Lineární algebra

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1) $\det(AB) = \det A \cdot \det B$

$\det(\underbrace{A \cdot A^{-1}}_{\det E = 1}) = \det A \cdot \det(A^{-1}) = 1$

2) $\det(A^{-1}) = \dfrac{1}{\det A}$

3) $\det A = \det A^T$

Spočtěte $det(AB^{-1}A^T)$, jestliže

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 3 & -1 \\ -1 & 2 & 2 \end{pmatrix} \begin{matrix} 1 & -1 \\ -2 & 3 \\ -1 & 2 \end{matrix}$$

$$\det(AB^{-1}A^T) \overset{1}{=} \det A \cdot \det(B^{-1}) \cdot \det(A^T) \overset{2,3}{=} \frac{(\det A)^2}{\det B} = 16$$

$\det A = 4$

$\det B = 1 = (6 - 1 - 8) - (-6 - 2 + 4)$

$= -3 - (-4) = 1$

$$\det A = 1 - 2 = -1 \neq 0$$

$$i\begin{pmatrix} - \cdots a_{ij} & \\ \vdots & \\ j & \end{pmatrix} \qquad (-1)^{i+j}|A_{ij}|$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Spočtěte inverzní matice k maticím

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A}\left(\text{Alg.Dop}\right)^T = -\begin{pmatrix} (-1)^{1+1}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} -2 & 0 \\ 0 & 1 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} \end{pmatrix}^T$$

$$= -\begin{pmatrix} 1 & +2 & -2 \\ +1 & 1 & -1 \\ -1 & -2 & 1 \end{pmatrix}^T = \begin{pmatrix} -1 & -1 & 1 \\ -2 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det B = ad - bc \neq 0$$

$$B^{-1} = \frac{1}{ad-bc} \begin{pmatrix} (-1)^{1+1} d & (-1)^{1+2} (-c) \\ (-1)^{2+1}(-b) & (-1)^{2+2} a \end{pmatrix}^T = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$XA - XB = B$

$X(A-B) = B$

$(A-B)^T X^T = B^T$

$X = B(A-B)^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}$

$\left((A-B)^T \mid B^T\right) \sim \left(E \mid X^T\right), \quad X =$

Vyjádřete z rovnice $XA = XB + B$ neznámou matici X.

Inverzní matice spočtěte pomoci determinantu a matice algebraických doplňku a rovnici vyřešte.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

$A - B = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}$

$\det(A-B) = -1(-1)^{1+2} \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + 1(-1)^{1+3} \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} = 1$

$(A-B)^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & +1 \\ -1 & +1 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$Ax = b$

A je typu $n \times n$

$\det A \neq 0$

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$A_1 = (b, a_2, \dots, a_n)$

$A_2 = (a_1, b, a_3 \dots, a_n)$

$A_n = (a_1, a_2, \dots a_{n-1}, b)$

$x_i = \dfrac{\det A_i}{\det A}$

Řešte soustavu Cramerovym pravidlem $\begin{pmatrix} 1 & 4 & | & -10 \\ 3 & -1 & | & 9 \end{pmatrix}$.

$A = \begin{pmatrix} 1 & 4 \\ 3 & -1 \end{pmatrix}, \det A = -1 - 12 = -13$

$\det A_1 = \begin{vmatrix} -10 & 4 \\ 9 & -1 \end{vmatrix} = 10 - 36 = -26$

$\det A_2 = \begin{vmatrix} 1 & -10 \\ 3 & 9 \end{vmatrix} = 9 + 30 = 39$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$x_1 = \dfrac{\det A_1}{\det A} = \dfrac{-26}{-13} = 2, \quad x_2 = \dfrac{\det A_2}{\det A} = \dfrac{39}{-13}$

$$\begin{cases} \text{pro } a \neq 0, a \neq \frac{1}{2} & \begin{pmatrix} 2/(2a-1) \\ 0 \\ -1/(2a-1) \end{pmatrix} \\ \text{pro } a = 0 & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \operatorname{span}\left(\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right) \\ \text{pro } a = \frac{1}{2} & \text{neex} \end{cases}$$

Řešte v závislosti na parametru $a \in \mathbb{R}$ soustavu

$$\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 2 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right).$$

1) $\det A = \begin{vmatrix} a & 1 & 1 \\ 0 & a & 0 \\ 1 & 0 & 2 \end{vmatrix}$ $R_2 - R_3$ $= a(-1)^{2+2} \begin{vmatrix} a & 1 \\ 1 & 2 \end{vmatrix} = a(2a-1) \neq 0$

pro $a \neq 0$ a $2a - 1 \neq 0$, $a \neq \frac{1}{2}$

pro $a \neq 0$ a $a \neq \frac{1}{2}$

$\det A_1 = \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 2 \\ 0 & 0 & 2 \end{vmatrix} = 2a$, $\det A_2 = \begin{vmatrix} a & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{vmatrix} = 0$ $1 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}$

$R_2 = R_3$

$$\det A_3 = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 0 \\ 1 & 0 & 0 \end{vmatrix} = -a$$

$$\begin{pmatrix} \frac{2}{2a-1} \\ 0 \\ \frac{-1}{2a-1} \end{pmatrix} \quad \text{pro } a \neq 0,\ a \neq \frac{1}{2}$$

2) $\underline{a=0}$

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array} \begin{array}{l} \\ R_2 \\ R_1 \\ R_3 - R_2 \end{array}$$

$x_3 = t$

$x_2 = 1-t$

$x_1 = -2t$

$t \in \mathbb{R}$

$$\underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\substack{x_3=0 \\ Ax=b}} + \operatorname{span}\Bigg(\underbrace{\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}}_{\substack{x_3=1 \\ Ax=0}}\Bigg)$$

$$\begin{pmatrix} -2t \\ 1-t \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}} \quad \underline{\underline{t \in \mathbb{R}}}$$

3) $a = \frac{1}{2}$

$$\left(\begin{array}{ccc|c} \frac{1}{2} & 1 & 1 & 1 \\ 1 & \frac{1}{2} & 2 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \begin{array}{l} R_1 - \frac{1}{2}R_3 \\ R_2 - R_3 \\ R_3 \end{array}$$

$\operatorname{rank} A = 2 \neq \operatorname{rank}(A|b) = 3$ řeš. neex.

$x_2 = 1$

$\frac{1}{2}x_2 = 0$

?!

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right) \begin{array}{l} R_3 \\ R_1 \\ 2R_2 - R_1 \end{array}$$

Napište všechna řešení soustavy v závislosti na parametru $a \in \mathbb{R}$

$$\left(\begin{array}{ccc|c} a^2 & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right).$$

$A_3 = A$

$$\det A = \begin{vmatrix} a^2-1 & 0 & 0 \\ 0 & a-1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \\ R_3 \end{array} = (a^2-1)(a-1) = (a-1)^2(a+1) = 0$$

pro $a = 1$
nebo $a = -1$

1) pro $a = 1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$x_1 + x_2 + x_3 = \cancel{1}\,0$$

$$x_1 = 1 - x_2 - x_3$$

$$x_2 = t$$

$$x_3 = s$$

$$\begin{pmatrix} 1-t-s \\ t \\ s \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$t, s \in \mathbb{R}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \operatorname{span}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$t = s = 0$

$Ax = b$

$Ax = 0$

$t = 1$, $s = 0$

$t = 0$, $s = 1$

2) $a = -1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1\;0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$-2x_2 = 0$ $\quad x_2 = 0$

$x_1 + x_3 = 1$, $x_3 = t$

$x_1 = 1 - t$

$$\begin{pmatrix} 1-t \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad t \in \mathbb{R}$$

$A = A_3$

3) pro $a \neq \pm 1$

$$\det A_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & 1 \end{vmatrix} = \underline{0}$$

$$\underbrace{\det A_3 = \begin{vmatrix} a^2 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & 1 \end{vmatrix} = \det A}$$

$$\det A_2 = \begin{vmatrix} a^2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \underline{0}$$

$$1 = \frac{\det A_3}{\det A} \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

pro $a \neq \pm 1$ $\qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

pro $a = 1$ $\qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \operatorname{span}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$

pro $a = -1$ $\qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \operatorname{span}\left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$

neex $\qquad\qquad$ $\det A \neq 0$

Najděte všechny hodnoty parametrů $a, b, c \in \mathbb{R}$, pro které má soustava jediné řešení

$$\left(\begin{array}{ccc|c} b+c & a+c & a+b & 2 \\ a & b & c & 23 \\ 1 & 1 & 1 & 11 \end{array}\right).$$

pro $a = b = c = 0$ $\qquad$ $0 = 2$, $0 = 23$, $x_1 + x_2 + x_3 = 11$

$$\begin{vmatrix} b+c & a+c & a+b \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = (b+c)\begin{vmatrix} b & c \\ 1 & 1 \end{vmatrix} - (a+c)\begin{vmatrix} a & c \\ 1 & 1 \end{vmatrix} + (a+b)\begin{vmatrix} a & b \\ 1 & 1 \end{vmatrix}$$

$$= (b+c)(b-c) - (a+c)(a-c) + (a+b)(a-b)$$

$$= b^2 - c^2 - (a^2 - c^2) + a^2 - b^2 = 0$$

Řešte v závislosti na parametru $a \in \mathbb{R}$ soustavu

$$\left(\begin{array}{ccc|c} a+1 & 1 & 3 & 1 \\ 8 & 2 & a+3 & 2 \\ 3 & 1 & 2 & -1 \end{array}\right).$$