Lineární algebra

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$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad \alpha\vec{y} = \begin{pmatrix} \alpha y_1 \\ \alpha y_2 \\ \alpha y_3 \end{pmatrix}$

$\left\lfloor \frac{x_1 y_1 + x_2 y_2 + x_3 y_3}{x_1^2 + x_2^2 + x_3^2} \right\rfloor$

$y_1 x_2 \quad y_1 x_3 \quad y_2 x_3$

Ověřte, že $\langle \vec{x} | \vec{y} \rangle = 2x_1y_1 + x_1y_2 + 2x_1y_3 + x_2y_1 + 2x_2y_2 + x_2y_3 + 2x_3y_1 + x_3y_2 + 5x_3y_3$

je skalární součin v $\mathbb{R}^3$.

$\langle x|y\rangle = \langle y|x\rangle$?

$\langle x|x\rangle = \underline{2x_1^2 + 2x_1x_2 + 4x_1x_3} + 2x_2^2 + 2x_2x_3 + 5x_3^2 \geq 0$?

$= 2(x_1^2 + x_1x_2 + 2x_1x_3) + \underline{2x_2^2 + 2x_2x_3 + 5x_3^2}$

$= 2\left(x_1^2 + 2x_1\left(\frac{1}{2}x_2\right) + 2x_1x_3 + \left(\frac{1}{2}x_2\right)^2 + x_3^2 + 2\frac{1}{2}x_2x_3\right)$

$- \frac{1}{2}x_2^2 - 2x_3^2 - \cancel{2x_2x_3} + 2x_2^2 + \cancel{2x_2x_3} + 5x_3^2$

$= 2\left(x_1 + \frac{1}{2}x_2 + x_3\right)^2 + \frac{3}{2}x_2^2 + 3x_3^2 \geq 0$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

$a = x_1$

$b = \frac{1}{2} x_2$

$c = x_3$

$\langle x | x \rangle = 0$ iff.

$x_1 + \frac{1}{2} x_2 + x_3 = 0 \Rightarrow x_1 = 0$

$x_2 = 0$

$x_3 = 0$

$x = 0$

$$G = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix} \end{array}$$

$G = G^T$

$\det G_1 = 2 > 0$

$\det G_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0$

$\det G_3 = 9 > 0$

Ať L je lin. prostor nad $\mathbb{R}$.
Zobrazení $\langle -|-\rangle : L \times L \to \mathbb{R}$ říkáme skalární součin, pokud pro libovolné vektory $\vec{x}, \vec{y} \in L$ platí

- komutativita: $\langle \vec{x}|\vec{y}\rangle = \langle \vec{y}|\vec{x}\rangle$
- linearita ve druhé složce: zobrazení $\langle \vec{x}|-\rangle : L \to \mathbb{R}$ je lineární
- pozitivní definitnost: $\langle \vec{x}|\vec{x}\rangle \geq 0$, $\langle \vec{x}|\vec{x}\rangle = 0$ iff $\vec{x} = \vec{o}$

$\forall y, z \in L$
$\forall \alpha \in \mathbb{R}$

$\langle x|\alpha y\rangle = \alpha \langle x|y\rangle$
$\langle x|y+z\rangle = \langle x|y\rangle + \langle x|z\rangle$

$\langle e_1 | e_2 \rangle = 1$

$\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - i$

Gramova matice: $\boxed{\langle \vec{x} | \vec{y} \rangle = \vec{x}^T G \vec{y}}$

- $G = (g_{ij})_{i,j=1\dots n}$, kde $g_{ij} = \langle \vec{e}_i | \vec{e}_j \rangle$
- symetrická: $G^T = G$ a determinanty všech $G_k = (g_{ij})_{i,j=1\dots k}$, $1 \le k \le n$, jsou kladné
- pozitivně definitní: $G = R^T R$, kde R je regulární
- vlástní čísla G jsou reálná a kladná

4. $\langle x|x\rangle = x_1^2 + 2x_1x_3 + 2x_3^2 = (x_1+x_3)^2 + x_3^2 \geq 0$

$\langle x|x\rangle = 0$ iff $x_1+x_3=0$, $x_3=0$ $\quad x_1=x_3=0$

$G = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

$\langle e_2|e_2\rangle = 0$, $e_2 \neq 0$

$\det G = 0$

Zjistěte, zda daná operace je skalárním součinem na $\mathbb{R}^3$

1. $\langle \vec{x}|\vec{y}\rangle = x_1y_1 + \underbrace{x_1y_2 - x_2y_1} + 2x_2y_2 + 3x_3y_3$ není
2. $\langle \vec{x}|\vec{y}\rangle = x_1y_1 + x_2y_2 - x_3y_3$ není
3. $\langle \vec{x}|\vec{y}\rangle = 2(x_1y_1 + x_2y_2 + x_3y_3)$
4. $\langle \vec{x}|\vec{y}\rangle = x_1y_1 + x_1y_3 + x_3y_1 + 2x_3y_3$ není

1. $\langle x|y\rangle \neq \langle y|x\rangle$, protože $\langle e_1|e_2\rangle = 1$, $\langle e_2|e_1\rangle = -1$

$G = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ není sym. $\quad G \neq G^T$

2. $\langle e_3|e_3\rangle = -1$,

$\langle x|x\rangle = x_1^2 + x_2^2 - x_3^2$

$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\quad \det G = -1$

$\langle \vec{x} | \vec{y} \rangle =$

$$2x_1y_1 + \cancel{x_1y_2} + 2x_1y_3 + \cancel{x_2y_1} + \cancel{2x_2y_2} + \cancel{x_2y_3} + 2x_3y_1 + \cancel{x_3y_2} + 5x_3y_3$$

Pro vektory $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ a $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ spočtěte $\langle \vec{u} | \vec{v} \rangle$, $||\vec{u}||$, $||\vec{v}||$, $\cos\varphi$, $d(\vec{u}, \vec{v})$.

$x_2 = y_2 = 0$

Ověřte platnost nerovnosti Cauchy-Schwarz-Bunyakovsky

$|\langle u | v \rangle| \leq ||u|| \cdot ||v||$

$18 \leq \sqrt{30} \cdot \sqrt{11}$

$18^2 = 324 \leq 30 \cdot 11 = 330$

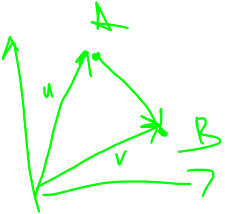
Spočtěte $proj_{\vec{v}}(\vec{u})$ a $rej_{\vec{v}}(\vec{u})$

$\langle u | v \rangle = 2 + 2 + 4 + 10 = 18$

$$(1,0,2) \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = (1,0,2) \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix} = 4 + 0 + 14 = 18$$

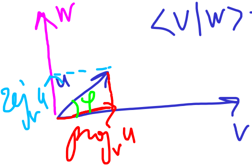
$$\|u\| = \sqrt{\langle u|u\rangle} = \sqrt{2+4+4+20} = \sqrt{30}$$

$$\|v\| = \sqrt{\langle v|v\rangle} = \sqrt{2+2+2+5} = \sqrt{11}$$



$$\cos\varphi = \frac{\langle u|v\rangle}{\|u\|\cdot\|v\|} = \frac{18}{\sqrt{30}\cdot\sqrt{11}}$$

$$d(u,v) = \|u-v\| = \left\| \begin{pmatrix}0\\0\\1\end{pmatrix} \right\| = \sqrt{5}$$



$$\langle v|w\rangle = 0$$

$$proj_v u = \|u\|\cdot\cos\varphi\cdot\frac{v}{\|v\|}$$

$$= \cancel{\|u\|}\,\frac{\langle u|v\rangle}{\cancel{\|u\|}\cdot\|v\|}\cdot\frac{v}{\|v\|} =$$

$$= \frac{\langle u|v\rangle}{\langle v|v\rangle}\cdot v = \frac{18}{11}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}\frac{18}{11}\\0\\\frac{18}{11}\end{pmatrix}$$

$$rej_v u = u - proj_v u =$$

$$= \begin{pmatrix}1\\0\\2\end{pmatrix} - \begin{pmatrix}18/11\\0\\18/11\end{pmatrix} = \begin{pmatrix}-7/11\\0\\4/11\end{pmatrix}$$

(c_1, c_2, c_3) je ortogonální báze

$\langle c_1|c_2\rangle = 0$
$\langle c_1|c_3\rangle = 0$
$\langle c_2|c_3\rangle = 0$

$\left(\frac{c_1}{\|c_1\|}, \frac{c_2}{\|c_2\|}, \frac{c_3}{\|c_3\|}\right)$ je ortonorm. báze

$\langle \vec{x}|\vec{y}\rangle =$
$2x_1y_1 + x_1y_2 + 2x_1y_3 + x_2y_1 + 2x_2y_2 + x_2y_3 + 2x_3y_1 + x_3y_2 + 5x_3y_3$

Proveďte Gram-Schmidtův ortogonalizační proces na bázi $(\vec{e_1}, \vec{e_2}, \vec{e_3})$

Najděte ortonormální bázi.

$\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right)$

1) $c_1 = e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$

2) $c_2 = e_2 - \alpha c_1$
$= e_2 - \mathrm{proj}_{c_1} e_2$
$= e_2 - \frac{1}{2} e_1 = \begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}$

$\langle c_1|c_2\rangle = 0$
$\langle c_1|e_2 - \alpha c_1\rangle = 0$
$\langle c_1|e_2\rangle - \alpha\langle c_1|c_1\rangle = 0, \quad \alpha = \frac{\langle c_1|e_2\rangle}{\langle c_1|c_1\rangle}$

3) $c_3 = e_3 - \alpha c_1 - \beta c_2$ $\qquad \langle c_3 | c_1 \rangle = 0$

$\langle c_3 | c_2 \rangle = 0$

$= e_3 - \mathrm{proj}_{\mathrm{span}(c_1, c_2)} e_3$

$= e_3 - \frac{\langle e_3 | c_1 \rangle}{\langle c_1 | c_1 \rangle} c_1 - \frac{\langle e_3 | c_2 \rangle}{\langle c_2 | c_2 \rangle} c_2 = e_3 - e_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\langle e_3 | e_1 \rangle = 2$ $\qquad \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \right\rangle = 2 \cdot 1 \left(-\frac{1}{2}\right) + 1 \cdot 1 = 0$

$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$ ortogonální báze

$\left\langle \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \right\rangle = \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + 2$

$= \frac{3}{2}$

$\left(\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \right)$ ortonorm. báze

$\left\langle \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \middle| \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle = 2 - 2 - 2 + 5$

$= 3$

$B=(b_1, b_2, b_3)$ $\quad \langle b_i | b_j \rangle = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

Nechť $\langle - | - \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ je skalární součin na $\mathbb{R}^3$ a

$B = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$ je jeho ortonormální báze.

Najděte hodnotu $\left\langle \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\rangle = \langle \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 | \beta_1 b_1 + \beta_2 b_2 + \beta_3 b_3 \rangle$

$= \alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3 =$

$\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3$

$= (\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = (2, -2, 2) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$= 0 - 2 + 2 = 0$

$\left(\begin{array}{ccc|cc} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{array} \right) \sim \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 2 & 1 \end{array} \right) \begin{array}{l} R_2 \\ R_1 - R_2 \\ R_3 - (R_1 - R_2) \end{array}$

$$\operatorname{coord}_B \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} \qquad \operatorname{coord}_B \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$G_K = (B^{-1})^T B^{-1} \qquad G_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$G_K = (T_{K\to B})^T \underbrace{G_B}_{E} T_{K\to B}$$

Nalezněte skalární součin, pro než je báze $B = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$ ortonormální.

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = T_{B\to K}, \quad B^{-1} = T_{K\to B} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

$$G=\begin{pmatrix}0&1&-1\\1&-1&1\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&-1&0\\-1&1&1\end{pmatrix}=\begin{pmatrix}2&-2&-1\\-2&3&1\\-1&1&1\end{pmatrix}$$

$$\left\langle \begin{pmatrix}0\\2\\0\end{pmatrix} \middle| \begin{pmatrix}1\\0\\2\end{pmatrix}\right\rangle = (0,2,0)\begin{pmatrix}2&-2&-1\\-2&3&1\\-1&1&1\end{pmatrix}\begin{pmatrix}1\\0\\2\end{pmatrix}=(0,2,0)\begin{pmatrix}0\\0\\1\end{pmatrix}=0$$

$$\langle x|y\rangle = x^T G y = 2x_1y_1-2x_1y_2-x_1y_3$$
$$-2x_2y_1+3x_2y_2+x_2y_3$$
$$-x_3y_1+x_3y_2+x_3y_3$$

$$(x_1,x_2,x_3)\,G\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$$