Lineární algebra

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$v \neq 0 \qquad f(v) = \lambda v$

Najděte vlastní čísla a vlastní vektory lineárního zobrazeni

$f : \mathbb{R}^{\leq 1}[x] \to \mathbb{R}^{\leq 1}[x] \; : \; f(a+bx) = 2a + (a+3b)x.$

$B = (1, x) \qquad A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$

$1 \mapsto 2 + x$

$x \mapsto 3x$ vl. vektor x (vl. číslo 3)

$\det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) = 0$

$\lambda_1 = 2, \; \lambda_2 = 3$

vl. čísla

$\lambda_1 = 2$ $\quad A - 2E = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ $\quad p_1(x) = 1 - x$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\lambda_2 = 3$ $\quad A - 3E = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$ $\quad p_2(x) = x$

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$A \sim D$ $\quad \exists$ regul. P $\quad$ P^{-1} : $\quad A = PDP^{-1}$

$$P^{-1}AP = D$$

$C = (1-x, \; x)$ $\quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $\quad \underline{\underline{P = T_{C \to B}}} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

$B = (1, x)$

$der: \mathbb{R}^{\leq 3}[x] \to \mathbb{R}^{\leq 3}[x]$

$ax^3+bx^2+cx+d \mapsto 3ax^2+2bx+c$

$B=(x^3, x^2, x, 1)$ $\qquad x^3 \mapsto 3x^2$

$$A=\begin{pmatrix} 0&0&0&0\\ 3&0&0&0\\ 0&2&0&0\\ 0&0&1&0 \end{pmatrix}$$

$$\det(A-\lambda E)=\begin{vmatrix} -\lambda&0&0&0\\ 3&-\lambda&0&0\\ 0&2&-\lambda&0\\ 0&0&1&-\lambda \end{vmatrix}=\lambda^4=0$$

$$v=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$(A-\lambda E)v=0$

$A-0\cdot E=A$

pro $\lambda=0$ vl. vektor je 1

$$J=\begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix}$$

$$J_1=\begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix}$$

$$J_2=\begin{pmatrix} 0&1&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix}$$

$B = (x^2, x, 1)$

1) $f(ax^2 + bx + c) = 2(ax^2 + bx + c)$

Najděte nějake zobrazení $f : \mathbb{R}^{\leq 2}[x] \to \mathbb{R}^{\leq 2}[x]$ tak, aby polynom $x^2 + x + 1$ byl jeho vlastním vektorem příslušným vlastnímu číslu 2.

$\boxed{f(x^2 + x + 1) = 2(x^2 + x + 1)}$

$C = (x^2 + x + 1, x, 1)$

2) $\begin{cases} f(x^2 + x + 1) = 2(x^2 + x + 1) \\ f(x) = 0 \\ f(1) = 0 \end{cases}$

ANO v bázi $\left(\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}-1\\0\\1\end{pmatrix}\right)$ je $D=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$

Rozhodněte, zda matice $A = \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$ je podobná diagonální matici, tj. zda existuje regulárni matice P taková, že $A = PDP^{-1}$, kde D je diagonální.

$$\det(A-\lambda E) = \begin{vmatrix}-\lambda&0&1\\0&1-\lambda&0\\1&0&-\lambda\end{vmatrix} = (1-\lambda)\begin{vmatrix}-\lambda&1\\1&-\lambda\end{vmatrix} = (1-\lambda)(\lambda^2-1)$$

$$= -(\lambda-1)^2(\lambda+1)$$

$$\lambda_{1,2}=1,\quad \lambda_3=-1$$

pro $\lambda_{1,2} = 1$, $A - E = \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \end{array}$ $\quad x - z = 0 \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

jsou LN

pro $\lambda_3 = -1$ $\quad A + E = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ $\quad \begin{matrix} x + z = 0 \\ 2y = 0 \end{matrix} \quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$J = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & -1 \end{array}\right)$$

$$N = A - E = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \qquad \begin{array}{l} \operatorname{rank} N = 1 \\ \operatorname{def} N = 2 \end{array}$$

$$N^2 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{pmatrix} = -2N$$

i	$d_i = \operatorname{def}(N^i)$	počet buněk $i \times i$
0	0	
1	2	$2\cdot 2 - 0 - 2 = 2$
2	2	

1×1

Jordanův tvar

$$\begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_n \end{pmatrix}$$

$$J = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

Najděte Jordanův tvar matice $A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$$\det(A - \lambda E) = \begin{vmatrix} 3-\lambda & 1 & -1 \\ 0 & (2-\lambda) & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (2-\lambda)\big((3-\lambda)(1-\lambda)+1\big)$$

$$= (2-\lambda)(3-4\lambda+\lambda^2+1)$$

$$= (2-\lambda)^3 = 0, \quad \lambda_{1,2,3} = 2$$

vl. v. v_1, v_2, v_3

$Av = 2v$

$B = (v_1, v_2, v_3)$

$f(v_2) = v_1 + 2v_2$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$f(v_1)$ $f(v_2)$ $f(v_3)$

$$A-2E = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$x+y-z=0$$

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$N = A - \lambda E$ je nilpotentní $\exists k$ $N^k = 0$, $N^{k-1} \neq 0$

$k = \mathrm{nil}(N)$

$d_i = \mathrm{def}(N^i)$ $i = 0, 1, \dots, k+1$, $d_{k+1} = d_k$

Počet Jord. buněk rozměru $i \times i$ je

$$2d_i - d_{i-1} - d_{i+1}, \quad i = 1, \dots, k$$

$$N = A - 2E = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \end{array}$$

$\mathrm{rank}\, N = 1$

$\mathrm{def}\, N = 2$

$$N^2 = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\mathrm{rank}\, N^2 = 0$

$\mathrm{def}\, N^2 = 3$

i	$d_i = \deg N_i$	počet buněk $i \times i'$
0	0	$2 \cdot 2 - 0 - 3 = 1$
1	2	$2 \cdot 3 - 2 - 3 = 1$
2	3	
3	3	

$N^0 = E$

$$\det A = (1-\lambda)(1-\lambda)(4-\lambda) - 4 = (1-2\lambda+\lambda^2)(4-\lambda) - 4$$
$$= 4 - 8\lambda + 4\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 - 4 = -\lambda(\lambda^2 - 6\lambda + 9) = -\lambda(\lambda-3)^2$$

Najděte Jordanův tvar matice $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -4 \\ -1 & 0 & 4 \end{pmatrix}$

$$\det(A - \lambda E) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & -4 \\ -1 & 0 & 4-\lambda \end{vmatrix} = -\lambda(\lambda-3)^2$$

$\lambda_1 = 0 \quad \lambda_{2,3} = 3$

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$J = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 3 & 1 \\ 0 & 0 & 3 \end{array}\right)$$

2 buňky 1×1

1 buňka 2×2

$$N = A - 3E = \begin{pmatrix} -2 & -1 & 0 \\ 0 & -2 & -4 \\ -1 & 0 & 1 \end{pmatrix}$$

rank $N = 2$

def $N = 1$

$$\begin{pmatrix} -2 & -1 & 0 \\ 0 & -2 & -4 \\ -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix} \begin{array}{l} R_3 \\ \frac{1}{2}R_2 \\ R_1 - 2R_3 \end{array}$$

$-x + z = 0$

$y + 2z = 0$

$$v_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$N^2 = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 1 & 1 & 1 \end{pmatrix}$$

rank $N^2 = 1$

def $N^2 = 2$

$$N^3 = -3N^2$$

i	$d_i = \mathrm{def}(N^i)$	počet buněk $i \times i$
0	0	
1	1	$2 \cdot 1 - 0 - 2 = 0$
2	2	$2 \cdot 2 - 1 - 2 = 1$
3	2	

$$\begin{pmatrix} -2 & -1 & 0 \\ 0 & -2 & -4 \\ -1 & 0 & 1 \end{pmatrix} \left(\begin{array}{cc|c} -2 & -1 & 0 \\ 0 & -2 & -4 \\ -1 & 0 & 1 \end{array}\right) = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 1 & 1 & 1 \end{pmatrix}$$