \cdot q^n$ for some fixed $a, q \in I\!R$.
**Fact.** Arithmetic sequence is such that $a_{n+1} = a_n + q$, while for geometric sequence we have $a_{n+1} = a_n \cdot q$.
**Definition.** Let $\{a_n\}$ be a sequence.
We say that it is **bounded above**, if $\exists K \in I\!R\ \forall n$: $a_n \leq K$.
We say that it is **bounded below**, if $\exists k \in I\!R\ \forall n$: $a_n \geq k$.
We say that it is **bounded**, if is shora i zdola omezen (tedy $\exists k, K\ \forall n$: $k \leq a_n \leq K$).
We say that it is **increasing**, if $\forall n$: $a_{n+1} > a_n$.
We say that it is **not decreasing**, if $\forall n$: $a_{n+1} \geq a_n$.
We say that it is **decreasing**, if $\forall n$: $a_{n+1} < a_n$.
We say that it is **not increasing**, if $\forall n$: $a_{n+1} \leq a_n$.
We say that it is **monotone**, if it has one of these four properties.
**Fact.**
Every increasing sequence is not decreasing.
Every decreasing sequence is not increasing.
A sequence cannot be decreasing and increasing at the same time.
Constant sequences $a_n = c$ are not increasing and not decreasing at the same time.

**Definition.** Let $\{a_n\}$ be a sequence.
We say that $L \in I\!R^*$ is the **limit** of $\{a_n\}$, if $\forall$ neighbourhood $U = U(a)\ \exists N \in I\!N\ \forall n \geq N$: $a_n \in U(a)$.
We also say that "$a_n$ goes to $L$ (for $n$ going to infinity)". In symbols: "$\lim\limits_{n\to\infty}(a_n) = L$" or "$a_n \to L$ for $n \to \infty$".
If we can find such $L$, we say that the **limit exists**, otherwise, we say that the **limit does not exist** .
If we can find such $L$, and $L = \pm\infty$, then we say that the sequence has **infinite limit** or **diverges to infinity**.
If we can find such $L$, and $L \in I\!R$, then we say that the sequence has **proper limit**, or that the sequence $\{a_n\}$ is **convergent** or that "$\{a_n\}$ **converges** to $L$ (for $n$ going to infinity)". If such $L \in I\!R$ does not exists, then we say that $\{a_n\}$ is **divergent** or that **diverges**.

"Epsilon–delta"- definition:
$a_n \to L \in I\!R \iff \forall \varepsilon > 0\ \exists N \in I\!N\ \forall n \geq N$: $|a_n - L| < \varepsilon$.
$a_n \to \infty \iff \forall K \in I\!R\ \exists N \in I\!N\ \forall n \geq N$: $a_n > K$.
$a_n \to -\infty \iff \forall k \in I\!R\ \exists N \in I\!N\ \forall n \geq N$: $a_n < k$.
**Theorem.**
If the limit of a sequence exists, then it is unique.
**Theorem.**
If a sequence converges, then it is bounded.
**Theorem.** If a sequence is monotone, then its limit exists.
If a sequence is monotone and bounded, then it converges.
**Definition.**
$e = \lim\limits_{n\to\infty}\big((1 + \frac{1}{n})^n\big)$.
Euler number is approximately $e \sim 2.718281828\dots$
**Fact.**
For $c \in I\!R$ it holds $\lim\limits_{n\to\infty}\big((1 + \frac{c}{n})^n\big) = e^c$.

**Fact.** (limit of a geometric sequence $\{q^n\}$)
$|q| < 1 \implies q^n \to 0$
$q = 1 \implies q^n = 1 \to 1$
$q > 1 \implies q^n \to \infty$
$q \leq -1 \implies \lim(q^n)$ does not exists.

**Undetermined forms:** $\infty - \infty$, $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\frac{x}{0}$, $0^0$, $1^\infty$, $\infty^0$.
**Theorem.** Let $a_n \to A$, $b_n \to B$, where $A, B \in I\!R^*$.
Then $(a_n + b_n) \to (A + B)$, $(a_n - b_n) \to (A - B)$, $(a_n \cdot b_n) \to (A \cdot B)$, $\big(\frac{a_n}{b_n}\big) \to \frac{A}{B}$, ${a_n}^{b_n} \to A^B$, if the result has a meaning, i.e. it is not an undetermined form.
In symbols:
$\lim(a_n \pm b_n) = \lim(a_n) \pm \lim(b_n)$, $\lim(a_n \cdot b_n) = \lim(a_n) \cdot \lim(b_n)$,
$\lim\big(\frac{a_n}{b_n}\big) = \frac{\lim(a_n)}{\lim(b_n)}$, $\lim({a_n}^{b_n}) = \lim(a_n)^{\lim(b_n)}$, if on the right hand side with have a meaningful result.

**Theorem.** Let $a_n \to L$, where $L \in I\!R$.
If the function $f$ is continuous at $L$, then $\lim\big(f(a_n)\big) = f(L)$.
In symbols: $\lim\big(f(a_n)\big) = f\big(\lim(a_n)\big)$.
If $\lim\limits_{x\to\infty}(f)$ has a meaning, then we can indicate the result as $f(\infty)$, in this case, the theorem holds also for $L=\infty$, similarly for $L=-\infty$.

**Theorem.** (Squeeze Theorem)
Let $\exists N$ so that the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are such that $a_n \le b_n \le c_n$ for every $n \ge N$. If $a_n \to L$ and $c_n \to L$, then also $b_n \to L$.

**Theorem.**
Let $f$ be a function on a certain $\langle n_0, \infty)$. We define $a_n = f(n)$ for $n \ge n_0$.
If $f$ is bounded (above, below) on $\langle n_0,\infty)$, also the sequence $\{a_n\}_{n=n_0}^{\infty}$ is bounded (above, below).
If $f$ is increasing (decreasing, not increasing, not decreasing) on $\langle n_0,\infty)$, also the sequence $\{a_n\}_{n=n_0}^{\infty}$ is increasing (decreasing, not increasing, not decreasing).
If $\lim\limits_{x\to\infty}\big(f(x)\big) = L$, then $\lim\limits_{n\to\infty}(a_n) = L$.

**Definition.** Let $\{a_n\}$ be a given sequence.
Its **subsequence** is a sequence $\{a_{n_k}\}$ for any possible choice of indexes (infinitely many are possible) such that $n_1 < n_2 < n_3 < \dots$.

**Theorem.**
If a sequence has a limit, then all its subsequences must have the same limit.

## Definitions and claims—Series of real numbers.

**Definition.** A **series** is a symbol $\sum\limits_{k=n_0}^{\infty} a_k = a_{n_0} + a_{n_0+1} + a_{n_0+2} + \dots$,
where $n_0 \in Z\!\!Z$, $a_k \in I\!R$ (series of real numbers).

**Definition.** Let $\sum\limits_{k=n_0}^{\infty} a_k$ be a series.
We define its **partial sums** by $s_N = \sum\limits_{k=n_0}^{N} a_k$ for $N \ge n_0$.
We say that the given series **converges** if $\{s_N\}_{N=n_0}^{\infty}$ converges.
We say that the given series **converges to** $A$, denoted $\sum\limits_{k=n_0}^{\infty} a_k = A$, if $\lim\limits_{N\to\infty}(s_N) = A$.
We say that the given series **diverges** if $\{s_N\}_{N=n_0}^{\infty}$ diverges.
We say that the given series **diverges to** $\infty$, denoted $\sum\limits_{k=n_0}^{\infty} a_k = \infty$, if $\lim\limits_{N\to\infty}(s_N) = \infty$.
We say that the given series **diverges to** $-\infty$, denoted $\sum\limits_{k=n_0}^{\infty} a_k = -\infty$, if $\lim\limits_{N\to\infty}(s_N) = -\infty$.

**Example.** $\sum\limits_{k=1}^{\infty}\frac{1}{2^k}$: $s_1 = \frac12$, $s_2 = \frac12+\frac14=\frac34$, $s_3 = \frac12+\frac14+\frac18=\frac78$, induction: $s_N = 1-\frac{1}{2^N}$, hence $s_N \to 1$ and $\sum\limits_{k=1}^{\infty}\frac{1}{2^k} = 1$ (series converges).

**Example.** $\sum\limits_{k=1}^{\infty} 1$: $s_1 = 1$, $s_2 = 1+1=2$, $s_3 = 1+1+1=3$, induction: $s_N = N$, hence $s_N \to \infty$ and $\sum\limits_{k=1}^{\infty} 1 = \infty$ (series diverges).

**Example.** $\sum\limits_{k=0}^{\infty}(-1)^k$: $s_0 = 1$, $s_1 = 1-1=0$, $s_2 = 1-1+1=1$, induction: $s_N = \begin{cases} 1, & N \text{ even;} \\ 0, & N \text{ odd,}\end{cases}$ thus $\lim\limits_{N\to\infty}(s_N)$ DNE and $\sum\limits_{k=0}^{\infty}(-1)^k$ diverges.

## Definitions and claims— Summing up a series.

**Definition.** Let $a,q \in I\!R$. The series $\sum\limits_{k=n_0}^{\infty} a\,q^k$ is called a **geometric series**.

**Fact.** (i) For $N \in I\!N_0$ we have $\sum\limits_{k=0}^{N} q^k = \dfrac{1-q^{N+1}}{1-q}$;
for $N \in I\!N$, $N \ge n_0$ we have $\sum\limits_{k=n_0}^{N} q^k = q^{n_0}\dfrac{1-q^{N+1-n_0}}{1-q} = \dfrac{q^{n_0}-q^{N+1}}{1-q}$.

(ii) We have $\sum\limits_{k=0}^{\infty} q^k \begin{cases} = \frac{1}{1-q}, & |q|<1; \\ = \infty \text{ (diverges)}, & q \geq 1; \\ \text{diverges}, & q \leq -1. \end{cases}$

More generally, $\sum\limits_{k=n_0}^{\infty} q^k = \dfrac{q^{n_0}}{1-q}$ for $|q|<1$.

**Definition.** Let $a,q \in I\!R$. The series $\sum\limits_{k=n_0}^{\infty} (a+qk)$ is called an **arithmetic series**.

**Fact.** (i) For $N \in I\!N_0$ we have $\sum\limits_{k=0}^{N} (a+qk) = (N+1)a + \frac{1}{2}N(N+1)q$.
(ii) An arithmetic series converges only if $a=q=0$.

Summing up a series: we can sum up directly only two kinds:
1) geometric series (might be in disguise):

**Example.** $\sum\limits_{k=2}^{\infty} \dfrac{5\cdot 3^{k-1}}{2^{2k+1}} = \sum\limits_{k=2}^{\infty} \dfrac{5\cdot 3^{-1}\cdot 3^k}{2^1\cdot (2^2)^k} = \frac{5}{6}\sum\limits_{k=2}^{\infty} \dfrac{3^k}{4^k} = \frac{5}{6}\sum\limits_{k=2}^{\infty} \big(\frac{3}{4}\big)^k = \frac{5}{6}\big(\frac{3}{4}\big)^2 \sum\limits_{k=0}^{\infty} \big(\frac{3}{4}\big)^k$
$= \langle\!\langle \big|\frac{3}{4}\big| < 1 \rangle\!\rangle = \frac{5}{6}\big(\frac{3}{4}\big)^2 \frac{1}{1-\frac{3}{4}} = \frac{15}{8}$.

Note: For a geometric series $\sum\limits_{k=n_0}^{\infty} q^k = q^{n_0} \sum\limits_{k=0}^{\infty} q^k$ is true in general.

Or substitution: $\sum\limits_{k=2}^{\infty} \big(\frac{3}{4}\big)^k = \left\langle\!\!\left\langle \begin{matrix} n=k-2 \implies k=n+2 \\ 2\mapsto 0,\ \infty\mapsto\infty \end{matrix} \right\rangle\!\!\right\rangle = \sum\limits_{n=0}^{\infty} \big(\frac{3}{4}\big)^{n+2} = \big(\frac{3}{4}\big)^2 \sum\limits_{n=0}^{\infty} \big(\frac{3}{4}\big)^n$.
This an be used for any series, sometimes I use notation $\langle\!\langle k-2 \mapsto k^* \rangle\!\rangle$.
2) telescopic series (might be in disguise):

**Example.** $\sum\limits_{k=3}^{\infty} \dfrac{1}{k(k-1)} = \sum\limits_{k=3}^{\infty} \big(\frac{1}{k-1} - \frac{1}{k}\big) = \big(\frac{1}{2}-\frac{1}{3}\big) + \big(\frac{1}{3}-\frac{1}{4}\big) + \big(\frac{1}{4}-\frac{1}{5}\big) + \dots$.

Induction: $s_N = \frac{1}{2} - \frac{1}{N} \to \frac{1}{2}$, hence $\sum\limits_{k=3}^{\infty} \dfrac{1}{k(k-1)} = \frac{1}{2}$.

Remark: formulas for finite sums:
$\sum\limits_{k=1}^{n} 1 = n$, $\sum\limits_{k=1}^{n} k = \frac{1}{2}n(n+1)$, $\sum\limits_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$, etc.

**Theorem.** Let series $\sum\limits_{k=n_0}^{\infty} a_k$, $\sum\limits_{k=n_0}^{\infty} b_k$ converge.

Then also the series $\sum\limits_{k=n_0}^{\infty} (a_k+b_k)$ converges and $\sum\limits_{k=n_0}^{\infty} (a_k+b_k) = \sum\limits_{k=n_0}^{\infty} a_k + \sum\limits_{k=n_0}^{\infty} a_k$.

For $c \in I\!R$ also $\sum\limits_{k=n_0}^{\infty} (c\,a_k)$ converges and $\sum\limits_{k=n_0}^{\infty} (c\,a_k) = c\Big(\sum\limits_{k=n_0}^{\infty} a_k\Big)$.

## Definitions and claims—-Convergence of series.

**Theorem.** Let $n_0 < n_1$, consider a series $\sum\limits_{k=n_0}^{\infty} a_k$. $\sum\limits_{k=n_0}^{\infty} a_k$ converges if and only if $\sum\limits_{k=n_1}^{\infty} a_k$ converges.

Then we also have $\sum\limits_{k=n_0}^{\infty} a_k = \sum\limits_{k=n_0}^{n_1-1} a_k + \sum\limits_{k=n_1}^{\infty} a_k$.

If we are only interested in convergence of a series and not its sum (if it exists), then we leave out the index specification.

**Theorem.** (**necessary condition** for convergence) If a series $\sum a_k$ converges, then necessarily $\lim\limits_{k\to\infty}(a_k) = 0$.

Equivalently: If $\lim\limits_{k\to\infty}(a_k) = 0$ is not true, then the series $\sum a_k$ necessarily diverges.

**Theorem.** Consider a series $\sum a_k$. If $a_k \geq 0$ for all $k$, then either $\sum a_k$ converges, or $\sum a_k = \infty$.

## Definitions and claims—- Tests for series with non-negative numbers.

**Theorem.** (**integral test**) Let $f \geq 0$ be a non-increasing function on $[n_0,\infty)$ for $n_0 \in Z\!\!\!Z$.

The series $\sum\limits_{k=n_0}^{\infty} f(k)$ converges if and only if $\int\limits_{n_0}^{\infty} f(x)\,dx$ converges.

Moreover we then have $\int\limits_{n_0}^{\infty} f(x)\,dx \leq \sum\limits_{k=n_0}^{\infty} f(k) \leq f(n_0) + \int\limits_{n_0}^{\infty} f(x)\,dx$.

**Example.** $\sum\limits_{k=3}^{\infty}\dfrac{1}{k\ln^2(k)}$: $\int\limits_{x=3}^{\infty}\dfrac{dx}{x\ln^2(x)}=\left|\begin{array}{l} y=\ln(x)\\ dy=\frac{dx}{x}\end{array}\right|=\int\limits_{x=\ln(3)}^{\infty}\dfrac{dy}{y^2}<\infty$. Therefore the series converges.

Moreover, $\sum\limits_{k=3}^{\infty}\frac{1}{k\ln^2(k)}\in\left[\frac{1}{\ln(3)},\frac{1}{3\ln^2(3)}+\frac{1}{\ln(3)}\right]\sim[0.91,1.19]$.

Trick: $\sum\limits_{k=3}^{\infty}\frac{1}{k\ln^2(k)}=\sum\limits_{k=3}^{9}\frac{1}{k\ln^2(k)}+\sum\limits_{k=10}^{\infty}\frac{1}{k\ln^2(k)}$

$\in\left[\sum\limits_{k=3}^{9}\frac{1}{k\ln^2(k)}+\int\limits_{10}^{\infty}\frac{dx}{x\ln^2(x)},\sum\limits_{k=3}^{9}\frac{1}{k\ln^2(k)}+\frac{1}{10\ln^2(10)}+\int\limits_{10}^{\infty}\frac{dx}{x\ln^2(x)}\right]\sim[1.059,1.078]$.

**Corollary.** ($p$-**test**) $\sum\frac{1}{k^p}$ converges if and only if $p>1$.

**Example.** $\sum\limits_{k=1}^{\infty}\dfrac{1}{k^2}=\dfrac{\pi^2}{6}$.

$\sum\limits_{k=1}^{\infty}\dfrac{1}{k}=\infty$ (**harmonic series**).

**Theorem.** (**comparison test**) Let $\exists n_0$ so that $0\le a_k\le b_k$ for $k\ge n_0$.
If $\sum b_k$ converges, then also $\sum a_k$ converges.
If $\sum a_k$ diverges, then also $\sum b_k$ diverges (i.e. $\sum a_k=\infty\implies\sum b_k=\infty$).
Remark: Symbolically (and roughly) $a_k\le b_k\implies\sum a_k\le\sum b_k$.

**Theorem.** (**limit comparison test**) Let $\exists n_0\in\mathbb{Z}$ so that $a_k\ge 0, b_k\ge 0$ for $k\ge n_0$.
If $a_k\sim b_k$, i.e. $\lim\limits_{k\to\infty}\left(\frac{a_k}{b_k}\right)=A>0$, then $\sum a_k\sim\sum b_k$, i.e. $\sum a_k$ converges if and only if $\sum b_k$ converges.

**Example.** $\sum\frac{1}{k^2+1}$: $0\le\frac{1}{k^2+1}\le\frac{1}{k^2}$ and $\sum\frac{1}{k^2}$ converges, therefore by CT also $\sum\frac{1}{k^2+1}$ converges.
Remark: Also IT and LCT would work here.

**Example.** $\sum\frac{1}{2k^2-1}$: $\frac{1}{2k^2-1}\ge\frac{1}{2k^2}\ge 0$, $\sum\frac{1}{2k^2}=\frac{1}{2}\sum\frac{1}{k^2}$ converges, but the inequality goes the wrong way, so no conclusion possible.
Guess $\frac{1}{2k^2-1}\sim\frac{1}{2k^2}$ for large $k$, confirm: $\lim\limits_{k\to\infty}\left(\frac{\frac{1}{2k^2-1}}{\frac{1}{2k^2}}\right)=1>0$,
$\sum\frac{1}{2k^2}=\frac{1}{2}\sum\frac{1}{k^2}$ converges, hence by LCT also $\sum\frac{1}{2k^2-1}$ converges.

**Example.** $\sum\frac{1}{k\ln^2(k)}$: Two comparisons seem reasonable, $\frac{1}{k^2}\le\frac{1}{k\ln^2(k)}\le\frac{1}{k}$, but both are in the wrong direction, so nothing here.
Limit comparison: No candidate, $\frac{1}{k\ln^2(k)}\sim\frac{1}{k}$ or $\frac{1}{k\ln^2(k)}\sim\frac{1}{k^2}$ definitely not true.
Thus comparison tests won't help.

**Theorem.** Let $a_k\ge 0$ for all $k$.
**ratio test**: (i) If $\exists q<1$ and $\exists n_0\in\mathbb{Z}$ such that $\forall k\ge n_0$: $\frac{a_{k+1}}{a_k}\le q$, then $\sum a_k$ converges.
(ii) If $\exists n_0\in\mathbb{Z}$ such that $\forall k\ge n_0$: $\frac{a_{k+1}}{a_k}\ge 1$, then $\sum a_k$ diverges ($=\infty$).
**limit ratio test**: Let $\lambda=\lim\limits_{k\to\infty}\left(\frac{a_{k+1}}{a_k}\right)$, assuming that the limit converges.
(i) If $\lambda<1$, then $\sum a_k$ converges.
(ii) If $\lambda>1$, then $\sum a_k$ diverges ($=\infty$).
**root test**: (i) If $\exists q<1$ and $\exists n_0\in\mathbb{Z}$ such that $\forall k\ge n_0$: $\sqrt[k]{a_k}\le q$, then $\sum a_k$ converges.
(ii) If $\exists n_0\in\mathbb{Z}$ such that $\forall k\ge n_0$: $\sqrt[k]{a_k}\ge 1$, then $\sum a_k$ diverges ($=\infty$).
**limit root test**: Let $\varrho=\lim\limits_{k\to\infty}\left(\sqrt[k]{a_k}\right)$, assuming that the limit converges.
(i) If $\varrho<1$, then $\sum a_k$ converges.
(ii) If $\varrho>1$, then $\sum a_k$ diverges ($=\infty$).

**Example.** $\sum\frac{k!}{2^k}$: Limit ratio test $\lambda=\lim\limits_{k\to\infty}\left(\frac{a_{k+1}}{a_k}\right)=\lim\limits_{k\to\infty}\left(\frac{(k+1)!}{k!}\frac{2^k}{2^{k+1}}\right)=\lim\limits_{k\to\infty}\left(\frac{1}{2}(k+1)\right)=\infty>1$.
Thus $\sum\frac{k!}{2^k}$ diverges.

**Example.** $\sum\frac{2}{\ln^k(k+1)}$: Limit root test $\varrho=\lim\limits_{k\to\infty}\left(\sqrt[k]{a_k}\right)=\lim\limits_{k\to\infty}\left(\frac{\sqrt[k]{2}}{\ln(k+1)}\right)=\frac{1}{\infty}=0<1$.
Thus $\sum\frac{2}{\ln^k(k+1)}$ converges.

**Example.** $\sum\left(\frac{k}{k+1}\right)^k$: Limit root test $\varrho=\lim\limits_{k\to\infty}\left(\sqrt[k]{a_k}\right)=\lim\limits_{k\to\infty}\left(\frac{k}{k+1}\right)=1$, no conclusion.
Similarly limit ratio fails. Integral test without chance, comparison as well.
But: $a_k=\left(1-\frac{1}{k+1}\right)^k\to e^{-1}\neq 0$, hence $\sum\left(\frac{k}{k+1}\right)^k$ diverges.

**Definitions and claims—- Tests for alternating series.**

**Theorem.** (**Alternating series test** or **Leibniz test**) Let $b_k \geq 0$ for all $k$ and let $\{b_k\}$ be non-increasing. The series $\sum(-1)^k b_k$ converges if and only if $\lim\limits_{k\to\infty}(b_k) = 0$.

**Example.** $\sum \frac{(-1)^k}{k}$: $b_k = \frac{1}{k} \geq 0$ is decreasing and $\to 0$, hence $\sum \frac{(-1)^k}{k}$ converges (compare with harmonic series).

**Definitions and claims—- Absolute convergence of series.**

**Definition.** We say that a series $\sum a_k$ **converges absolutely** if the series $\sum |a_k|$ converges.

**Theorem.** If a series $\sum a_k$ converges absolutely, then it also converges and we have $\Big|\sum\limits_{k=n_0}^{\infty} a_k\Big| \leq \sum\limits_{k=n_0}^{\infty} |a_k|$.

But not the other way around! Recall that $\sum \frac{(-1)^k}{k}$ converges, but $\sum\big|\frac{(-1)^k}{k}\big| = \sum \frac{1}{k} = \infty$.

**Definition.** We say that a series **converges conditionally** if it converges, but not absolutely.

Thus there are three possibilities now:

—$\sum a_k$ converges, $\sum |a_k|$ converges: absolute convergence (the second implies the first here)

—$\sum a_k$ converges, $\sum |a_k|$ diverges: conditional convergence

—$\sum a_k$ diverges, $\sum |a_k|$ diverges (the first implies the second)

**Example.** conditional convergence: $\sum \frac{(-1)^k}{k}$; absolute convergence: $\sum \frac{(-1)^k}{k^2}$; divergence: $\sum(-1)^k$.

**Example.** $\sum \frac{\sin(k)}{2^k}$: We do not know how to investigate this series directly. Its terms are not non-negative, therefore the tests won't work. We can't use AST, since the series is not alternating. The necessary condition won't help either, since $a_k \to 0$.

Thus we try the absolute convergence and hope that it will come out true, so that we have some conclusion: $\sum\big|\frac{\sin(k)}{2^k}\big| = \sum \frac{|\sin(k)|}{2^k} \leq \sum \frac{1}{2^k}$, this converges, therefore by comparison test also $\sum\big|\frac{\sin(k)}{2^k}\big|$ converges, hence $\sum \frac{\sin(k)}{2^k}$ converges absolutely.

**Example.** $\sum(-1)^k \frac{2^k}{k^3}$: absolute: $\sum\big|(-1)^k \frac{2^k}{k^3}\big| = \sum \frac{2^k}{k^3}$, ratio test: $\frac{a_{k+1}}{a_k} = 2\big(\frac{k}{k+1}\big)^3 \to 2 = \lambda > 1$,
thus $\sum\big|(-1)^k \frac{2^k}{k^3}\big|$ diverges, hence $\sum(-1)^k \frac{2^k}{k^3}$ does not converge absolutely. But we do not know whether it by itself does converge (then it would be conditional convergence) or not.

However, $\frac{2^k}{k^3} \to \infty$, thus $a_k = (-1)^k \frac{2^k}{k^3} \not\to 0$, so the series diverges.

**Theorem.** Consider a series $\sum\limits_{k=2n_0}^{\infty} a_k$.

If $\sum a_k$ converges absolutely, then also $\sum a_{2k}$ and $\sum a_{2k+1}$ converge and
$\sum\limits_{k=2n_0}^{\infty} a_k = \sum\limits_{k=n_0}^{\infty} a_{2k} + \sum\limits_{k=n_0}^{\infty} a_{2k+1}$.

Not true for conditional convergence, see $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k}$.

**Theorem.** Consider a series $\sum\limits_{k=n_0}^{\infty} a_k$.

If $\sum a_k$ converges absolutely, then for every choice of signs $\varepsilon_k = \pm 1$ also $\sum \varepsilon_k a_k$ converges.

If $\sum a_k$ converges conditionally, then there is a choice of signs $\varepsilon_k = \pm 1$ such that $\sum \varepsilon_k a_k = \infty$.

**Definition.** Consider a series $\sum\limits_{k=n_0}^{\infty} a_k$.

By a **rearrangement** of $\sum\limits_{k=n_0}^{\infty} a_k$ we mean any series $\sum\limits_{k=n_0}^{\infty} a_{\pi(k)}$, where $\pi$ is an arbitrary bijective mapping of $\{n_0, n_0+1, n_0+2, \dots\} \subset \mathbb{Z}$ onto $\{n_0, n_0+1, n_0+2, \dots\}$, i.e. $\pi$ is a permutation of $\{n_0, n_0+1, n_0+2, \dots\}$.

**Theorem.** Consider a series $\sum\limits_{k=n_0}^{\infty} a_k$.

If $\sum a_k$ converges absolutely, then also all its rearrangements $\sum a_{\pi(k)}$ converge and we have $\sum\limits_{k=n_0}^{\infty} a_{\pi(k)} = \sum\limits_{k=n_0}^{\infty} a_k$.

If $\sum\limits_{k=n_0}^{\infty} a_k$ converges conditionally, then $\forall c \in \mathbb{R} \cup \{\pm\infty\}$ there exists its rearrangement such that $\sum\limits_{k=n_0}^{\infty} a_{\pi(k)} = c$.

**Definitions and claims—- Sequences and series of functions.**

**Definition.** By a **sequence of functions** we mean an ordered set
$\{f_k\}_{k=n_0}^{\infty} = \{f_{n_0}, f_{n_0+1}, f_{n_0+2}, \dots\}$, where $f_k$ are functions.

Remark: Given a sequence of functions $\{f_k\}_{k=n_0}^{\infty}$ and $x \in \bigcap D(f_k)$, then $\{f_k(x)\}$ is a standard sequence of real (complex) numbers.

**Definition.** Let $\{f_k\}_{k\geq n_0}$, $f$ be functions on a set $M$.
We say that $\{f_k\}$ **converges (pointwise)** to $f$ on $M$, denoted $f_k \to f$ or $f = \lim\limits_{k\to\infty}(f_k)$,
if $\forall x \in M$: $\lim\limits_{k\to\infty}\bigl(f_k(x)\bigr) = f(x)$.

**Example.** Consider $f_k(x) = \arctan(kx)$. Then $\lim\limits_{k\to\infty}\bigl(f_k(x)\bigr) = \begin{cases} 0, & x = 0; \\ \frac{\pi}{2}, & x > 0; \\ -\frac{\pi}{2}, & x < 0. \end{cases}$

**Definition.** Let $\{f_k\}_{k\geq n_0}$, $f$ be functions on a set $M$.
We say that $\{f_k\}$ **converges uniformly** to $f$ on $M$, denoted $f_k \rightrightarrows f$,
if $\forall \varepsilon > 0 \exists N_0 \in \mathbb{N}$ such that $\forall k \geq N_0 \forall x \in M : |f(x) - f_k(x)| < \varepsilon$.

**Theorem.** Let $f_k \rightrightarrows f$ on $M$.
(i) If all $f_k$ are continuous on $M$, then also $f$ is continuous there.
(ii) If all $f_k$ have a derivative on $M$, then also $f$ has it there and $f' = \lim\limits_{k\to\infty}(f_k')$ on $M$.
(iii) If all $f_k$ have antiderivative on $M$, then also $f$ has it there and $\int_{x_0}^{x} f\,dx = \lim\limits_{k\to\infty}\bigl(\int_{x_0}^{x} f_k\,dx\bigr)$ for $\overline{x_0, x} \subseteq M$.

**Definition.** A **series of functions** is a symbol $\sum\limits_{k=n_0}^{\infty} f_k = f_{n_0} + f_{n_0+1} + f_{n_0+2} + \dots$, where $f_k$ are functions.

Remark: Given a series of functions $\sum f_k$ and $x \in \bigcap D(f_k)$, then $\sum f_k(x)$ is a standard series of real (complex) numbers.

**Definition.** Consider a series of functions $\sum\limits_{k=n_0}^{\infty} f_k$.
The **region of convergence** of this series is the set $\bigl\{x \in \bigcap D(f_k);\ \sum f_k(x) \text{ converges}\bigr\}$. By defining $f(x) = \sum\limits_{k=n_0}^{\infty} f_k(x)$ we then obtain a function $f$ on this set called the **sum of the series**, denoted $\sum\limits_{k=n_0}^{\infty} f_k = f$.
The **region of absolute convergence** of this series is the set
$\bigl\{x \in \bigcap D(f_k);\ \sum f_k(x) \text{ converges absolutely}\bigr\}$.
We say that this series **converges uniformly** to $f$ on $M$, denoted $\sum f_k \rightrightarrows f$ on $M$, if the sequence of partial sums $\Bigl\{\sum\limits_{k=n_0}^{N} f_k(x)\Bigr\}$ converges uniformly to $f$ on $M$.

**Theorem.** Consider series of functions $\sum f_k$ and $\sum g_k$.
If $\sum\limits_{k=n_0}^{\infty} f_k = f$ on $M$ and $\sum\limits_{k=n_0}^{\infty} g_k = g$ on $M$, then $\forall a, b \in \mathbb{R}$: $\sum\limits_{k=n_0}^{\infty}(af_k + bg_k) = af + bg$ on $M$.

**Theorem.** (Weierstrass criterion) Let $f_k$ for $k \geq n_0$ be functions on $M$. Let $a_k \geq 0$ satisfy $\forall x \in M \forall k \geq n_0$: $|f_k(x)| \leq a_k$.
If $\sum a_k$ converges, then $\sum f_k$ converges uniformly on $M$.

**Example.** $\sum x^k = \frac{1}{1-x}$ on $(-1, 1)$, but the convergence is not uniform. It will be uniform if we restrict our attention to $[-\varrho, \varrho]$ for $\varrho \in (0, 1)$.

**Theorem.** Let $\sum f_k \rightrightarrows f$ on $M$.
(i) If all $f_k$ are continuous on $M$, then also $f$ is continuous there.
(ii) If all $f_k$ have a derivative on $M$, then also $f$ has it there and $f' = \sum\limits_{k=n_0}^{\infty} f_k'$ on $M$.
(iii) If all $f_k$ have an antiderivative on $M$, then also $f$ has it there and $\int_{x_0}^{x} f\,dx = \sum\limits_{k=n_0}^{\infty} \int_{x_0}^{x} f_k\,dx$ for $\overline{x_0, x} \subseteq M$.

None of this is true in general for ordinary (pointwise) convergence.